Mike's book | k field.

Review of algebraic geometry (Appendix A)

a. Affine algebraic schemes.

A.1. k field. A a k-algebra.

$X = $ "spm"(A) maxi'l ideals of A. $\quad \mathfrak{a} \triangleleft A$, $Z(\mathfrak{a}) = \{\mathfrak{m} \mid \mathfrak{m} \supset \mathfrak{a}\}$

Then
- $Z(0) = X$, $\cdot\ Z(A) = \phi$.
- $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$.
- $Z(\sum \mathfrak{a}_i) = \bigcap Z(\mathfrak{a}_i)$ for any family!

eg if $\mathfrak{m} \notin Z(\mathfrak{a}) \cup Z(\mathfrak{b})$ $\exists f \in \mathfrak{a}, g \in \mathfrak{b}$ s.t $f \notin \mathfrak{m}$ & $g \notin \mathfrak{m} \Rightarrow f \cdot g \notin \mathfrak{m}$. ($\mathfrak{m}$ is prime)

But $f \cdot g \in \mathfrak{a}\mathfrak{b} \setminus \mathfrak{m} \Rightarrow \mathfrak{m} \notin Z(\mathfrak{a}\mathfrak{b})$ □

Closed sets $Z(\mathfrak{a})$ form a topology, called the Zariski topology. Write $\mathrm{spm}(A)$ for X w/ it.

A.2. $S \subset \mathrm{spm}(A)$, $I(S) := \bigcap\{\mathfrak{m} \mid \mathfrak{m} \in S\}$.

$$I(Z(\mathfrak{a})) \overset{\text{def}}{=} \bigcap\{\mathfrak{m} \mid \mathfrak{m} \mid \mathfrak{m} \supset \mathfrak{a}\}$$

this is almost Nullstellensatz! (see prop 13.11 CA) $\longrightarrow = \sqrt{\mathfrak{a}}$ (or $\mathrm{rad}(\mathfrak{a})$) $\forall \mathfrak{a} \triangleleft A$.

$$\left\{\begin{matrix}\text{radical ideals} \\ \text{of } A\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{closed subsets} \\ \text{of } X\end{matrix}\right\}.$$

$\xrightarrow{\ Z(-)\ }$, $\xleftarrow{\ I(-)\ }$

ALSO: $Z(I(S)) = \overline{S}$

$\{\text{max'l ideals}\} \longleftrightarrow \{\text{points of } X\}$, $X = \mathrm{spm}(A)$.

A.3. $f \in A$, $D(f) := \{\mathfrak{m} \mid f \notin \mathfrak{m}\} = Z((f))^c$.

$\mathfrak{a} \triangleleft A$. $\mathfrak{a} = (f_1, \dots, f_n)$ (by Hilbert basis thm A is Noetherian) ($f_i \in A$). Then

$$X \setminus Z(\mathfrak{a}) = D(f_1) \cup \dots \cup D(f_n).$$

The $D(f)$ are basis of Zariski topology.

A.4. $\alpha: A \longrightarrow B$ hom of f.g. k-algebras, $\mathfrak{m} \triangleleft B$ maximal. Then $\alpha^{-1}(\mathfrak{m})$ is max'l (this is false for rings, $i: \mathbb{Z} \hookrightarrow \mathbb{Q}$, (0) not max'l in $\mathbb{Z}$). B is f.g $\Rightarrow B/\mathfrak{m}$ is f.g. k alg

$\boxed{\therefore B/\mathfrak{m} \supset k \text{ is a finite field extension}}$

This is by Zariski lemma. $A \xrightarrow{\alpha} B \xrightarrow{\pi} B/\mathfrak{m}$.

$$\begin{matrix} & B/\mathfrak{m} \\ A' & \big| <\infty \\ <\infty & k \end{matrix}$$

$A' = \pi(\alpha(A)) \subset B/\mathfrak{m}$.

Let $x, y \in \pi(\alpha(A))$. Sup $xy = 0 \bmod \mathfrak{m}$.

$xy \in \mathfrak{m} \Rightarrow x \in \mathfrak{m}$ or $y \in \mathfrak{m}$.

$\pi(\alpha(A))$ is integral domain since $B/\mathfrak{m}$ is a field.

(08) Lemma: R f.d. F-v.space. R integral domain. Then R is a field. "Pf": let $r \in R$, $r \neq 0$. pd $r \in R^\times$.

$r = \sum_{i=1}^{n} c_i r_i$ $\quad c_i \in F$. Cannot conclude.

Try #2: Consider $r \in R$, $r \neq 0$, and take $\{1, r, r^2, \dots\}$

This is an infinite set $\therefore$ is dependent $/F$.

$\therefore \sum^{m} c_i r^i = 0$ $\forall m$. $\therefore c_0 = r(\dots)$.

$c_0 \neq 0 \Rightarrow r \in R^\times$

$c_0 = 0 \Rightarrow 0 = r(\dots) \Rightarrow (\dots) = 0$ and ~~repeat~~ (induction on degree)

Proof 2. $0 \neq r \in R$. $\quad R \xrightarrow{\varphi} R$, $x \longmapsto rx$. injective map. F-linear

$(rx = ry \Rightarrow r(x-y) = 0 \Rightarrow x = y)$.

R f.d $\Rightarrow \varphi$ bijective F-linear transformation $\Rightarrow \varphi$ has inverse

better: φ surj $\Rightarrow 1 = rx$ for some x □.

A' is a field.

finite field ext.

$$\begin{matrix} B/\mathfrak{m} \\ \big| <\infty \\ A' = \alpha(A) \bmod \mathfrak{m}. \\ \big| <\infty. \\ k. \end{matrix}$$

$$\underbrace{A/\alpha^{-1}(\mathfrak{m})}_{= \mathfrak{m}'} \xrightarrow[\overline{\alpha}]{\sim} \alpha\left(A/\alpha^{-1}(\mathfrak{m})\right) \subset B/\mathfrak{m}$$

$\boxed{\text{f.d.}}$ surj ✓.

$a \bmod \mathfrak{m}' \longmapsto \alpha(a) \bmod \alpha(\mathfrak{m}')$, well def! ✓.

$A \xrightarrow{\alpha} B$. $\quad a - a' \in \mathfrak{m} \Rightarrow \alpha(a - a') \in \alpha(\mathfrak{m}')$

$\alpha(a) - \alpha(a') \in \alpha(\mathfrak{m}')$

$\underbrace{\alpha^{-1}(\mathfrak{m})}_{\text{ideal.}} \longmapsto \mathfrak{m}$. inj?

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[rr, bend right, "f"'] & B \arrow[r, two heads, "\pi"] & B/\mathfrak{m} \\ \alpha^{-1}(\mathfrak{m}) \arrow[u, hook] & \mathfrak{m} \arrow[u, hook] & \end{tikzcd}$$

$\alpha(\alpha^{-1}(\mathfrak{m})) \subset \mathfrak{m}$. $\quad \alpha^{-1}(\mathfrak{m}) \subset \ker f$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B/\mathfrak{m} \\ A/\mathfrak{m}' \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

universal property.

$\alpha(a) \bmod \alpha(\mathfrak{m}') = \alpha(b) \bmod \alpha(\mathfrak{m}')$

$\Rightarrow \alpha(a - b) \in \alpha(\mathfrak{m}') = \alpha(\alpha^{-1}(\mathfrak{m})) \subset \mathfrak{m}$.

pd $a - b \in \mathfrak{m}'$, but $\alpha(a - b) \subset \mathfrak{m} \Rightarrow a - b \in \alpha^{-1}(\mathfrak{m}) = \mathfrak{m}'$ □.

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$R/m \supset k$

(☆) That is because a ideal I of a k-algebra R is a k-subspace of R as a v.s. Then R/I is a v.s $/k$ and hence contains k.

Lemma used (⋆): $\phi: R \to S$ ring homo, $J \lhd S$ then

$$\phi^*: \frac{R}{\phi^{-1}(J)} \hookrightarrow \frac{S}{J} \quad \text{is injective!}$$

$$a + \phi^{-1}(J) \longmapsto \phi(a) + J$$

Proof: easy!

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \arrow[r, "\pi"] & B/\mathfrak{m} \\ \alpha^{-1}(\mathfrak{m}) \arrow[u, phantom, "\cup"] & \mathfrak{m} \arrow[u, phantom, "\cup"] & \alpha(A) = A' \arrow[u, dash] \\ & & K \arrow[u, dash] \end{tikzcd}$$

$B/\mathfrak{m}$ field. $< \infty$

$\frac{\alpha(A)}{\mathfrak{m}}$ field by lemma (⋆)

$$\frac{\alpha(A)}{\mathfrak{m}} \cong \frac{A}{\alpha^{-1}(\mathfrak{m})} \text{ (by Lemma (⋆))}$$

Daniel Allcock notes!

