Theory of Stochastic Processes
Vol. 3 (19), no. 3–4, 1997, pp. 393–402

194

UDC 519.21

EUGENE SENETA AND TUHAO CHEN

A SEQUENTIALLY REJECTIVE TEST PROCEDURE

We present a refinement of Holm's (1979) simple multiple test procedure of the sequentially rejective type. This paper is centred on the use of degree two upper bounds for unions of events, while retaining the key property (1) below. In particular, an adaptive procedure is proposed.

1. The Setting

Consider the simple hypotheses $H_1, H_2, \ldots, H_n$ in a multiple test problem, where the respective test statistics $X_1, \ldots, X_n$ are univariate, with X_i having continuous distribution function F_i when its hypothesis H_i is true. Suppose also that each test is upper tail. The sequel is concerned with a simple sequentially rejective multiple test procedure following ideas in Holm (1979). In particular we require that the following property hold: If the set $\{H_i, i \in I\}$ is the set of true hypotheses (where I may be any non-null proper subset of $\{1, \ldots, n\}$), then

$$P(H_i, i \in I, \text{ are accepted}) \geq 1 - \alpha \tag{1}$$

for pre-specifed size of test α.

We focus on the random variable $R_i = 1 - F_i(X_i)$, $i = 1, \ldots, n$, and the corresponding ordered sequence $R_{t_1} < R_{t_2} < \ldots < R_{t_n}$. Denote the hypotheses corresponding to the ordered values by H_{t_i}, $i = 1, \cdots, n$. If H_i is true, $R_i = 1 - F_i(X_i)$ is the p-value of the test for H_i, and is well-known to have uniform distribution on $[0, 1]$.

2. The "Bonferroni" Adjustment

When $I = \{1, \ldots, n\}$, that is when all H_i, $i = 1, \ldots, n$ are assumed true, then for given α,

$$\mathsf{P}(R_{t_1} \leq \alpha/n) = \mathsf{P}\left(\bigcup_{i=1}^{n}\{R_i \leq \alpha/n\}\right) \leq \sum_{i=1}^{n} \mathsf{P}(R_i \leq \alpha/n) = n\alpha/n = \alpha.$$

1991 *AMS Mathematics Subject Classification.* 62F03, 60E15.
Key words and phrases. simultaneous hypotheses, P-values, Hunter's inequality.
The work of the second author was done in part at the University of Sydney.

We have used Boole's (first Bonferroni) inequality, and then the uniformity of distribution of R_i under H_i. Thus the compound hypothesis that all H_i, $i = 1, \dots, n$, are true may be accepted at α level of testing if $R_{t_1} > \alpha/n$, and rejected if $R_{t_1} \leq \alpha/n$. This is a well-known procedure, though often formulated in terms of X_i's rather than R_i's.

3. The Simple Sequentially Rejective Procedure

The preceding approach has been refined by Holm (1979) to produce a subset, possibly null, of the hypotheses $\{H_i, i = 1, \dots, n\}$ to be rejected (with the complement, possibly null, to be accepted), the procedure satisfying requirement (1) for arbitrary fixed I. This procedure consists in examining if $R_{t_1} \leq \alpha/n$; if this inequality is not satisfied, H_i, $i = 1, \dots, n$, are all accepted (as above). Otherwise the corresponding hypothesis H_{t_1} is rejected and discarded along with its test statistic X_{t_1}, and the process is restarted with the remaining $(n-1)$ test statistics and hypotheses. Thus one now examines whether $R_{t_2} \leq \alpha/(n-1)$; if the inequality is not satisfied accept $H_{t_2}, \dots, H_{t_n}$. Otherwise further discard H_{t_2} and its test statistic X_{t_2}. Continue in this way. To summarize, if $R_{t_1} \leq \alpha/n, R_{t_2} \leq \alpha/(n-1), \dots, R_{t_{p-1}} \leq \alpha/(n-p+2)$, then at step p the remaining hypotheses are $H_{t_p}, \dots, H_{t_n}$ and the inequality next to be checked is $R_{t_p} \leq \alpha/(n-p+1)$. The process may at most run until a decision is made on the basis of whether $R_{t_n} \leq \alpha$ or not.

