Last time: Conservative vector fields

Let **F** be the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = \langle y \cos xy + 2xy, xcosxy + 2e^{2y} + x^2 \rangle.$$

Find f such that $\nabla f = F$; check your work when you're done.

(a) I'm done, I found f.

(b) It is not possible to find f; it must be that **F** is not conservative.

(c) I don't know what to do.

Solution:

We want f such that $\nabla f = \langle y \cos xy + 2xy, xcosxy + 2e^{2y} + x^2 \rangle$.

Step 1: We must have $f_x(x, y) = y \cos xy + 2xy$.

This tells us that $f(x, y) = \sin xy + x^2 y + h(y)$.

Step 2: From step 1, we get that $f_y(x, y) = x \cos xy + x^2 + h'(y)$.

But we also want to ensure that $f_y(x, y) = xcosxy + 2e^{2y} + x^2$, so we must have $h'(y) = 2e^{2y}$.

Step 3: We conclude that $h(y) = e^{2y} + k$ (we can take $k = 0$).

Step 4: So we can take $f(x, y) = \sin xy + x^2 y + e^{2y}$.

Step 5: We double check our answer.

Announcements:

Midterm 2 is next Tuesday, March 12.

Next Wednesday, March 13, there will be lecture as usual.

But next Friday, March 15, there will be no lecture.

Last time:

Theorem

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path $\Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves C.

Theorem (A)

If $\mathbf{F}$ is a vector field on D, and D is open and connected, then $\mathbf{F}$ is conservative $\Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.

Is the converse true?

That is, if we know $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, can we conclude that $\mathbf{F}$ is conservative?

Answer: Not always.

We need some conditions on the domain D.

Example: simply connected sets

Which of the following sets are open and simply connected?

1. $\mathbb{R}^2$
2. $\{(x, y) \mid (x, y) \neq (0, 0)\}$

(a) Only (1).

(b) Only (2).

(c) Both (1) and (2).

(d) Neither (1) nor (2).

(e) I don't know.

Correct answer: (a)

Is **F** conservative?

Let $\mathbf{F} = \langle P, Q \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}.$

Is $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$? Is **F** conservative?

(a) Yes and Yes

(b) Yes and No

(c) No and Yes

(d) No and No

(e) I don't know

Solution

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{-1}{x^2+y^2} + \frac{-(-y)(2y)}{(x^2+y^2)^2} \\ &= \frac{-x^2-y^2}{(x^2+y^2)^2} + \frac{2y^2}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2}.\end{aligned}$$

$$\begin{aligned}\frac{\partial Q}{\partial x} &= \frac{1}{x^2+y^2} + \frac{-(x)(2x)}{(x^2+y^2)^2} \\ &= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2}.\end{aligned}$$

Solution

So $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, but we can't use Theorem B, because D is not simply connected:

$$D = \{(x, y) \mid (x, y) \neq (0, 0)\}.$$

So Theorem B doesn't give us information about whether or not **F** is conservative.

However, if we compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $C = \{x^2 + y^2 = 1\}$ the unit circle, we get -2π, which is not 0.

So by method A', **F** is not conservative.