Zariski lemma: Let k be a field and K a field extension which is finitely generated as a k-algebra. Then K is algebraic over k. In particular, K is a f.d. v.s. over k.

Partial proof: k infinite and $K = k(x)$ simple transcendental. Let A be a k-algebra f.g. with generators $f_1, \dots, f_m \in K$. Can choose $c \in k$ (if k is finite have to find an irreducible coprime ones polynomial in $k[x]$, since they are infinitely many [Euclid's proof of infinitude of primes]) c out of the poles of f_i. Of course

$$\frac{1}{x-c} \in k(x) \setminus A \quad \text{—✗—}$$

(A is smaller than $k(x)$)

For general case, r first do it for transcendence degree one in K □

'Weak' Nullstellensatz. Let $k = \bar{k}$. $\mathfrak{m} \lhd R = k[x_1, \dots, x_n]$ maximal then $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in k$. As a consequence a family of polynomial functions on k^n with no common zeros generates all R.

Proof. $R/\mathfrak{m} \supset k$ finite field extension (Zariski lemma).

$\therefore R/\mathfrak{m} = k$. Consider the natural map

$R/\mathfrak{m} \supset k$ (⋆)

$$R \longrightarrow R/\mathfrak{m} = k$$
$$x_i \longmapsto a_i \in k. \quad (x_i - a_i) \subseteq \mathfrak{m}$$

But $I = (x_1 - a_1, \dots, x_n - a_n)$ is maximal since $R/I \cong k$.

$\therefore \mathfrak{m} = I$.

Nullstellensatz. Let $k = \bar{k}$, $g, f_1, \dots, f_m \in R = k[x_1, \dots, x_n]$, regarded as polynomial functions on k^n. If $g \in I(Z((f_1, \dots, f_m)))$ then $g \in \mathrm{rad}((f_1, \dots, f_m))$.

(or $I := (f_1, \dots, f_m)$ $\quad I(Z(I)) \subset \mathrm{rad}(I) = \sqrt{I}$).

Furthermore, equality happens.

Proof. Probably no one will ever improve on the trick of Rabinowitsch. The polynomials $f_1, \dots, f_m$ and $x_{n+1} g - 1$ have no common zeroes in k^{n+1}, so by the weak Nullstellensatz

$$1 = p_1 f_1 + \dots + p_m f_m + p_{m+1}(x_{n+1} g - 1)$$

where $p_i \in k[x_1, \dots, x_{n+1}]$. Take the homomorphism

$$k[x_1, \dots, x_{n+1}] \longrightarrow k(x_1, \dots, x_n)$$
$$x_{n+1} \longmapsto 1/g$$

$$1 = p_1(x_1, \dots, x_n, 1/g) f_1 + \dots + p_m(x_1, \dots, x_n, 1/g) f_m$$

$\exists r$ st $g^r \in (f_1, \dots, f_m)$. (clearing denominators) □

Hilbert basis theorem: If R is a Noetherian ring, then $R[x]$ is a Noetherian ring.

Proof (left-Noetherian case)

Let $\mathfrak{a} \subseteq R[x]$ be a left-ideal suppose is not f.g. By AC $\exists \{f_0, f_1, \dots\}$ family of polynomials in $\mathfrak{a}$ such that if $\mathfrak{b}_n = (f_0, \dots, f_{n-1})$ then $f_n \in \mathfrak{a} \setminus \mathfrak{b}_n$ is choosen with minimal degree. Then $\deg f_i \leq \deg f_{i+1}$. Let a_n be the leading coef of f_n. Let $\mathfrak{b} = (a_0, \dots, a_m, \dots) \subseteq R$. Since R Noetherian the chain of ideals $(a_0) \subseteq (a_0, a_1) \subseteq \dots$ stops i.e. $\mathfrak{b} = (a_0, \dots, a_{N-1})$ $\forall N$. Then

$$a_N = \sum_{i<N} u_i a_i, \quad u_i \in R. \text{ Consider}$$

$$g = \sum_{i<N} u_i x^{\deg(f_N) - \deg(f_i)} f_i \in \mathfrak{b}_N$$

$\deg(g) = f_N$ with leading term a_N.

$$f_N - g \in \mathfrak{a} \setminus \mathfrak{b}_N \text{ but } \deg(f_N - g) < \deg(f_N) \quad \text{—✗—}$$

□

We have well defined map

$$\alpha^*: \mathrm{Spm}(B) \longrightarrow \mathrm{Spm}(A)$$
$$\mathfrak{m} \longmapsto \alpha^{-1}(\mathfrak{m})$$

α^* is continuous since $(\alpha^*)^{-1}(D(f)) = D(\alpha(f))$

Proof. $\alpha^{-1}(\tilde{m})$

$\Leftrightarrow$ $\alpha^{-1}(\tilde{m})$

$f \in \alpha^{-1}(m)$, $\alpha(f) \in \alpha\alpha^{-1}(m) \subset m$. ✓

$\alpha(f) \in m \Rightarrow f \in \alpha^{-1}(m)$

More detail: $x \in \mathrm{Spm}(B)$

$x \notin (\alpha^*)^{-1}(D(f)) \Leftrightarrow \alpha^*(x) \in Z(f) \Leftrightarrow \alpha^{-1}(x) \in Z(f)$

$\Leftrightarrow \alpha^{-1}(x) \supset (f) \Leftrightarrow x \supset (\alpha(f)) \Leftrightarrow x \in Z(\alpha(f))$

$\Leftrightarrow x \notin D(\alpha(f))$

A.S. $S \subset A$ multiplicative. $S^{-1}A$ localization by $S (\ni 1)$.

$f \in A$, $S_f := \{1, f, f^2, \dots\}$.

$$A_f := S_f^{-1}A \simeq A[T]/(1-Tf).$$ From Atiyah-Macdonald book?

Prop 3.3. S^{-1} is exact, i.e. if $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact at M, f, g maps of A-modules, where M', M, M'' are A-modules, then

$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact at $S^{-1}M$ (maps of $S^{-1}A$-modules). ($S^{-1}M', S^{-1}M, S^{-1}M''$ are $S^{-1}A$-modules)

Proof. $0 = S^{-1}(0) = S^{-1}(g \circ f) = S^{-1}(g \circ f) = S^{-1}g \circ S^{-1}f$.

$\therefore \mathrm{Im}(S^{-1}f) \subseteq \ker(S^{-1}g)$. Let $\frac{m}{s} \in \ker(S^{-1}g)$, then $\frac{g(m)}{s} = 0$ in $S^{-1}M''$ $\exists t \in S$ s.t. $tg(m) = 0$

but $tg(m) = g(tm) = 0 \Rightarrow tm \in \ker(g) = \mathrm{Im} f$

$\therefore \exists m' \in M'$ s.t. $f(m') = tm$, but

$S^{-1}(f)\left(\frac{m'}{st}\right) = \frac{tm}{st} = \frac{m}{s}$ $\therefore \frac{m}{s} \in \mathrm{Im}(S^{-1}(f))$ □

Prop 3.8 Let M be an A-module. TFAE:

(i) $M = 0$

(ii) $M_p = 0$ $\forall p \triangleleft A$ prime.

(iii) $M_m = 0$ $\forall m \triangleleft A$ maximal.

Proof (i) ⇒ (ii) ⇒ (iii) is obvious.

Say (iii) and suppose $M \neq 0$. Let $x \in M - \{0\}$

$a := \mathrm{Ann}(x) \triangleleft A$, $a \neq (1)$ $\exists m$ maximal s.t. $a \triangleleft m$. Consider $\frac{x}{1} \in M_m$ $\therefore \exists s \in A - m$ s.t. $sx = 0$ but $s \in \mathrm{Ann}(x) \subset m$ —×—.

Prop 3.9 Let $\phi: M \to N$ be an A-module homomorphism. TFAE.

(i) ϕ is injective (resp. surj)

(ii) $\phi_p: M_p \to N_p$ inj (resp. surj) $\forall p$ prime of A. — As A_p-modules!

(iii) $\phi_m: M_m \to N_m$ " " $\forall m$ max'l of A.