The argument for justifying (1), given by Holm (1979) is as follows, (where m is the number of elements in I, and H_i, $i \in I$, recall, are the hypotheses being assumed true): firstly,

$$\begin{aligned}(2)\qquad \bigcap_{i\in I}\left\{R_i > \frac{\alpha}{m}\right\} &\Rightarrow \left\{\text{Every hypothesis } H_i \text{ with } R_i > \frac{\alpha}{m} \text{ is accepted}\right\} \\ &\Rightarrow \{H_i, i \in I, \text{ are accepted}\}.\end{aligned}$$

Next, since $H_i, i \in I$, are assumed true

$$(3)\qquad \mathsf{P}\left(\bigcap_{i\in I}\left\{R_i > \frac{\alpha}{m}\right\}\right) = 1 - \mathsf{P}\left(\bigcup_{i\in I}\left\{R_i \leq \frac{\alpha}{m}\right\}\right) \geq 1 - \alpha$$

by Boole's inequality as before. Thus (1) holds.

4. An Extended Sequentially Rejective Procedure

The essence of our extension is to use degree two inequalities; these are sharper than Boole's, in that probabilities of joint events are utilized. The use of such inequalities in such settings is not new (see the survey: Seneta, 1993) although the application has not been widespread, because one needs to be able to calculate the probabilities of joint events.

The theme of this paper is to replace $\alpha/(n-p+1)$ by $\Delta(p) = \alpha/(n-p+1) + \beta(p)$, $p = 1, \dots, n-1$ in the procedure of §3 above where $\beta(p) > 0$, while retaining the property (1).

With $\alpha, R_1, R_2, \cdots, R_n$ as in §1, $0 < \alpha < 1$, $V(p)$ any fixed subset $\{v_1, \cdots, v_p\}$ of p elements of $\{1, 2, \cdots, n\}$, and $\Pi(V(p))$ the set of all permutations of the set $V(p)$, for $p = 1, 2, \cdots, n$

$$(4) \qquad \beta(p) = \frac{1}{(n-p+1)} \min_{V(n-p+1)} \left\{ \max_{\Pi(V(n-p+1))} \left\{ \sum_{i=2}^{n-p+1} \max_{1 \le s \le i-1} p(v_i, v_s) \right\} \right\}$$

with

$$p(v_i, v_s) = \mathsf{P}\left(R_{v_i} \le \frac{\alpha}{n-p+1},\ R_{v_s} \le \frac{\alpha}{n-p+1} \right),$$

where the probabilities are calculated under the assumption that all H_i, $i = 1, \cdots, n$, are true. We shall always interpret $\sum_{i=2}^{1}$ as zero, so here $\beta(n) = 0$.

For $p = 1, 2, \cdots, n$ let

$$(5) \qquad \Delta(p) = \frac{\alpha}{n-p+1} + \beta(p).$$

The test procedure is now analogous to that described in §3 with $\Delta(p)$ defined by (5), beginning with a check of whether $R_{t_1} \le \Delta(1)$. The general step p: if hypotheses $H_{t_1}, \cdots, H_{t_{p-1}}$ have all been discarded, the inequality next to check is

$$R_{t_p} \le \Delta(p).$$

Note that $\Delta(p) > \alpha/(n-p+1)$. Thus the test procedure is more powerful than Holm's, at the cost of obtaining the $\beta(p)$'s, providing $\Delta(p)$, $p = 1, 2, \cdots, n$, is strictly increasing.

