MATH 241 - INTEGRATING VECTOR FIELDS ON SURFACES. (§16.7) Mon. 15 April 2019

Preliminary stuff:
- choose a problem from 2018 midterm to review
- announcements about the exam.

	function $f(x,y,z)$	vector field $\vec{F}(x,y,z)$
curve $C \subset \mathbb{R}^3$	$\int_C f\,ds = \int_a^b f(\vec{r}(t))\lvert\vec{r}\,'(t)\rvert\,dt$	$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot\vec{T}\,ds$
surface $S \subset \mathbb{R}^3$	$\iint_S f\,dS = \iint_D f(\vec{r}(u,v))\lvert\vec{r}_u\times\vec{r}_v\rvert\,dA$	?? Today: find out!

Remarks [see slides]

1) doesn't depend on choice of parametrization

depends on parametrization only up to orientation.

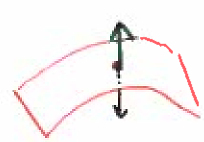
2) For curves we looked at the tangent vector (and its length).

For surfaces, there are infinitely many directions for tangent vectors

But there's a unique line of normal vectors "$\{\vec{r}_u\times\vec{r}_v\}$"

3) For curves, the choice of orientation tells us whether we take tangent vector pointing to the left or the right

For surfaces a choice of orientation will tell us whether to take the normal vector pointing up or down.



Definition: Let S be a surface in $\mathbb{R}^3$.
S is orientable if it has two sides
- inside & outside / top & bottom.
- you can paint them two different colours.

<u>Definition</u>: S is oriented if we choose one of the sides.

$\Rightarrow$ then we always know to take the unit normal vector pointing out of the chosen side; we write $\vec{n}$.

Ⓘ Take a strip of paper, and tape the ends together to form a cylinder. Is it orientable?

Ⓘ Now tape the ends together with a half-twist. This is a Möbius strip. Is it orientable?

Let $\vec{F}$ be the (continuous) velocity field of a fluid flow on $\mathbb{R}^3$.

Let S be an oriented surface, with $\vec{n}$ the unit normal vector.

<u>Definition</u>. the flux of $\vec{F}$ across S is

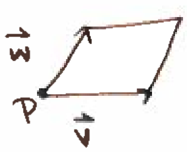
$$\iint_S \vec{F}\cdot d\vec{S} := \iint_S \vec{F}\cdot\vec{n}\, dS$$

<u>Note</u>: if $-S$ is S with the opposite orientation, then

$$\iint_{-S} \vec{F}\cdot d\vec{S} = -\iint_S \vec{F}\cdot d\vec{S}.$$

<u>Claim</u>: The flux is the (signed) volume of fluid flowing across S in unit time.

<u>Why?</u> Cover S with tiny patches S_{ij} and approximate each S_{ij} by parallelograms.



with $\vec{n} = \dfrac{\vec{v}\times\vec{w}}{|\vec{v}\times\vec{w}|}$ over all of S_{ij}

$\vec{F} \approx$ constant $\vec{F}(P)$ over S_{ij}

$\Rightarrow$ flux across S_{ij} is $\underbrace{\vec{F}\cdot\vec{n}}_{\text{component of } \vec{F} \text{ in the direction of } \vec{n}}\ \Delta A$.

Let S be a smooth oriented surface with unit normal vector $\vec{n}$.

Parametrize S by $\vec{r}(u,v)$, $(u,v)\in D\subset\mathbb{R}^2$.

$\vec{r}_u\times\vec{r}_v$ is perpendicular to S at each point.

<u>Def</u>. $\vec{r}$ is positively oriented if $\dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|} = \vec{n}$

and negatively oriented if $\dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|} = -\vec{n}$.

Theorem Let $\vec{F}$ be a continuous vector field on S.

If $\vec{r}_u \times \vec{r}_v$ is positively oriented, then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$

proof:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v\, dA. \qquad \square$$

Example: Let S be $\{z = \sqrt{x^2+y^2},\ 1 \le z \le 3\}$, oriented downward, and let $\vec{F} = \langle x, y, z^2 \rangle$.

Step 1: parametrize S:

$$\vec{r}(u,\theta) = \langle u\cos\theta, u\sin\theta, u \rangle, \quad 0 \le \theta \le 2\pi, \quad 1 \le u \le 3.$$

Step 2: Find $\vec{r}_u \times \vec{r}_\theta$ & compare to $\vec{n}$.

- $\vec{r}_u = \langle \cos\theta, \sin\theta, 1 \rangle$; $\vec{r}_\theta = \langle -u\sin\theta, u\cos\theta, 0 \rangle$

$$\Rightarrow \vec{r}_u \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -u\sin\theta & u\cos\theta & 0 \end{vmatrix} = \langle -u\cos\theta, -u\sin\theta, u \rangle$$

$\uparrow$ points upward, not downward.

So $\vec{r}_u \times \vec{r}_\theta$ is negatively oriented

Step 3: Calculate $\iint_S \vec{F} \cdot d\vec{S}$:

$$\iint_S \vec{F} \cdot d\vec{S} = -\iint_D \vec{F}(\vec{r}(u,\theta)) \cdot (\vec{r}_u \times \vec{r}_\theta)\, dA$$

$$= -\int_0^{2\pi} \int_1^3 \langle u\cos\theta, u\sin\theta, u^2 \rangle \cdot \langle -u\cos\theta, -u\sin\theta, u \rangle\, du\, d\theta$$

$$= -\int_0^{2\pi} \int_1^3 -u^2 + u^3\, du\, d\theta \quad \cdots$$

Example: Let S be the graph of a function $f: D \to \mathbb{R}$, oriented upward.

Let $\vec{F}$ be a continuous vector field on S.

Find $\iint_S \vec{F}(x,y,z) \cdot d\vec{S}$.

Step 1: parametrize S:

$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle, \quad (u,v) \in D.$

Step 2: Calculate $\vec{r}_u \times \vec{r}_v$ and compare to $\vec{n}$.

$\vec{r}_u = \langle 1, 0, f_x \rangle, \quad \vec{r}_v = \langle 0, 1, f_y \rangle$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle .$$

$\boxed{2}$ Is this positively or negatively oriented?

Step 3: Calculate $\iint_S \vec{F} \cdot d\vec{S}$.