Proof (i) ⇒ (ii):

$0 \to M \xrightarrow{\phi} N$ is exact by 3.3

$0 \to M_p \xrightarrow{\phi_p} N_p$ is exact. (ii) ⇒ (iii) is obvious

(iii) ⇒ (i) Let $M' = \ker\phi$, then

$0 \to M' \to M \to N$ is exact,

hence by (3.3) $0 \to M'_m \to M_m \xrightarrow{\phi_m} N_m$ is exact $\forall m$. Hence $M'_m = 0$ $\forall m$ $\overset{(3.8)}{\Longrightarrow}$ $M' = 0$.

$0 \to M \to N$ is exact ⇒ ϕ is injective.

(for surjectivity just reverse all the arrows)

"Being 0 or being injective map or being flat is a local property!!"

Let D be a basic open of $\mathrm{Spm} A$.

$S_D := A - \bigcup_{m \in D} m$ ⇝ multiplicative subset

If $D = D(f)$ define

$$S_f^{-1}A \xrightarrow{\varphi} S_D^{-1}A$$
$$\frac{a}{f^k} \longmapsto \frac{a}{f^k}$$

it is well-defined since $S_f \subset S_D$

EXTREMELY IMPORTANT EXERCISE!!
Prove the map above is a bijection

One solution:

Step 1: Note that $S_f^{-1}A$ and $S_D^{-1}A$ are both $S_f^{-1}A$-modules and the map above is an $S_f^{-1}A$-modules homomorphism.

Step 2:

$$\left\{\begin{matrix}\text{maximal ideals} \\ \text{of } S_f^{-1}A\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{max'l ideals } m \\ \text{of } A \text{ s.t.} \\ m \cap S_f = \emptyset\end{matrix}\right\}$$

$$m \cdot S_f^{-1}A \longleftarrow\!\!\!\shortmid\; m$$

$A \xrightarrow{\varphi_0} S_f^{-1}A$, $\varphi_0^{-1}(\tilde{m}) \subseteq A$, $\tilde{m} \subseteq S_f^{-1}A$; since $A, S_f^{-1}A$ are both f.g. k-algebras $\varphi_0^{-1}(\tilde{m})$ is maximal. [S_f is f.g. as a monoid ⇒ $S_f^{-1}A$ f.g. k-algebra]

Step 3: $\varphi: S_f^{-1}A \longrightarrow S_D^{-1}A$ is injective (bijective)
iff $\forall m \triangleleft A$ maximal s.t. $m \cap S_f = \emptyset$
(Prop 3.8)
$$\varphi_{\tilde m} : (S_f^{-1}A)_{\tilde m} \longrightarrow (S_D^{-1}A)_{\tilde m} \text{ is injective (bijective)}$$
where $\tilde m := m\cdot(S_f^{-1}A) \triangleleft S_f^{-1}A$.
Such an m_0 does not contain $f \in S_f$.
Since $f \notin m_0$, $f \in A\setminus m_0$, $A\setminus m_0$ is multiplicative hence $S_f \subset A\setminus m_0$.
Moreover, $S_D = A \setminus \bigcup_{f\notin m} m = \bigcap_{f\notin m} A\setminus m$
$$\Rightarrow A\setminus m_0 \supset S_D \supset S_f .$$
Therefore $(S_f^{-1}A)_{\tilde m} \cong (A\setminus m)^{-1}A$ and
$(S_D^{-1}A)_{\tilde m} \cong (A\setminus m)^{-1}A$.

In particular, $S_f^{-1}A$ and $S_D^{-1}A$ are isomorphic as $S_f^{-1}A$-modules. Then $\varphi_{\tilde m}$ is an isomorphism, hence φ is isomorphism □

If D, D' are both basic open sets and $D' \subset D$ then $S_{D'} \supset S_D$ and there is a canonical map $S_D^{-1}A \longrightarrow S_{D'}^{-1}A$. (*)

A.6. There is a unique sheaf $\mathcal{O}_X$ of k-algebras on $X = Spm(A)$ s.t. $\mathcal{O}_X(D) = S_D^{-1}A$ for every basic open set D of X, and the restriction map $\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D')$ is the map (*) above for every pair $D' \subset D$ of basic open subsets. Note that, for $f \in A$
$$A_f := S_f^{-1}A \simeq S_{D(f)}^{-1}(A) =: \mathcal{O}_X(D(f)).$$
This shows $\mathcal{O}_X(D(f))$ is well defined (does not depend on the choice of f).

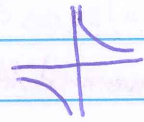
Example: $X = \mathbb{A}^1_k = Spm(k[x])$, $A = k[x]$
$$\mathcal{O}_X(D(x)) = k[x]_x \simeq k[x, x^{-1}]$$
$$D(x) = \{m \in X \mid x \notin m\} = \{(x-\alpha) \mid \alpha \neq 0\}$$
$$D(x) = \mathbb{A}^1_k - \{0\}$$

Maximal ideals in $k[x]$ are $(x-\alpha)$, $\alpha \in k^{\times}$.
(0) is prime not maximal.

A.7. By a k-ringed space we mean a topological space endowed with a sheaf of k-algebras. An affine algebraic scheme over k is a k-ringed space isomorphic to $Spm(A)$ for some k-algebra A. A morphism (or regular map) of affine algebraic schemes over k is a morphism of k-ringed spaces (it is automatically a morphism of locally ringed spaces).

A.8. The functor $A \rightsquigarrow Spm(A)$ is a contravariant equivalence to the category of aff alg schemes /k from the category of k-algebras, with quasi-inverse $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$. In particular, for all k-alg A, B,
$$Hom(A,B) \simeq Hom(Spm(B), Spm(A)).$$

e.g. $A = k[x,y]/(xy-1)$



$X = Spec\, A$
$X \subseteq \mathbb{A}^2_k$
$Y = Spec\, B$
$\mathcal{O}_X = A$, $\mathcal{O}_Y = B$.
$B = k[z]_z \simeq k[z, z^{-1}]$.
$$\varphi: A \longrightarrow B, \quad x \mapsto z, \quad y \mapsto z^{-1}.$$
$m \triangleleft A$ is $m = (x-\alpha, y-\beta)$
$m \supset (xy-1) \Rightarrow \alpha\beta = 1$.

Def: A morphism of ringed spaces $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair (f, φ), where $f: X \to Y$ continuous, and $\varphi: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is morphism of sheaves on Y.
(Recall $U \subset Y$ open, $(f_*\mathcal{O}_X)(U) := \mathcal{O}_X(f^{-1}(U))$).
i.e. for $U \subset V \subset Y$ we have:
$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "\varphi_V"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_X(f^{-1}(V)) \arrow[d] \\ \mathcal{O}_Y(U) \arrow[r, "\varphi_U"] & \mathcal{O}_X(f^{-1}(U)) \end{tikzcd}$$

A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ s.t. all stalks of $\mathcal{O}_X$ are local rings. (Almost never $\mathcal{O}_X(U)$ is a local ring for every $U \subset X$ open).
$$\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}_X(U) \underset{\text{affine?}}{=} (\mathcal{O}_X \setminus m_x)^{-1}\mathcal{O}_X .$$

A morphism of locally ringed spaces is a mor of r.spaces s.t. $\forall x \in X$ $\varphi_x : \mathcal{O}_{Y, f(p)} \longrightarrow \mathcal{O}_{X,p}$ is a local homomorphism. This is automatic for k-algebras.
A k-algebra ⇒ stalks of A^m k-algebras too.

eg. $A = \mathcal{O}_X$ $X = \operatorname{Spec} \mathbb{R}[x,y]/(xy-1)$

$Y = \operatorname{Spec} \mathbb{R}[z,z^{-1}]$ $\mathcal{O}_Y = B$.

$\varphi: A \xrightarrow{\cong} B$ let $\mathfrak{m} = (x-1, y-1) \supset (xy-1)$

$x \longmapsto z$

$y \longmapsto z^{-1}$

$\varphi_{\mathfrak{m}}: A_{\mathfrak{m}} \longrightarrow B_{\mathfrak{m}}$ → map of $A\backslash\mathfrak{m}$ modules.

$\frac{a}{s} \longmapsto \frac{\varphi(a)}{s}$ $s \in A\backslash\mathfrak{m}$.