This last requirement is automatically satisfied if one uses a generally smaller but simpler value of $\beta(p)$ viz.

$$\frac{n-p}{n-p+1} \min_{i<j} \mathsf{P}\left(R_i \le \frac{\alpha}{n-p+1},\ R_j \le \frac{\alpha}{n-p+1} \right).$$

To see this note that then, for $p = 1, 2, \cdots, n$,

$$\Delta(p) = \frac{\alpha}{n-p+1} + \left(1 - \frac{1}{n-p+1}\right) \min_{i<j} \mathsf{P}\left(R_i \le \frac{\alpha}{n-p+1},\ R_j \le \frac{\alpha}{n-p+1} \right)$$

so for $p = 1, 2, \cdots, n-1$

$$\Delta(p+1) \ge \frac{\alpha}{n-p} + \left(1 - \frac{1}{n-p}\right) \min_{i<j} \mathsf{P}\left(R_i \le \frac{\alpha}{n-p+1},\ R_j \le \frac{\alpha}{n-p+1} \right)$$

and

$$\begin{aligned}
&\Delta(p+1) - \Delta(p) \\
&\quad \ge \left[\frac{1}{n-p} - \frac{1}{n-p+1}\right] \left[\alpha - \min_{i<j} \mathsf{P}\left(R_i \le \frac{\alpha}{n-p+1},\ R_j \le \frac{\alpha}{n-p+1} \right)\right] \\
&\quad \ge \left[\frac{1}{n-p} - \frac{1}{n-p+1}\right] \left[\alpha - \mathsf{P}\left(R_i \le \frac{\alpha}{n-p+1} \right)\right] \\
&\quad > 0,
\end{aligned}$$

since

$$\mathsf{P}\left(R_i \le \frac{\alpha}{n-p+1}\right) = \frac{\alpha}{n-p+1} \le \frac{\alpha}{2}, \qquad p = 1, 2, \cdots, n-1.$$

Several more powerful variants of Holm's procedure (e.g. Simes', Hochberg's, Hommel's) are referenced in a paper of Wright (1992); for an earlier monographic discussion, especially of Holm's procedure, see Hochberg and Tamhane (1987). These variants, like Holm's procedure, have the positive feature of being based on probabilities of single events (degree 1; ours involves probabilities of intersections and so is of degree 2); but also the feature that property (1) has not been theoretically verified. We now show that our modification satisfies (1).

We first note Hunter's (1976) inequality, in the non-graph-theoretic form obtained by Margaritescu (1986; for a simple iterative proof see Seneta, 1988), for any set of events $A_1, A_2, \cdots, A_k$:

$$\mathsf{P}\left(\bigcup_i A_i\right) \le \sum_{i=1}^{k} \mathsf{P}(A_i) - \max_{\Pi} \sum_{i=2}^{k} \max_{1 \le s \le i-1} \mathsf{P}\left(A_i \bigcap A_s\right) \tag{6}$$

where Π is the set of all permutations of the subscript set $\{1, 2, \cdots, k\}$.

Theorem. *If the set $\{H_i, i \in I\}$ is the set of true hypotheses (where I may be any non-null subset of $\{1, \cdots, n\}$) then $\mathsf{P}(H_i, i \in I$ are accepted$) \ge 1 - \alpha$ for any prespecified size of test α, providing $\Delta(p)$, $p = 1, 2, \cdots, n$ is strictly increasing, where $\beta(p)$ is defined by (4).*

Proof. Let n be the cardinality of the set of hypotheses I being assumed true, and put $\alpha(m) = m\Delta(n - m + 1) = \alpha + m\beta(n - m + 1)$. Then

$$\bigcap_{i \in I} \left\{R_i > \frac{\alpha(m)}{m}\right\} \Rightarrow \left\{\text{Every hypothesis } H_i \text{ with } R_i > \frac{\alpha(m)}{m} \text{ is accepted}\right\}$$
$$\Rightarrow \{H_i, i \in I, \text{ are accepted }\}$$

since $\Delta(p)$ is non-decreasing with p (by Lemma 1).

