Last time: change of coordinates in three-dimensions

Theorem

Let T be a transformation from $D \subset \mathbb{R}^3$ to $\mathbb{R}^3$ such that

- *D and $T(D)$ are "nice";*
- *$\frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$ and T is continuous on D except possibly on the boundary.*

Suppose f is a continuous function on $T(D)$. Then

$$\iiint_{T(D)} f(x,y,z)\,dV_{xyz} = \iiint_D f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV_{uvw}.$$

Participants Needed for a Research Study!

Message from PhD student Emily Teitelbaum

My dissertation research seeks to contribute to higher education professionals' understanding of racism from the perspective of White American students.

The study will involve three 1-hour, in-person interviews with me. Participants who complete interviews will receive a $10 gift card for each interview.

Are you:

- A full-time undergraduate student enrolled at UIUC;
- Self-identifying as White and American;
- At least 18 years old;
- Willing to be audio-recorded?

Please contact Emily Teitelbaum at erteite2@illinois.edu if interested!

Practice with parametrizing surfaces

We just parametrized the sphere $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. If we only want to parametrize the lower half of the sphere, $S_h = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \leq 0\}$, we should keep the same function $\mathbf{r}(\phi, \theta)$, but we should change the domain so that

(a) ϕ is in $[0, \frac{\pi}{2}]$;

(b) ϕ is in $[\frac{\pi}{2}, \pi]$;

(c) θ is in $[0, \pi]$;

(d) θ is in $[\pi, 2\pi]$;

(e) I don't know.

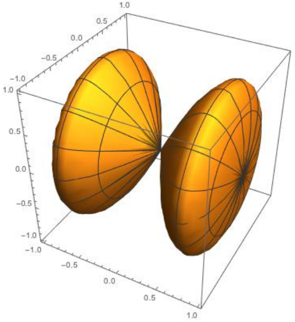
The correct answer is (b).

Let $D = \{(u, v) \mid 0 \leq u \leq 2\pi,\ -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}\}$.

Define $\mathbf{r} : D \to \mathbb{R}^3$ by

$$\mathbf{r}(u, v) = (\sin v, \cos u \sin 2v, \sin u \sin 2v).$$

For any choice of $v = v_0$, the function $u \mapsto \mathbf{r}(u, v_0)$ defines a curve on the surface S. What do these curves look like?



(a) The curves which form circles on S.

(b) The curves on S which pass through the origin $(0, 0, 0)$.

(c) None of these.

(d) Both of these.

(e) I don't know.

Practice with tangent planes

Find one or more partners. Parametrize the paraboloid $z = x^2 + y^2$.

Use the parametrization to find the tangent plane to the paraboloid at $(1, 1, 2)$.

(a) We can't find a parametrization.

(b) We found a parametrization $\mathbf{r}(u, v)$, but we can't find (u_0, v_0) such that $\mathbf{r}(u_0, v_0) = (1, 1, 2)$.

(c) We found a parametrization and the point (u_0, v_0), but we can't find the normal vector to the tangent plane.

(d) We found the normal vector, but we can't remember how to write down the equation to the plane.

(e) We did it!

Solution

We parametrize the paraboloid by setting

$$x = u \qquad y = v \qquad z = u^2 + v^2.$$

This gives $\mathbf{r}(u, v) = \langle u, v, u^2 + v^2 \rangle$.

We want to find (u_0, v_0) such that $\mathbf{r}(u_0, v_0) = \langle 1, 1, 2 \rangle$, so we must solve

$$\langle u_0, v_0, u_0^2 + v_0^2 \rangle = \langle 1, 1, 2 \rangle.$$

We obtain

$$u_0 = 1; \qquad v_0 = 1.$$

Solution

So we have $\mathbf{r}(u,v) = \langle u, v, u^2 + v^2 \rangle$, $\mathbf{r}(1,1) = \langle 1,1,2 \rangle$. Now we need to compute $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$.

$$\begin{aligned}
\mathbf{r}_u(u,v) &= \langle 1, 0, 2u \rangle \\
\Rightarrow \mathbf{r}_u(1,1) &= \langle 1, 0, 2 \rangle. \\
\mathbf{r}_v(u,v) &= \langle 0, 1, 2v \rangle \\
\Rightarrow \mathbf{r}_v(1,1) &= \langle 0, 1, 1 \rangle.
\end{aligned}$$

So $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$ is given by

$$\begin{aligned}
\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix} &= \mathbf{i}(-2) - \mathbf{j}(2) + \mathbf{k}(1) \\
&= \langle -2, -2, 1 \rangle.
\end{aligned}$$

Solution

So the tangent plane is the plane through $P = (1, 1, 2)$ with normal vector $\langle -2, -2, 1 \rangle$.

This plane has equation

$$0 = -2(x - 1) - 2(y - 1) + (z - 2),$$

or equivalently

$$0 = 2x + 2y - z - 2.$$