$A_{\mathfrak{m}} = \{\frac{a}{s} \mid a \in A, s \notin \mathfrak{m}\}$ → it is local

$\mathfrak{m}A_{\mathfrak{m}} \lhd A_{\mathfrak{m}}$ maximal (unique) ideal

$\mathfrak{m}A_{\mathfrak{m}} = \{\frac{a}{s} \mid a \in \mathfrak{m}, s \notin \mathfrak{m}\}$

B is A-module. $B_{\mathfrak{m}} = \{\frac{b}{s} \mid s \notin \mathfrak{m}\}$ $\frac{b}{s} = \frac{b'}{s'}$ iff $\exists s''$ s.t. $s''\cdot(s'\cdot b - s\cdot b') = 0$ (i.e. $\varphi(s'')\varphi(s')b - \varphi(s'')\varphi(s)b' = 0$.

$a \cdot b := \varphi(a)b$.

For l.s. R algebras we must have:

$- \varphi_{\mathfrak{m}}^{-1}(\mathfrak{m}B_{\mathfrak{m}}) = \mathfrak{m}A_{\mathfrak{m}}$.

$\mathfrak{m}B_{\mathfrak{m}} = \{a \cdot \frac{b}{s} \mid a \in \mathfrak{m}, s \notin \mathfrak{m}, b \in B_{\mathfrak{m}}\}$

$= \{\frac{\varphi(a)b}{s} \mid \text{''}\}$

$\varphi_{\mathfrak{m}}(\frac{a}{s}) = \frac{\varphi(a)}{s} \in \mathfrak{m}B_{\mathfrak{m}}$ since $a \in \mathfrak{m}$.

More explicit, $a = f(x,y) \bmod (xy-1)$.

$\varphi_{\mathfrak{m}}(\frac{x}{s}) = \frac{z}{s}$ $f \in (x-1, y-1)$

$\varphi_{\mathfrak{m}}(\frac{y}{s}) = \frac{z^{-1}}{s}$. $f = h_1(x-1) + h_2(y-1)$

$\varphi_{\mathfrak{m}}\left(\frac{xy(x-1) + h_2(y-1)}{s}\right) = \frac{\cancel{z z^{-1}}(z-1) + h_2(z^{-1}-1)}{s}$.

but $(z-1) + h_2(z^{-1}-1) \in \operatorname{ext}(\varphi(\mathfrak{m}))$ ✓

$\mathbb{R}[z,z^{-1}]_{\tilde{\mathfrak{m}}}$

$\tilde{\mathfrak{m}} =$ $e(\varphi(\mathfrak{m}))$ z

$\mathfrak{m} = (x-1, y-1)$ $\text{''} \; \mathbb{R}[z,z^{-1}]_{(z-1, z^{-1}-1)} \text{''}$.

$e(\varphi(\mathfrak{m}))$ $(z-1, z^{-1}-1)$

$=$ $\{\frac{a}{b} \mid b \notin (z-1, z^{-1}-1)\}$.

c.

$\cdots$ $\mathbb{R}[z] \longrightarrow \mathbb{R}[z,z^{-1}]$

$z \longmapsto z$ $\rightsquigarrow \mathbb{A}^1_{\mathbb{R}} - \{0\} \hookrightarrow \mathbb{A}^1_{\mathbb{R}}$.

$\tilde{\mathfrak{m}} \lhd \mathbb{R}[z,z^{-1}]$ is of the form

$(z-\alpha, z^{-1}-\beta)$ $\alpha\beta = 1$

$(z-\alpha, z^{-1} - \frac{1}{\alpha})$ $\alpha \neq 0$.

the map $\mathbb{R}[z] \xrightarrow{\varphi} \mathbb{R}[z,z^{-1}]$

$\varphi^{-1}(\tilde{\mathfrak{m}}) = (z-\alpha) \lhd \mathbb{R}[z]$.

$\mathbb{A}^1 - \{0\} \longrightarrow \mathbb{A}^1$.

$(\alpha, \frac{1}{\alpha}) \longmapsto \alpha$ regular!

$X = \operatorname{Specm}(A)$

Check! $\mathcal{O}_{X,x} \cong A_{\mathfrak{m}_x} := \{\frac{f}{g} \mid g(x) \neq 0\}$ $g \notin \mathfrak{m}_x$.

$\{(f, \underset{x}{U}) \mid f = \frac{f_1}{f_2}, \; f_2(p) \neq 0 \; \forall p \in U\}$

$(f, \underset{x}{V}) = (f', \underset{x}{V'})$ iff $f|_{U\cap V} = f'|_{U\cap V}$.

$(f, U_x) \longmapsto \frac{f_1}{f_2}$ $f_2(x) \neq 0$.

$f = \frac{f_1}{f_2}$ inverse map:

$\frac{f}{g} \longmapsto (\frac{f}{g}, \underset{x}{D(g)})$.

$f_2 \neq 0 \; \forall p \in U_x$

$x \in U_x$ $g(x) \neq 0$.

$f_2(x) \neq 0$

$(f, U_x) \longmapsto \frac{f_1}{f_2} \longmapsto \left(\frac{f_1}{f_2}, D(f_2)\right)$

$\frac{f_1}{f_2}, D(f_2) \supset U_x$ but $\left.\frac{f_1}{f_2}\right|_{U_x} = \left.\frac{f_1}{f_2}\right|_{D(f_2)}$

A9. Let M be an A-module. There is a unique sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules on $X = \operatorname{Specm}(A)$ s.t. $\mathcal{M}(D) = S_D^{-1}M$ for every basic D open, and the restriction map $\mathcal{M}(D) \longrightarrow \mathcal{M}(D')$ is the canonical map $S_D^{-1}M \longrightarrow S_{D'}^{-1}M$, $D' \subset D$

Def A sheaf of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ on X s.t. for every $U \subset X$ $\mathcal{F}(U)$ is $\mathcal{O}_X(U)$-module. And for $V \subset U$ the restrictions

$s \in \mathcal{O}_X(U)$ $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is compatible with

$f \in \mathcal{F}(U)$

$\operatorname{res}(s f) = \operatorname{res}(s) \cdot \operatorname{res}(f)$ $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$

A morphism of sheaves of $\mathcal{O}_X$-modules is $\mathcal{F}, \mathcal{G}$ a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ s.t. $\forall U \subseteq X$ the map $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is a $\mathcal{O}_X(U)$-module homomorphism.

A sheaf of $\mathcal{O}_X$-modules is said to be coherent if it is isomorphic to $\tilde{M}$ for some f.g. A-module M. The functor $M \mapsto \tilde{M}$ is an equivalence to the category of coherent $\mathcal{O}_X$-modules from the category of f.g. A-modules with quasi-inverse $\mathcal{M} \mapsto \mathcal{M}(X)$.

$$\left\{\begin{matrix}\text{f.g. projective}\\ A\text{-modules}\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{locally free } \mathcal{O}_X\text{-modules}\\ \text{of finite rk}\end{matrix}\right\}$$

{trivial bundles} $\longleftrightarrow$ {free modules}

(locally free module is st localization to any maximal ideal is a free module)

N projective and f.g. module $\iff$ N locally free.

A.10. For fields $K \supset k$, the Zariski topology on K^n induces that on k^n. Proof:

(a) every closed S of k^n is $T \cap k^n$ for some closed T of K^n

Let $S = Z(f_1, \ldots, f_m)$ $f_i \in k[X_1, \ldots, X_n]$

Then $S = k^n \cap \{\text{zero set of } f_1, \ldots, f_m \text{ in } K^n\}$

(b) $T \cap k^n$ is closed for every closed subset T of K^n.

Let $T = Z(f_1, \ldots, f_m)$ $f_i \in K[X_1, \ldots, X_n]$

Let $(e_j)_{j \in J}$ be a k-basis of K.

$f_i = \sum e_j f_{ij}$ (finite sum) then

$$Z(f_i) \cap k^n = \left\{\begin{matrix}\text{zero set of the family}\\ (f_{ij})_{j \in J} \text{ in } k^n\end{matrix}\right\}$$

for each i, and so $T \cap k^n = Z(f_{ij})$ in k^n

b. Algebraic schemes

A.11. Let $(X, \mathcal{O}_X)$ be a k-ringed space. An open subset U of X is said to be affine if $(U, \mathcal{O}_X|_U)$ is an affine algebraic scheme over k.

An algebraic scheme over k is a k-ringed space $(X, \mathcal{O}_X)$ that admits a finite cover by open affine subsets. A morphism of alg. schemes (also called a regular map) is a morphism of k-ringed spaces.

