MATH 241 - LECTURE 26 - INTEGRATING IN ~~POLAR~~ CYLINDRICAL AND SPHERICAL COORDINATES - Wed. March 27, 2019

$\boxed{1}$ Let E be the solid bounded by the cylinder $x^2+y^2=1$, the paraboloid $z=1-x^2-y^2$, and the plane $z=2$.

Find D, $u_1(x,y)$, $u_2(x,y)$ such that

$$E=\left\{(x,y,z) \;\middle|\; \begin{array}{l}(x,y)\in D\\ u_1(x,y)\le z\le u_2(x,y)\end{array}\right\}.$$

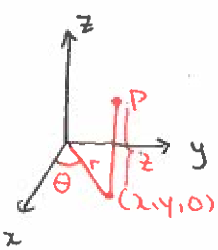
[see slide for answer]

· TODAY: TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES. (§§15.7, 15.8)

Note that the region $D=\{(x,y) \mid x^2+y^2\le 1\}$ is easier to deal with in polar coordinates:

$$D=\left\{(r,\theta) \;\middle|\; \begin{array}{l}0\le\theta\le 2\pi\\ r\le 1\end{array}\right\}.$$

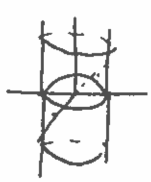
Definition: the cylindrical coordinates of a point $P=(x,y,z)$ are (r,θ,z) where



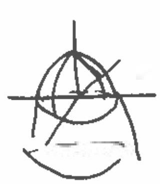
- $r\cos\theta=x$, $r\sin\theta=y$
- $z=z$.

Examples

- The set $\{r=1\}$ describes the cylinder of radius 1.



- The set $z=1-r^2$ describes the paraboloid $z=1-x^2-y^2$



from first question above
↓
So $E=\left\{(r,\theta,z) \;\middle|\; \begin{array}{l}0\le\theta\le 2\pi\\ 0\le r\le 1\\ 1-r^2\le z\le 2\end{array}\right\}$

INTEGRATING IN CYLINDRICAL COORDINATES.

Suppose E is of the form $\left\{ (r,\theta,z) \;\middle|\; \begin{array}{l} \alpha \le \theta \le \beta \\ g_1(\theta) \le r \le g_2(\theta) \\ h_1(r,\theta) \le z \le h_2(r,\theta) \end{array} \right\}$ where $\beta - \alpha \le 2\pi$.

$D = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} \alpha \le \theta \le \beta \\ g_1(\theta) \le r \le g_2(\theta) \end{array} \right\}$.

<u>Theorem:</u>

$$\iiint_E f(x,y,z)\,dV = \iint_D \underbrace{\left(\int_{\text{bound}}^{\text{bound}} f(x,y,z)\,dz \right)}_{\text{function } G \text{ of } (x,y)} dA$$

$$= \iint_D G(x,y)\,dA$$

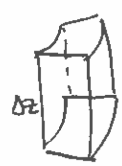
$$= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} G(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$\boxed{= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta}$$

- Recall – the extra factor of r comes from

$A\Big($  $\Big) \approx r\,\Delta\theta\,\Delta r$.

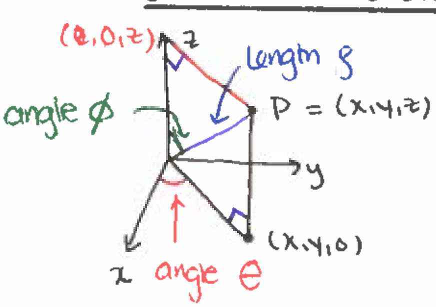
or equivalently

$V\Big($  $\Big) \approx r\,\Delta\theta\,\Delta r\,\Delta z$ (polar box).

<u>Example:</u> Set up the integral to find the moment of ~~inte~~ inertia about the z-axis of a solid with shape E and constant density ρ_0.

[See slides]

SPHERICAL COORDINATES:



- ρ is the length of $\langle x,y,z \rangle$

$\Rightarrow \rho = \sqrt{x^2+y^2+z^2} \ge 0$.

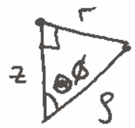
- ϕ (phi) is the angle between $\langle x,y,z \rangle$ and $\langle 0,0,1 \rangle$

$\Rightarrow 0 \le \phi \le \pi$.

$\cos\phi = z/\rho$

θ is the cylindrical coordinate: $\cos\theta = \frac{y}{x}$

Note from the triangle



that if $r = \sqrt{x^2+y^2}$ (as in cylindrical coordinates)

then $r = \rho \sin\phi$.

So given the three values (ρ, θ, ϕ) we can find x, y, z:

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$

$$y = r\sin\theta = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\theta.$$

Example: $B = \{(\rho, \theta, \phi) \mid \rho = 1\}$

consists of all points at distance 1 from $(0,0,0)$.

$\Rightarrow$ B is the unit sphere

(easier equation to work with than $\{x^2+y^2+z^2 \leq 1\}$)

On the earth:

- ρ is altitude (up/down)
- ϕ is latitude (north/south)
- θ is longitude (east/west)

Example: Let $E = \{(x,y,z) \mid 1 \leq x^2+y^2+z^2 \leq 4,\ z \geq 0\}$.

- In spherical coordinates,

$$\left\{\begin{array}{l} 0 \leq \theta \leq 2\pi \\ 1 \leq \rho \leq 2 \\ 0 \leq \phi \leq \pi \end{array}\right\}$$

INTEGRATING IN SPHERICAL COORDINATES.

- we work with a "spherical wedge"

- this has
  - width $\Delta\rho$
  - length $r\Delta\theta = \rho\sin\phi\,\Delta\theta$
  - height $\rho\,\Delta\phi$

$\Rightarrow$ volume $\approx \rho^2\sin\phi\,\Delta\rho\,\Delta\theta\,\Delta\phi$

Theorem: For $B = \left\{ (\rho, \theta, \phi) \;\middle|\; \begin{array}{l} a \le \rho \le b \\ \alpha \le \theta \le \beta \\ c \le \phi \le d \end{array} \right\}$ and f on B continuous,

we have
$$\iiint_B f\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\;d\rho\,d\theta\,d\phi.$$

Example: Find the average value of x, y, z ~~volume~~ of $E = \left\{ \begin{array}{l} 0 \le \theta \le 2\pi \\ 1 \le \rho \le 2 \\ 0 \le \phi \le \pi \end{array} \right\}$

$$\text{average of } f = \frac{\iiint_E f\,dV}{\iiint_E dV}$$

1) $$\iiint_E dV = \int_0^\pi \int_0^{2\pi} \int_1^2 \rho^2\sin\phi\;d\rho\,d\theta\,d\phi$$
$$= \int_0^\pi \int_0^{2\pi} \left[\tfrac{1}{3}\rho^3\right]_1^2 \sin\phi\;d\theta\,d\phi$$
$$= 2\pi \int_0^\pi \left[\tfrac{7}{3}\sin\phi\right] d\phi$$
$$= \frac{14\pi}{3} \left[-\cos\phi\right]_0^\pi$$
$$= \frac{28\pi}{3}.$$

2) $$\iiint_E x\,dV = \int_0^\pi \int_0^{2\pi} \int_1^2 (\rho\sin\phi\cos\theta)\,\rho^2\sin\phi\;d\rho\,d\theta\,d\phi \ldots$$