Now

$$\begin{aligned}\mathsf{P}\left(\bigcap_{i \in I} \{R_i > \frac{\alpha(m)}{m}\}\right) &= 1 - \mathsf{P}\left(\bigcup_{i \in I} \left\{R_i \le \frac{\alpha(m)}{m}\right\}\right) \\ &\ge 1 - \sum_{i=1}^{m} \mathsf{P}\left(R_{w_i} \le \frac{\alpha(m)}{m}\right) \\ &\quad + \max_{\Pi(I)} \left\{\sum_{i=2}^{m} \max_{1 \le s \le i-1} p'(w_i, w_s)\right\}\end{aligned}$$

with

$$p'(w_i, w_s) = \mathsf{P}\left(R_{w_i} \le \frac{\alpha(m)}{m},\ R_{w_s} \le \frac{\alpha(m)}{m}\right)$$

by (6), writing $I = \{w_1, w_2, \cdots, w_m\}$. Since $\beta(n-m+1) \geq 0$ and since $\Delta(n-m+1) = \alpha(m)/m < 1$

$$\mathsf{P}\left(\bigcap_{i\in I}\left\{R_i > \frac{\alpha(m)}{m}\right\}\right) \geq 1 - \sum_{i=1}^{m} \mathsf{P}\left(R_{w_i} \leq \frac{\alpha(m)}{m}\right) + \max_{\Pi(I)}\left\{\sum_{i=2}^{m} \max_{1\leq s\leq i-1} \mathsf{P}\left(R_{w_i} \leq \frac{\alpha}{m},\ R_{w_j} \leq \frac{\alpha}{m}\right)\right\}.$$

Since I is a possible $V(m)$ and since the hypotheses in I are being assumed true,

$$\begin{aligned}\mathsf{P}\left(\bigcap_{i\in I}\{R_i > \frac{\alpha(m)}{m}\}\right) &\geq 1 - m\alpha(m)/m + m\beta(n-m+1)\\ &= 1 - \alpha - m\beta(n-m+1) + m\beta(n-m+1)\\ &= 1 - \alpha.\end{aligned}$$

□

A parallel argument goes through with the modified value of β. This is needed in §5.

A simple algorithm for determining a permutation of indices $\{1, 2, \cdots, k\}$ of events $A_1, \cdots, A_k$ which maximizes

$$\sum_{i=2}^{k} \max_{1\leq s\leq i-1} \mathsf{P}\left(A_i \bigcap A_s\right)$$

is that of Jarník (1930; see Seneta, 1993, for connection to work of Hunter, and of Stoline, 1983). Put $r_{ij} = \mathsf{P}(A_i \bigcap A_j)$ $\quad i, j = 1, \cdots, k$. Initially let $C = \phi$ (the empty set), $U = \{1, \cdots, k\}$. Take any subscript i_0 from U and place it in C, so now $C = \{i_0\}, U = \{1, 2, \cdots, k\} - \{i_0\}$. Then

1. Find the largest r_{ij} for $i \in C$, $j \in U$, and denote a corresponding pair of (i, j) by (i^*, j^*).
2. Redefine U and C by setting $U = U - \{j^*\}, C = C + \{j^*\}$. If $U \neq \phi$ go to 1; otherwise stop.

The final C, taking the initial i and j^*'s added in sequence, is an optimizing permutation.

To implement the algorithm in our setting, that is, to calculate $\beta(p)$, $p = 1, 2, \cdots, n-1$, we need to have available the values, when $H_1, H_2, \cdots, H_n$ are all assumed true

$$\mathsf{P}\left(R_i \leq \alpha/(n-p+1),\ R_j \leq \alpha/(n-p+1)\right), \qquad i < j,$$

that is: $n(n-1)^2/2$ in all, generally speaking.