The local ring at a point x of X is denoted by $\mathcal{O}_{X,x}$ or just $\mathcal{O}_x$, and the residue field at x is denoted by $\kappa(x)$. For example, if $X = \operatorname{Spm}(A)$ and $x = \mathfrak{m}$, then $\mathcal{O}_{X,x} = A_\mathfrak{m}$ and $\kappa(x) = A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} \overset{(*)}{\cong} A/\mathfrak{m}$.

Lemma (*): $I \trianglelefteq A$, $S \subset A$ mult.

$A \xrightarrow{\pi} A/I$ canonical proj. $T := \pi(S)$ then

$$\frac{S^{-1}A}{S^{-1}I} \xrightarrow{\sim} T^{-1}(A/I).$$

In particular, $\mathfrak{p} \trianglelefteq A$ prime then $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong \operatorname{Frac}(A/\mathfrak{p})$.

Proof: $\frac{a}{s} \bmod S^{-1}I \longmapsto \frac{\pi(a)}{\pi(s)}$.

well-defined: $\frac{a}{s} = \frac{a'}{s'} \bmod S^{-1}I$ $\quad \frac{as'-sa'}{ss'} = \frac{i}{s''}$ $\forall i, s''$, $i \in I$.

$s'''((as'-sa')s''-iss') = 0$ $\forall s''' \in S$. In particular $(as'-sa')s_0 \in I$ $\forall s_0 \in S \iff \pi(as'-sa')\pi(s_0) = 0$ in A/I

$\iff \frac{\pi(a)}{\pi(s)} = \frac{\pi(a')}{\pi(s')}$ in $T^{-1}(A/I)$ since $\pi(s_0) \in T$.

Injective: $s_0(as'-sa') - i = 0 \Rightarrow \frac{as'-sa}{1} = \frac{i}{s_0}$ in $S^{-1}A$

$\Rightarrow as'-sa = 0$ in $S^{-1}A/S^{-1}I \Rightarrow \frac{a}{s} = \frac{a'}{s'} \bmod S^{-1}I$.

Surjective: let $\frac{a \bmod I}{t} \in T^{-1}(A/I)$, $t = \pi(s)$ $\forall s \in S$

$\frac{a}{s} \bmod S^{-1}I \longmapsto \frac{\pi(a)}{\pi(s)} = \frac{a \bmod I}{t}$.

Ring homomorphism: Omitted $\square$

A regular map $\varphi: Y \to X$ induces a local hom of local rings $\mathcal{O}_{X,\varphi(y)} \longrightarrow \mathcal{O}_{Y,y}$ $\forall y \in Y$

A.12 A morphism $\varphi: Y \to X$ is said to be surjective (resp. injective, open, closed) if the map $|\varphi|: |Y| \to |X|$ of topological spaces is. φ is surj iff $\varphi(k^a): Y(k^a) \to X(k^a)$ is surjective.

A.13 Let X be an algebraic scheme over k, and let A be a k-algebra. By definition, a morphism of k-schemes $X \xrightarrow{f} \operatorname{Spm}(A) =: Y$ gives a morphism of sheafs of k-algebras $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$. then for every $V \subset Y$ open we have $\mathcal{O}_Y(V) \longrightarrow f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$ in particular,

$$\mathcal{O}_Y(Y) \cong A \longrightarrow \mathcal{O}_X(X).$$

A.14 Let X alg scheme $/k$. $|X|$ is a Noetherian topological space (i.e. open subsets satisfy a.c.c or closed satisfy d.c.c). Then $|X| = W_1 \cup \cdots \cup W_r$ for finitely many closed irred subsets W_i.
If $W_i \not\subset W_j$ then this decomposition is unique, and its elements are called irreducible components of X.
(From A.13: there is a natural isomorphism
$$\mathrm{Hom}(X, \mathrm{Spm}(A)) \simeq \mathrm{Hom}(A, \mathcal{O}_X(X)).)$$
A noetherian space has only f. many connected comp, each of which is open and closed, and it is a disjoint union of them. It is also quasi-compact.

A.15 The image of a reg map $\varphi: Y \to X$ of alg. schemes is constructible.

Def. A constructible set in a topological space is a finite union of locally closed sets (sets that are intersections of an open and a closed subset) (or sets that are relatively open in their closures)

Therefore contains a dense open subset of its closure. If φ dominant (image dense) then its image contains a dense open subset of X.

A.16 A regular map $\varphi: Y \to X$ of alg schemes is affine if for all $U \subset X$ affine open subschemes, $\varphi^{-1}(U)$ is an affine open subscheme of X. It suffices to check this in an open affine covering of X.

A.17 A regular map $\varphi: Y \to X$ of alg schemes$/k$ is finite if, for every open affine subscheme $U \subset X$, $\varphi^{-1}(U)$ is affine and $\mathcal{O}_Y(\varphi^{-1}(U))$ is a finite $\mathcal{O}_X(U)$-algebra. Suffices to check for a covering.

e.g. $\mathrm{Spm}(B) \xrightarrow{\alpha^*} \mathrm{Spm}(A)$ defined by $\alpha: A \to B$ is finite iff $A \to B$ finite

A.18 (Ext of the base field: ext of scalars). Let K be a field containing k. There is a functor $X \rightsquigarrow X_K$ from alg. sch$/k$ to alg sch$/K$.

e.g. $X = \mathrm{Spm}(A)$, then $X_K = \mathrm{Spm}(K \otimes A)$.

A.19 For an alg sch X/k, we let $X(R)$ the set of points of X w/coordinates in a k-algebra R,
$$X(R) := \mathrm{Hom}(\mathrm{Spm}(R), X)$$
e.g. $X = \mathrm{Spm}(A)$, $X(R) = \mathrm{Hom}(A, R)$.
For a ring R containing k, we let
$X(R) := \varinjlim X(R_i)$, where R_i runs over all f.g. k-subalgebras of R.
$X(R) = \mathrm{Hom}(A, R)$ again if $X = \mathrm{Spm}(A)$.
$R \rightsquigarrow X(R)$ is a functor from all k-alg to sets.

A.20. Let A k-alg. (f.g). Let $A_{k^a} = k^a \otimes A$.
If $\mathfrak{m} \triangleleft A_{k^a}$ is maximal then $\mathfrak{m} \cap A$ maximal since $A/\mathfrak{m} \cap A \hookrightarrow A_{k^a}/\mathfrak{m} = k^a$;
remember $A_{k^a}/\mathfrak{m} \supset k^a$ finite field extension by Zariski lemma since $A_{k^a}/\mathfrak{m}$ is f.g. k^a-alg. Then $A_{k^a}/\mathfrak{m} = k^a$, and by lemma (A.6) $A/\mathfrak{m} \cap A$ is a field. The map
$$\pi: \mathrm{Spm}(A_{k^a}) \longrightarrow \mathrm{Spm}(A)$$
$$\mathfrak{m} \longmapsto \mathfrak{m} \cap A$$
is surjective(1), continuous(2), and closed(3), and hence it is a quotient map. For general X alg sch$/k$ the projection $X_{k^a} \to X$ realises $|X|$ as a quotient of $|X_{k^a}|$(4).

(1) Every max ideal $\mathfrak{m}$ of A is ker of some k-alg homo $A \to k^a$ (C.A 13.2)
Proof: $A/\mathfrak{m} \supset k$ finite field ext.
$A \to A/\mathfrak{m} \subset k^a$.
This extends to $A_{k^a} \to k^a$ whose kernel contains $\mathfrak{m}$. (2) $\pi^{-1}(Z(f_1, \ldots, f_s)) = Z(f_1, \ldots, f_s)$.
(3) Very similar to A.10.
(4) Cor 2.3.12 of EGA IV.

A.21 X alg scheme. An $\mathcal{O}_X$-module $\mathcal{M}$ is coherent if for every open aff U, $\mathcal{M}|_U$ is coherent.
It suffices to check for a covering of X.
A sheaf $\mathcal{I}$ of ideals in $\mathcal{O}_X$ is coherent if its restriction to open aff U is the subsheaf

of $\mathcal{O}_X|_U$ defined by an ideal in the ring $\mathcal{O}_X(U)$.
($\mathcal{F}$ sheaf U open, $V \subset U$ open $\mathcal{F}|_U(V) = \mathcal{F}(V)$,
if Z closed $i: Z \to X$ inclusion then
$\mathcal{F}|_Z := i^{-1}\mathcal{F}$).

$f: X \to Y$, $\mathcal{F}$ on X $f_*(\mathcal{F})$ on Y
def. by $f_*(\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$.
direct image sheaf.

