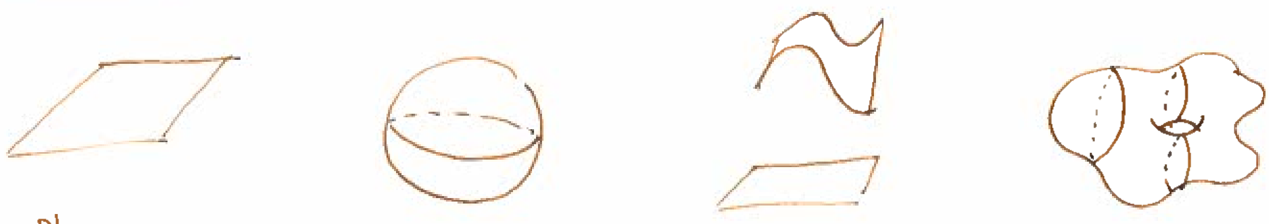
Definition: A surface is a set in $\mathbb{R}^3$ that
- looks locally like $\mathbb{R}^2$

  $\Updownarrow$
- can be made out of sheet metal.

Examples:



$f: D \to \mathbb{R}$, $D \subset \mathbb{R}^2$

Parametrization:

Recall: a curve is parametrized by a vector-valued function
$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle \qquad a \le t \le b.$$
* needs one variable t.

A surface is parametrized by two variables
$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad (u,v) \in D \subset \mathbb{R}^2.$$

Note: $\vec{r}$ should be 1-1, except possibly on the boundary of D.

Example: parametrize the plane through P containing $\vec{v}$ and $\vec{w}$: $\vec{r}(u,v) = P + u\vec{v} + v\vec{w}$.

Example: parametrize the sphere of radius 1 around $\vec{0}$.

$\Rightarrow$ in spherical coordinates, $\rho = 1$.

take as parameters ϕ, θ

$\Rightarrow$ $x(\phi,\theta) = \sin\phi\cos\theta$, $y(\phi,\theta) = \sin\phi\sin\theta$ $z(\phi,\theta) = \cos\phi$.

$$D = \left\{(\phi,\theta) \;\middle|\; \begin{array}{l} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}\right\}.$$

☑ Change the domain

Example: parametrize the graph of $f: \mathbb{R}^2 \to \mathbb{R}$.

$\Gamma_f = \{(x,y,z) \mid z = f(x,y)\}$.

$\Rightarrow$ $x = u$, $y = v$, $z = f(u,v)$. $\qquad -\infty < u,v < \infty$.

Example Parametrize the cylinder $y^2 + z^2 = 4$.

$(x,y,z) = (x, 2\cos\theta, 2\sin\theta)$ cylindrical coordinates

↳ parameters (x,θ) $-\infty < x < \infty$

$0 \leq \theta \leq 2\pi$.

$\boxed{2}$ Let $D = \left\{(u,v) \;\middle|\; \begin{array}{l} -\frac{\pi}{2} \leq v \leq \frac{\pi}{2} \\ 0 \leq u \leq 2\pi \end{array} \right\}$, and define $\vec{r}: D \to \mathbb{R}^3$ by

$\vec{r}(u,v) = (\sin v, \cos u \sin 2v, \sin u \sin 2v)$

Remarks:
- this is a surface of revolution
- the curves $\vec{r}(u_0,v)$ and $\vec{r}(u,v_0)$ are called grid curves.

§ TANGENT PLANES.

- Let $P = \vec{r}(u_0,v_0)$ be a point in S.
  - it lies on the grid curve C_1 given by $\vec{r}(u_0,v)$ and C_2 given by $\vec{r}(u,v_0)$.

    the tangent lines to these curves at P have direction vector $\vec{r}_v(u_0,v_0)$ and $\vec{r}_u(u_0,v_0)$.

Definition: we say that S is smooth at P if

$$\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0) \neq \vec{0}.$$

In that case, the vectors $\vec{r}_u(u_0,v_0)$, $\vec{r}_v(u_0,v_0)$ determine a unique plane $\mathbb{P}$ containing both these vectors and the point P

- with normal vector $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$.

Definition: this plane is the tangent plane to S at P.

Example Consider S parametrized by $D = [0,2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$\vec{r}(u,v) = (\sin v, \cos u \sin 2v, \sin u \sin 2v)$ as above.

Find the tangent plane to S at $P = (\frac{1}{\sqrt{2}}, -1, 0)$

Step 1: Find (u_0, v_0) with $\vec{r}(u_0,v_0) = P = (\frac{1}{\sqrt{2}}, -1, 0)$

- $\sin v = \frac{1}{\sqrt{2}} \Rightarrow v = \frac{\pi}{4}$.
- $\cos u \sin 2v = \cos u \sin\frac{\pi}{2} = \cos u = -1$

$\Rightarrow u = \pi$.

(check: $\sin u \sin 2v = \sin \pi \sin\frac{\pi}{2} = 0$ ☺).

Step 2: Find $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0). = \vec{n}$

- $\vec{r}_u(u,v) = \langle 0, -\sin u \sin 2v, \cos u \sin 2v \rangle$

$\Rightarrow \vec{r}_u(u_0,v_0) = \langle 0, 0, -1 \rangle$

- $\vec{r}_v(u,v) = \langle \cos v, 2\cos u \cos 2v, 2\sin u \cos 2v \rangle$

$\Rightarrow \vec{r}_v(u_0,v_0) = \langle \frac{1}{\sqrt{2}}, 0, 0 \rangle$

- $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{vmatrix} = \vec{i}\cdot 0 - \vec{j}\left(\frac{1}{\sqrt{2}}\right) + \vec{k}\cdot 0$

$= \langle 0, -\frac{1}{\sqrt{2}}, 0 \rangle$

Step 3: Find the equation of the plane with $\vec{n} = \langle 0, -\frac{1}{\sqrt{2}}, 0 \rangle$, passing through $P = (\frac{1}{\sqrt{2}}, -1, 0)$:

$$-\frac{1}{\sqrt{2}}(y+1) = 0$$

i.e. $y = -1$. (check with picture)

__Example__ Parametrize the paraboloid $z = x^2 + y^2$.

$\boxed{\text{ii}}$ Find the tangent plane to the paraboloid at $(1,1,2)$.

[Solution on slides]