5. Error Rate and Power

We have constructed $\beta(p)$ in (4) in accordance with Hunter's inequality (6), and this has been used in the proof above, since it is at least as sharp as the degree 2 inequalities of Sobel-Uppuluri-Galambos, and of Kounias (see Seneta, 1988). In the case of exchangeable events $A_1, A_2, \cdots, A_k$, all 3 inequalities degenerate to

$$\mathsf{P}\left(\bigcup_i A_i\right) \le k\mathsf{P}(A_1) - (k-1)\mathsf{P}\left(A_1 \bigcap A_2\right).$$

To fix ideas in this Section assume, when all of $H_1, H_2, \cdots, H_n$ are true, that $X_1, X_2, \ldots, X_n$ are exchangeable. Then from (4) for $p = 1, 2, \cdots, n$

$$\begin{aligned}\beta(p) &= \frac{n-p}{n-p+1}\mathsf{P}\left(R_1 \le \frac{\alpha}{n-p+1}, R_2 \le \frac{\alpha}{n-p+1}\right)\\ &\le \frac{n-p}{n-p+1}\mathsf{P}\left(R_1 \le \frac{\alpha}{n-p+1}\right) = \frac{(n-p)\alpha}{(n-p+1)^2}.\end{aligned}$$

The quantity

$$\frac{(n-p)}{(n-p+1)^2} = \frac{1}{n-p+1}\left(1 - \frac{1}{n-p+1}\right)$$

increases with $p = 1, 2, \cdots, n-1$ obtaining its maximum value of $1/4$ at $p = n-1$. Thus for $p = 1, 2, \cdots, n-1$

$$\beta(p) \le \frac{(n-p)\alpha}{(n-p+1)^2} \le \frac{\alpha}{4}.$$

In the extreme case that the exchangeable test statistics (given $H_1, \cdots, H_n$) are in fact independent (or nearly), the increase in power due to our modified procedure will be minute, since

$$\beta(p) = \frac{n-p}{n-p+1}\left[\mathsf{P}\left(R_1 \le \frac{\alpha}{n-p+1}\right)\right]^2 = \frac{(n-p)\alpha^2}{(n-p+1)^3} \le \frac{\alpha^2}{8}$$

which is 0.0003125 at $\alpha = 0.05$.

As in the parallel confidence interval setting (Stoline, 1983), the degree 2 correction will be of most use in the case of tight correlation. A significant improvement over Holm's procedure may thus be expected only when the pairwise correlation of the X_i's is close to unity (assuming $H_1, H_2, \cdots, H_n$ are true) in which case

$$\beta(p) = \frac{(n-p)\alpha}{(n-p+1)^2}$$

for each $p = 1, 2, \cdots, n$ approximately, and we explore this (other extreme) case further in a simple example with $n = 3$.

Suppose $Z \sim \mathcal{N}(0,1)$, and $X_1 = \mu_1 + Z$, $X_2 = \mu_2 + Z$, $X_3 = \mu_3 + Z$, where $\mu_1 \ge \mu_2 \ge \mu_3 \ge 0$, and $\mu_i = \mathsf{E}\,X_i$, $i = 1, 2, 3$, with $H_i : \mu_i = 0$, $i = 1, 2, 3$. Here

$$\Delta(1) = \frac{\alpha}{3} + \frac{2\alpha}{9} = \frac{5\alpha}{9}, \qquad \Delta(2) = \frac{3\alpha}{4}, \qquad \Delta(3) = \alpha,$$

and $R_1 \le R_2 \le R_3$.

$$\mathsf{P}\{H_i, i = 1,2,3 \text{ accepted } | H_i, i = 1,2,3 \text{ true}\} = \mathsf{P}\left(R_1 > \frac{5\alpha}{9}\right) = 1 - \frac{5\alpha}{9}$$

under our modified procedure, and $1 - \alpha/3$ under Holm's original procedure, with error rate of $5\alpha/9$ being substantially closer to the nominal rate α than $\alpha/3$. Hence the modified test is less conservative. Clearly the same argument holds if we take $I = \{1\}$ or $\{1,2\}$ or $\{1,3\}$ since $H_1 \Rightarrow H_2, H_3$.