$\mathcal{F}$ on Y define $f^{-1}(\mathcal{F})$ on X as
the sheafification of the presheaf

$$U \longmapsto \varinjlim_{V \supset f(U)} \mathcal{F}(V)$$

Let $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ mor of
ringed spaces. Let $\mathcal{F}$ an $\mathcal{O}_X$-module
then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module

$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ is $\underset{\overset{\|}{f_*\mathcal{O}_X(V)}}{\mathcal{O}_X(f^{-1}(V))}$-module

$f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$

Then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module.
Since we have $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ then
$f_*\mathcal{F}$ has structure of $\mathcal{O}_Y$-module is
called the direct image of $\mathcal{F}$ by f.
Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules, in particular
$U \subset Y$, $\mathcal{G}(U)$ is a $\mathcal{O}_Y(U)$-module.

$$f^{-1}(\mathcal{G})(V) = \varinjlim_{W \supset f(V)} \mathcal{G}(W)$$

$$f^{-1}(\mathcal{O}_Y)(V) = \varinjlim_{W \supset f(V)} \mathcal{O}_Y(W).$$

then $f^{-1}\mathcal{G}$ is a $f^{-1}\mathcal{O}_Y$-module.

Ex 1.18 Hartshorne:

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

(f^{-1} is left adjoint of f_*)

Then we have a morphism of sheaves of rings
on X:

$$f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X. \quad \left(\begin{array}{l}\text{then } \mathcal{O}_X \text{ is a} \\ f^{-1}\mathcal{O}_Y\text{-module}\end{array}\right)$$

We define $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

<u>Thm</u> There is a natural isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

i.e. f^* is the left adjoint of f_*.

A.22. Dictionary b/w spec & spm.
X scheme in sense of EGA, X_0 closed pts.

$$\left\{\begin{array}{c}\text{closed (resp. open,} \\ \text{constructible) subsets} \\ \text{of } X\end{array}\right\} \underset{\text{lattice iso}}{\overset{\sim}{\longleftrightarrow}} \left\{\begin{array}{c}\text{closed (resp. open,} \\ \text{constructible subsets} \\ \text{of } X_0\end{array}\right\}$$

$$S \overset{\sim}{\longmapsto} S \cap X_0.$$

<u>e.g</u> X connected iff X_0 is connected.
To recover X from X_0 add a point z for each
irred closed subset Z of X_0 which is not a point
$z \in U$ open iff $U \cap Z \neq \emptyset$.
The ringed spaces $(X, \mathcal{O}_X)$ and $(X_0, \mathcal{O}_X|_{X_0})$ have
the same lattice of open subsets and the same ring
for each open subset. A regular map $\varphi: Y \to X$
of alg schemes over k is surjective iff
it is surjective on closed points (DG, I, §3, 6.10).

c. Subschemes

A.23. Let X alg scheme $/k$. An open subscheme
of X is a pair $(U, \mathcal{O}_X|_U)$ with U open in X.
Is again an alg scheme $/k$.

A.24 Let $X = \mathrm{Spm}(A)$ aff sch $/k$, let $\mathfrak{a} \triangleleft A$
ideal. Then $\mathrm{Spm}(A/\mathfrak{a})$ is an alg scheme with
underlying top space $Z(\mathfrak{a})$

Let Y alg scheme $/k$ and $\mathcal{I}$ a coherent sheaf
of ideals in $\mathcal{O}_X$. $\mathrm{Supp}(\mathcal{O}_X/\mathcal{I}) =: Z$ is a
closed subset of X.

M. Brion

$\Lambda(V) := T(V) / (x\otimes x \mid x\in V)$

$T(V) :=$ tensor alg of $V := \bigoplus_{k=0}^{\infty} V^{\otimes k}$

Lectures on the geometry of flag varieties

Flag variety X. $\leftrightarrow$ complex projective alg variety X, homogeneous under a complex linear alg. gp.

§1. Grassmannians and flag varieties

§1.1. Grassmannians

The Grassmannian $Gr(d,n)$ is the set of d-dimensional linear subspaces of $\mathbb{C}^n (=V)$

Def Let V v.s. The k-th exterior power of V, denoted by $\Lambda^k(V)$ is the vector subspace of $\Lambda(V)$ spanned by the elements of the form

$x_1\wedge x_2 \wedge\cdots\wedge x_k$, $x_i\in V$

If $\{e_1,\dots,e_n\}$ basis of V then

$\{e_{i_1}\wedge\cdots\wedge e_{i_k} \mid 1\le i_1 \lneq \cdots \lneq i_k\le n\}$ is basis for $\Lambda^k(V)$ of cardinal $\binom{n}{k}<\infty$.

Let $E\in Gr(d,n)$, $E\le \mathbb{C}^n$ and $\{v_1,\dots,v_d\}$ a basis of E. Note $v_1\wedge\cdots\wedge v_d$ only depends on E up non-zero scalar.

e.g. $E=\mathbb{R}e_1\oplus\mathbb{R}e_2$ $\{e_1,e_2\}$ basis

$\{e_1+e_2, e_1-e_2\}$ basis too.

$(e_1+e_2)\wedge(e_1-e_2)$

$= \cancel{e_1\wedge e_1} - e_1\wedge e_2 + \underbrace{e_2\wedge e_1}_{-e_1\wedge e_2} - \cancel{e_2\wedge e_2}$

$= -2(e_1\wedge e_2)$ $\dim(\Lambda^2 E)=1$

Let $V=k\{e_1,e_2,e_3\}$, $\{e_1,e_2,e_3\}$ LI.

$E\subset V$. $\dim\Lambda^2 V = \binom{3}{2} = 3$

$= \operatorname{span}\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$

The point $[v_1\wedge\cdots\wedge v_d] =: L(E)$ in the projective space $\mathbb{P}(\Lambda^d\mathbb{C}^n)$ only depends on E

$L: Gr(d,n) \longrightarrow \mathbb{P}(\Lambda^d\mathbb{C}^n)$ is injective.

$E \longmapsto [v_1\wedge\cdots\wedge v_d]$

Im L = cone of decomposable d-vectors in $\mathbb{P}(\Lambda^d\mathbb{C}^n)$.

$Gr(d,n)$ is a subvariety of $\mathbb{P}(\Lambda^d\mathbb{C}^n)$.

This is called the Plücker embedding

* from Shafarevich book*

§4.1 Closed subsets of Projective space

Let V be a v.s. of dimension $n+1$ / k. The set of lines (1-dim v.s.) of V is called the n-dimensional projective space denoted by $\mathbb{P}(V)$ (or $\mathbb{P}^n$).

Coordinate ring $k[S_0,\dots,S_n] =: S$.

The Grassmannian variety $Gr(r,V)$ parametrises all r-dim s.s of V. To define it consider $\Lambda^r V$.

Let $L\in Gr(r,V)$ and $\{f_1,\dots,f_r\}$ basis of L and send it to $f_1\wedge\cdots\wedge f_r\in\Lambda^r V$. By passing to another basis this element is multiplied by the determinant of the coordinate change matrix.

e.g. $V=\mathbb{R}^3$

$\beta_1 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\} = \{f_1,f_2,f_3\}$.

$\beta_2 = \{e_1, e_2, e_3\}$.

$P_{21} = \begin{pmatrix}1&0&1\\0&1&-1\\0&1&1\end{pmatrix}$

$\det P_{21} = \begin{vmatrix}1&-1\\1&1\end{vmatrix} = 2$, $\det P_{12}=\frac{1}{2}$.

$f_1\wedge f_2\wedge f_3 = e_1\wedge(e_2+e_3)\wedge(e_1-e_2+e_3)$

$= e_1\wedge e_2\wedge e_3 + e_1\wedge e_3\wedge(-e_2)$

$= 2\, e_1\wedge e_2\wedge e_3$.

In particular, $P: Gr(r,V) \longrightarrow \mathbb{P}(V)$

$L \longmapsto P(L)$

is well-defined. It is injective too (use props of Λ^r).

If $\{e_i\}_{i=1}^n$ is a basis of V then $\{e_{i_1}\wedge\cdots\wedge e_{i_r} \mid 1\le i_1\lneq\cdots\lneq i_r\le n\}$ is a basis of $\Lambda^r V$. and

$$P(L) = \sum_{i_1<\cdots<i_r} P_{i_1\dots i_r}(e_{i_1}\wedge\cdots\wedge e_{i_r})$$

The homogeneous coordinates $P_{i_1\dots i_r}$ are called the Plücker coordinates of L.