Let us now suppose $\tilde{H}_i : \mu_i = \mu_i^0, i = 1,2,3$ where $\mu_1^0 > \mu_2^0 > \mu_3^0 > 0$, and consider

$$\begin{aligned}
&\mathsf{P}\left\{H_i, i = 1,2, \text{ rejected } | \tilde{H}_i, i = 1,2, \text{ true}\right\} \\
&= \mathsf{P}\left\{R_1 \le 5\alpha/9, R_2 \le 3\alpha/4 | \tilde{H}_i, i = 1,2, \text{ true}\right\} \\
&= \mathsf{P}\left\{X_1 \ge \Phi^{-1}\left(1 - \frac{5\alpha}{9}\right), \ X_2 \ge \Phi^{-1}\left(1 - \frac{3\alpha}{4}\right)\right\} \\
&= \mathsf{P}\left\{Z \ge \Phi^{-1}\left(1 - \frac{5\alpha}{9}\right) - \mu_1^0, Z \ge \Phi^{-1}\left(1 - \frac{3\alpha}{4}\right) - \mu_2^0\right\} \\
&= \mathsf{P}\left\{Z \ge \max\left(\Phi^{-1}(1 - \frac{5\alpha}{9}\right) - \mu_1^0, \ \Phi^{-1}\left(1 - \frac{3\alpha}{4}) - \mu_2^0\right)\right\}
\end{aligned}$$

and taking $\alpha = 0.05$

$$\mathsf{P}\left\{Z \ge \max(1.915 - \mu_1^0, 1.780 - \mu_2^0)\right\}. \tag{7}$$

With Holm's procedure we get

$$\begin{aligned}
&\mathsf{P}(Z \ge \max(\Phi^{-1}(1 - \frac{\alpha}{3}) - \mu_1^0, \Phi^{-1}(1 - \frac{\alpha}{2}) - \mu_2^0)) \\
&= \mathsf{P}(Z \ge \max(2.128 - \mu_1^0, 1.960 - \mu_2^0)).
\end{aligned} \tag{8}$$

Let us now take in (7) and (8), for simplicity $\mu_1^0 = 1.915$ and $\mu_2^0 = 1.780$. Then (7) evaluates to 0.5 while (8) evaluates to $\mathsf{P}(Z \ge \max(0.213, 0.180)) = 0.416$. The improvement in power is then $(0.5 - 0.416)/0.416 \approx 20\%$.

Of course the values $\mu_1^0 = 1.915$ and $\mu_2^0 = 1.780$ are quite far from the 0 value specified by H_1 and H_2. For μ_1^0 close to 0 (and hence μ_2^0 and μ_3^0 close to 0), the evaluations of (7) and (8) will approach the above error rates $5\alpha/9$ and $\alpha/3$.

The above example is simplistic for easy illustration; a "natural" area of application of the exchangeable case is that where $X_i = T_i$, $i = 1, \cdots, n$, and under the null hypotheses H_i, $i = 1, \cdots, n$, $\{T_1, T_2, \cdots, T_n\}$ have a multivariate t distribution with ν degrees of freedom and common pairwise correlation parameter ρ, as in Dunnett's tests (Hochberg and Tamhane, 1987; Seneta, 1993).

Below we display for $n = 3$, $\nu = 16$, $\rho = 0.5$, 0.7, 0.9, 1 and $\alpha = 0.05$, the values of $\Delta(p)$, $p = 1, 2$, when $X_i = T_i$, $i = 1, 2, 3$.

$\alpha/(n-p+1)$	ρ	0.5	0.7	0.9	1
0.017	$\Delta(1)$	0.019	0.020	0.023	0.028
0.025	$\Delta(2)$	0.028	0.030	0.033	0.037

The values in the last column are $5\alpha/9$ and $3\alpha/4$. Clearly, the power gain over Holm's procedure will not be great at the value $\rho = 0.5$, a standard value in Dunnett's test, but with high values of ρ, there is confirmation that the adjustment by $\beta(p)$ is worth using.