Not every point of $\mathbb{P}(\Lambda^r V)$ is of the form $P(L)$ for some r-dim subspace L.

In other words, not every $x \in \Lambda^r V$ is of the form $f_1 \wedge \dots \wedge f_r$ with $f_r \in V$.

Convolution: Let $u \in V^*$ be a vector of the dual space.

For $x \in \Lambda^1 V = V$ the convolution $u \lrcorner x$ is an element of $\mathbb{R}$, just $u(x) \in \mathbb{R}$.

For $x \in \Lambda^0 V = \mathbb{R}$ we set $u \lrcorner x$.

Let

$$u \lrcorner (x \wedge y) := (u \lrcorner x) \wedge y + (-1)^a (x \wedge (u \lrcorner y))$$

For $x \in \Lambda^a V$.

e.g. $V = \mathbb{R}^4$, $r = 2$, $u : \mathbb{R}^4 \longrightarrow \mathbb{R}$ "$u = \pi_2$".

$$v \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot v$$

$u \lrcorner v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot v$. $u \lrcorner \begin{pmatrix} 100 \\ -12 \\ 100 \\ 94 \end{pmatrix} = -12$.

$a = 1$

$u \lrcorner 3 = 0$ $\quad u \lrcorner (x \wedge y) = u(x)y - u(y)x$.

$$u \lrcorner \left(\begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right) = 4 \wedge \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix} \wedge 2 \right)$$

$$\cong 2 \begin{bmatrix} 0-2 \\ 4-4 \\ 0-1 \\ 4-0 \end{bmatrix} = 2 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -2 \\ +4 \end{bmatrix}$$

$a = 2$

$$u \lrcorner ((x \wedge y) \wedge z) = u \lrcorner (x \wedge y) \wedge z + ((x \wedge y) \wedge (u \lrcorner z))$$
$$= u(x)(y \wedge z) - u(y)(x \wedge z) + u(z)(x \wedge y)$$

Then $\left(u \lrcorner \Lambda^r V\right) \subset \Lambda^{r-1} V$. For $u_1, \dots, u_s \in V^*$ consider $u_1 \lrcorner (u_2 \lrcorner (\dots (u_s \lrcorner x) \dots))$ it depends only on x and $u_1 \wedge \dots \wedge u_s \in \Lambda^s V^*$ and it is denoted by $y \lrcorner x \in \Lambda^{r-s} V$.

e.g. $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^t$, $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^t$

$u_1' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^t$, $u_2' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^t$.

$\det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1$. $\quad u_1 \wedge u_2 = u_1' \wedge u_2' \in \Lambda^2 V^*$.

$\Lambda^2 V$

$$u_1 \lrcorner \left(u_2 \lrcorner \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) =: u_1 \wedge u_2 \lrcorner \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$u_2 \lrcorner \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$.

$u_1 \lrcorner 1 = 0$.

$\Lambda^2 V$, $u_1 \lrcorner \left(u_2 \lrcorner \begin{pmatrix} 3 \\ 8 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$$u_2 \lrcorner \left[\begin{pmatrix} 3 \\ 8 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \left(u_2 \lrcorner \begin{pmatrix} 3 \\ 8 \end{pmatrix}\right) \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \end{pmatrix} \wedge \left(u_2 \lrcorner \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 8 \\ 0 \end{pmatrix} - 0.$$

$u_1 \lrcorner \begin{pmatrix} 8 \\ 0 \end{pmatrix} = \boxed{8}$

$$u_1' \lrcorner \underbrace{\left(u_2' \lrcorner \begin{pmatrix} 3 \\ 8 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}_{\ast}$$

$$\ast = 19 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 16 \\ -8 \end{pmatrix}$$

$u_1' \lrcorner \ast = 16 - 8 = \boxed{8}$.

$$u_1 \lrcorner (u_2 \lrcorner (v_1 \wedge v_2))$$
$$= u_1 \lrcorner \left[u_2(v_1) v_2 - u_2(v_2) v_1\right]$$
$$= u_2(v_1) u_1(v_2) - u_2(v_2) u_1(v_1).$$

Useful criteria

$x \in \Lambda^r V$ totally decomposable $\Rightarrow$ $x \wedge x = 0$

Theorem: $x \in \Lambda^r V$ totally decomposable $\Longleftrightarrow$ $(y \lrcorner x) \wedge x = 0$ $\forall y \in \Lambda^{r-1} V^*$ element of the basis of $\Lambda^{r-1} V^*$

($\Rightarrow$) Clear.

Proof. It suffices to check for $y = u_{i_1} \wedge \dots \wedge u_{i_{r-1}}$ where $\{u_i\}$ is a basis of V^*. In particular can take u_i dual basis of the basis $\{e_i\}$ of V. ($u_i(e_j) = \delta_{ij}$).

e.g. $y \in \Lambda V^*$, $x \in \Lambda^2 V$, $x = f_1 \wedge f_2$

$(y \lrcorner x) \wedge x = \star \wedge (f_1 \wedge f_2) = 0$.

$$y \lrcorner (f_1 \wedge f_2) = (y \lrcorner f_1) \wedge f_2 - f_1 \wedge (y \lrcorner f_2)$$
$$= (y(f_1) f_2 - y(f_2) f_1) = \star$$

$v_i \in V$ all L.I. $\quad 0 \neq 2\, v_1 \wedge v_2 \wedge v_3 \wedge v_4$

$x = v_1 \wedge v_2 + v_3 \wedge v_4$ is indecomposable since $x \wedge x \neq 0$.

$x \in \Lambda^2 V$. let $y = f \in \overset{1}{\Lambda} V^*$. lets compute

$(y \lrcorner x) \wedge x$.

$[y \lrcorner (v_1 \wedge v_2 + v_3 \wedge v_4)] \wedge x$

$= [y(v_1) v_2 - y(v_2) v_1] \wedge v_3 \wedge v_4 +$

$+ [y(v_3) v_4 - y(v_4) v_3] \wedge v_1 \wedge v_2$.

$= y(v_1)\, \underbrace{v_2 \wedge v_3 \wedge v_4}_{} - \%\, \underbrace{v_1 \wedge v_3 \wedge v_4}_{}$

$+ \quad \%\, \underbrace{v_4 \wedge v_1 \wedge v_2}_{\text{all different!}} + \%\, \underbrace{v_3 \wedge v_1 \wedge v_2}_{}$.

$\Rightarrow y(v_1) = 0 = y(v_2) = y(v_3) = y(v_4)$

$\Rightarrow y = 0$ — X —. The idea of general proof is clear for $(\Leftarrow)$.

$\underset{\overset{r-1}{\Lambda} V^*}{y} \lrcorner \underset{\overset{r}{\Lambda} V}{x} \qquad y \lrcorner x \in \Lambda V = V$.

$(y \lrcorner x) \wedge x \in \Lambda^{r+1} V$

let $x \in \Lambda^2 V \quad V = \mathrm{span}\{e_1, \dots, e_4\} \leq \mathbb{R}^4$.

$x = p_{12}\, e_1 \wedge e_2 + p_{13}\, e_1 \wedge e_3 + p_{14}\, e_1 \wedge e_4 +$

$+ p_{23}\, e_2 \wedge e_3 + p_{24}\, e_2 \wedge e_4 + p_{34}\, e_3 \wedge e_4$.

$x \wedge x = 0 = (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$

$\Rightarrow p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. $(*)$

So the image of $Gr(2,4)$ on $\mathbb{P}(\Lambda^2 V) \cong \mathbb{P}^5$ satisfies $(*)$.

e.g. $f \in \Lambda^2 V$ decomposable

$f = f_1 \wedge f_2, \quad f_1, f_2 \in V$.

$f_1 = a e_1 + b e_2 + c e_3 + d e_4$

$f_2 = a' e_1 + b' e_2 + c' e_3 + d' e_4$.

$f_1 \wedge f_2 = \underbrace{(ab' - a'b)}_{p_{12} \text{ as before}}\, e_1 \wedge e_2 + \dots$

a big cancelation produces $(*)$.

Third way) Write the conditions of $(✡)$ for $y \in \left\{ \begin{matrix} u_1, u_2, \\ u_3, u_4 \end{matrix} \right\}$ ← 6 elements

$x = \sum_{i<j} p_{ij}\, e_i \wedge e_j$. is decomposable iff the following 4 conditions holds.

$(u_1 \lrcorner x) \wedge x = (u_2 \lrcorner x) \wedge x = \dots = 0$.