6. An Adaptive Variant

As noted in §4, the modification of Holm's procedure retains its essential structure and properties. The increase in power is (in general) at the cost of considerable calculation of (4) for each p, and the checking of strict monotonicity of $\Delta(p)$. Each $\beta(p)$ is not random.

We now present an adaptive variant where calculation at each step is substantially less, being determined by the joint outcome of all test statistics in the experiment, until the procedure stops. Using the ordered p-values $R_{t_i}, i = 1, \cdots, n$, observed, define the index sets $K(\cdot)$ by $K(p) = \{t_p, t_{p+1}, \cdots, t_n\}, p = 1, 2, \cdots, n$ (random sets for $p \geq 2$), and write

$$\gamma(p) = \max_{j \in K(p)} \sum_{i \in K(p)-\{j\}} \mathsf{P}\left(R_i \leq \frac{\alpha}{n-p+1}, \right.$$
$$\left. R_j \leq \frac{\alpha}{n-p+1} | H_s, s \in K(p), \text{ true}\right)$$

for $1 \leq p \leq n-1$, with $\gamma(n) = 0$.

The procedure is as follows. At Step 1: If $R_{t_1} \leq \alpha/n$, reject H_{t_1} and go to Step 2. If $R_{t_1} > \alpha/n$, check whether $R_{t_2} > \alpha/(n-1)$ or $R_{t_1} > (\alpha + \gamma(1))/n$ and if so accept $H_{t_1}, H_{t_2}, \cdots, H_{t_n}$ and stop; otherwise reject H_{t_1} and H_{t_2} and go to Step 3. At Step p, $p = 1, 2, \cdots, n-1$, if reached, check if $R_{t_p} \leq \alpha/(n-p+1)$, and if so reject H_{t_p} and go to Step $p+1$. If $R_{t_p} > \alpha/(n-p+1)$, check whether $R_{t_{p+1}} > \alpha/(n-p)$ or $R_{t_p} > (\alpha+\gamma(p))/(n-p+1)$, and if so accept $H_{t_p}, \cdots, H_{t_n}$ and stop; otherwise reject $H_{t_p}, H_{t_{p+1}}$ and go on to Step $(p+2)$, if $p \leq n-2$, or stop if $p = n-1$. If Step n is reached, check if $R_{t_n} \leq \alpha$, and if so reject H_{t_n}; otherwise accept H_{t_n}.

To show that (1) holds

Lemma. *Define*

$$\gamma = \max_{i \in I} \sum_{j \in I - \{i\}} \mathsf{P}\left(R_i \leq \frac{\alpha}{m}, R_j \leq \frac{\alpha}{m} | H_i, i \in I, \text{ true}\right)$$

where I and m are as in Section 1. At $m = 1, \gamma = 0$. Notice that $\gamma \leq \alpha$. Then

$$\bigcap_{i \in I} \{R_i > \frac{\alpha + \gamma}{m}\} \subseteq \{H_i, i \in I, \text{ are all accepted}\}.$$

Proof. For any fixed realization (sample point) of the experiment within

$$\bigcap_{i \in I} \{R_i > \frac{\alpha + \gamma}{m}\},$$

if the $R_i, i \in I$, are in fact the largest m values R_{t_i}, $i = n - m + 1, \cdots, n$, then $I = K(n - m + 1)$, and $\gamma = \gamma(n - m + 1)$, so H_i, $i \in I$, will all be accepted according to the procedure, by at most the $(n - m + 1)$th step. If the $R_i, i \in I$, are not the largest m values, then they are among the $m + k$ largest, for some $k, 1 \leq k \leq n - m$. Since for any $i \in I$

$$R_i > \frac{\alpha + \gamma}{m} \geq \frac{\alpha}{m + k}$$

then at worst at the $(n - (m + k) + 1)$th step, $H_i, i \in I$, will be accepted.

To prove that (1) holds, we follow the pattern of proof of the earlier theorem, beginning with

$$\mathsf{P}\,(H_i, i \in I, \text{are accepted} | H_i, i \in I, \text{ are true})$$
$$\geq \mathsf{P}\left(\bigcap_{i \in I} \left\{R_i > \frac{\alpha + \gamma}{m}\right\} | H_i, i \in I, \text{ are true}\right)$$

(by the Lemma)

$$= 1 - \mathsf{P}\left(\bigcup_{i \in I} R_i \leq \frac{\alpha + \gamma}{m}, i \in I, | H_i, i \in I, \text{ are true}\right)$$
$$\geq 1 - \sum_{i \in I} \mathsf{P}\left(R_i \leq \frac{\alpha + \gamma}{m} | H_i, i \in I, \text{are true}\right)$$
$$+ \max_{i \in I} \sum_{j \in I - \{i\}} \mathsf{P}\left(R_i \leq \frac{\alpha + \gamma}{m}, R_j \leq \frac{\alpha + \gamma}{m} | H_i, i \in I, \text{ are true}\right)$$

by Kounias' inequality:

(9)
$$\mathsf{P}\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} \mathsf{P}(A_i) - \max_{j=1,\cdots,k} \sum_{i \neq j}^{k} \mathsf{P}\left(A_i \bigcap A_j\right)$$
$$\geq 1 - (\alpha + \gamma) + \max_{i \in I} \sum_{j \in I - \{i\}} \mathsf{P}\left(R_i \leq \frac{\alpha}{m},\ R_j \leq \frac{\alpha}{m} | H_i, i \in I, \text{ are true}\right)$$

by the uniformity of distribution of each R_i, $i \in I$, since $(\alpha+\gamma)/m < 1$; so the left hand side is equal to

$$1-(\alpha+\gamma)+\gamma = 1-\alpha.$$

□

Needless to say, $\gamma(p)$ and γ could have been defined in terms of Hunter's Inequality (6) to give a sharper result with a little more calculation at each step in general, with coincidence for exchangeable events.

Bibliography

1. Hochberg, Y. and Tamhane, A. C., *Multiple Comparison Procedures*, Wiley, New York, 1987.
2. Holm, S., *A simple sequentially rejective multiple test procedure*, Scandinavian Journal of Statistics **6** (1979), 65–70.
3. Hunter, D., *An upper bound for the probability of a union*, Journal of Applied Probability **13** (1976), 597–603.
4. Jarník, V., *O jistém problému minimálním. [On a certain minimal problem.]*, Práce Moravské Přírodovedecké Spolecnosti v Brně (Acta Societ. Scient. Natur. Moravicae **6** (1930), 57–63.
5. Krishnaiah, P. R., Armitage, J. V., and Breiter, M. C., *Tables for the Probability Integrals of the Bivariate t Distribution*, ARL 69–0060, Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Wright–Patterson AFB, Ohio, 1969.
6. Margaritescu, E., *A note on Bonferroni's inequalities*, Biometrical Journal **28** (1986), 937–943.
7. Seneta, E., *Degree, iteration and permutation in improving Bonferroni-type bounds*, Australian Journal of Statistics **30A** (1988), 27–38.
8. Seneta, E., *Probability Inequalities and Dunnett's Test*, Multiple Comparisons, Selection, and Applications in Biometry (F. M. Hoppe, ed.), vol. Chapter 3, Marcel Dekker, New York, 1993, pp. 29–45.
9. Stoline, M. R., *The Hunter method of simultaneous inference and its recommended use for applications having large known correlation structures*, Journal of the American Statistical Association **78** (1983), 366–370.
10. Wright, S. P., *Adjusted P-values for simultaneous inference*, Biometrics **48** (1992), 1005–1013.

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail: seneta_e@maths.su.oz.au

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada
E-mail: chenth@icarus.math.mcmaster.ca