$u_1 \lrcorner x = / \; u_1 \lrcorner (e_i \wedge e_j) = u_1(e_i) e_j - u_1(e_j) e_i$.

$= u_1(e_i) e_j = e_j$ or $-u_1(e_i) e_i = -e_i$

$u_1 \lrcorner x = p_{12} e_2 + p_{13} e_3 + p_{14} e_4$

$(u_1 \lrcorner x)^{\wedge x} = p_{12}\, e_2 \wedge (p_{13}\, e_1 \wedge e_3 + p_{14}\, e_1 \wedge e_4 + p_{34}\, e_3 \wedge e_4) +$

$+ p_{13}\, e_3 \wedge (p_{24}\, e_2 \wedge e_4 + p_{12}\, e_1 \wedge e_2 + p_{14}\, e_1 \wedge e_4)$

$+ p_{14}\, e_4 \wedge (p_{12}\, e_1 \wedge e_2 + p_{13}\, e_1 \wedge e_3 + p_{23}\, e_2 \wedge e_3)$

$= (\cancel{-p_{12}p_{13}} + \cancel{p_{13}p_{12}})\, e_{1,2,3} + (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})\, e_{2,3,4}$

$+ (\text{ cancelation })\, e_{1,3,4} + (\text{ cancelation })\, e_{1,2,4}$

$= 0. \iff p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$

In general $(✡)$ for $\{u_i\}$ dual basis

$$\sum_{t=1}^{r+1} (-1)^t\, p_{i_1 \dots i_{r-1} j_t}\, p_{j_1 \dots \hat{j_t} \dots j_{r+1}} = 0$$

for all sequences $i_1, \dots, i_{r-1}, j_1, \dots, j_{r+1}$.

• How to reconstruct L from the coordinates $p_{i_1 \dots i_r}$ of $P(L)$? $\dim L = r$. $P(L) = (p_{i_1} : \dots : p_{i_r})$

A basis of L is given by $\{f_i\}$, where (WLOG $\sup p_{1 \dots r} \neq 0$)

possible error in Shafarevich → $f_i = e_i + \sum_{k>r} a_{ik} e_k$ for $i = 1, \dots, r$

$f_1 = e_1 + \sum_{k>r} a_{1k} e_k, \quad f_2 = e_2 + \sum_{k>2} a_{2k} e_k$. maybe r!

one gets maybe is ~~$(i+1)$~~! NO it is not!

$a_{ik} = (-1)^r\, p_{1 \dots \hat{i} \dots rk}$ (supposing WLOG $p_{1 \dots r} = 1$)

eg) $L = \langle e_1 + 2e_2, -e_2 \rangle < \mathbb{R}^4$.

$P(L) = [(e_1 + 2e_2) \wedge (-e_2)] = [-e_1 \wedge e_2 + 0]$ $\quad p_{12} = -1$. $\quad p_{12} = 1$.

$f_1 = e_1 + 0$

$f_2 = e_2 + 0$. ✓

$L = \langle 2e_1 + e_2, e_2 + e_3, 3e_4 \rangle \quad r = 3$.

$P(L) = [6 e_1 \wedge e_2 \wedge e_4 + 3 e_2 \wedge e_3 \wedge e_4]$

$p_{124} = 6, \quad p_{234} = 3$

The new V basis is

~~$f_1 = e_1 + a_{14} e_4$~~ $\quad a_{14} = P_{234} = 3$

~~$f_2 = e_2 + a_{24} e_4$~~ $\quad a_{24} = P_{134} = 0$

~~$f_3 = e_3 + a_{34} e_4$~~ $\quad a_{34} = P_{124} = 0$.

~~$f_1 = e_1 + 3e_4$~~ $\quad f_1' = 2e_1 + e_2$

~~$f_2 = e_2$~~ $\quad f_1' = e_2 + e_3$

~~$f_3 = e_3 + 6e_4$~~ $\quad f_3' = 3e_4$.

~~In this case $P_{123} = 0$ not working.~~

new example to force $P_{123} \neq 0$

$L = \langle 2e_1 + e_2,\ e_2 + e_4,\ 3e_3 \rangle$

$$P(L) = [6\, e_1 \wedge e_2 \wedge e_3 - 3\, e_2 \wedge e_3 \wedge e_4 \ -6\, e_1 \wedge e_3 \wedge e_4]$$

$$= [e_1 \wedge e_2 \wedge e_3 - \tfrac{1}{2} e_2 \wedge e_3 \wedge e_4 - \tfrac{1}{3} e_1 \wedge e_3 \wedge e_4]$$

$P_{123} = 1, \quad P_{234} = -\frac{1}{2}, \quad P_{134} = -1$

$f_1 = e_1 + a_{14} e_4 = e_1 - \frac{1}{2} e_4$

$f_2 = e_2 + a_{24} e_4 = e_2 - e_4$. $\quad a_{24} = (-1)^4 P_{134} = -1$.

$f_3 = e_3 + a_{34} e_4 = e_3$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & +1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & +1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & +1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

of $\dim = r$

Then a subspace V is determined by the coefficients a_{ik} for $i \in \{1, \dots, r\}$ and $k \in \{r+1, \dots, n\}$. Then the parameters are $r(n-r)$.

Then if $P_{i_1 \dots i_r} \neq 0$. it defines an open subset of $Gr(r, V)$ isomorphic to $\mathbb{A}^{r(n-r)}$ with coordinates a_{ik}.

• Back to M. Brion notes. The general linear group $G := GL_n(\mathbb{C})$ acts on the variety $X := Gr(d, n)$ via its natural action on $\mathbb{C}^n$.

If $L \subset \mathbb{C}^n$ a d-dim subspace.

And $g \in GL_n(\mathbb{C})$ $\quad X$ ∥

$$g \cdot L = \{ g(x) \mid x \in L \} = g(L) \in Gr(d, n)$$

$G \circlearrowright X$. transitive action

(X has a unique G-orbit)

$G \circlearrowright \mathbb{P}(\Lambda^d \mathbb{C}^n)$ since

$G \circlearrowright \Lambda^d \mathbb{C}^n$

$$g \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_d) = g(v_1) \wedge \dots \wedge g(v_d)$$

$P(g \cdot L) = g \cdot P(L)$ Then P is G-equivariant

$L = \langle f_1, \dots, f_d \rangle$, $P(L) = [f_1 \wedge \dots \wedge f_d]$

$$P(g \cdot L) = P(\langle g f_1, \dots, g f_d \rangle)$$

$$= [g f_1 \wedge \dots \wedge g f_d]$$

$$= g \cdot [f_1 \wedge \dots \wedge f_d] = g \cdot P(L).$$

Let $(e_1, \dots, e_n)$ std basis on $\mathbb{C}^n$

Isotropy group of $\langle e_1, \dots, e_d \rangle$ $= \mathrm{Stab}_G \langle e_1, \dots, e_d \rangle$

is $P := \left\{ \begin{pmatrix} \overbrace{*}^{d} & * \\ \underbrace{0}_{d} & * \end{pmatrix} \right\} \Big\} n-d$

P is max'l parabolic subgroup of G

parabolic = closed and s.t G/P is a projective variety.

Claim: $G/P = X = Gr(d, n)$.

Pf $P = \mathrm{Stab}_G(\langle e_1, \dots, e_d \rangle)$

$G/P \cong$ Orbit of $(\langle e_1, \dots, e_d \rangle) = X$ since the action is transitive

$gP \mapsto gE_{1,\dots,d}$ $\quad P = \mathrm{Stab}(E_{1,\dots,d})$

$$G/P \overset{\sim}{=} X = Gr(d,n) \subseteq \mathbb{P}(\Lambda^d \mathbb{C}^n)$$

$$\dim G/P = \dim G - \dim P = n^2 - (n^2 - d(n-d)) = d(n-d)$$

Let $I := (i_1, \dots, i_d)$ multi-index.
$1 \le i_1 < i_2 < \dots < i_d \le n$.

Let E_I the coordinate subspace of $\mathbb{C}^n$
i.e. $E_I = \langle e_{i_1}, \dots, e_{i_d} \rangle$

$E_{1,2,\dots,d} :=$ std coord. subspace of $\mathbb{C}^n$.

Def $T := \mathrm{diag}_n \mathbb{C} \subset GL_n(\mathbb{C})$.

(this is a maximal torus of G)

Prop: (1) T-fixed points in X: $X^T = \{E_I \mid I \text{ multi-index}\}$

Pf: Obviously if $t \in T$

$\boxed{\text{BETTER PROOF LATER!}}$

$t \bullet E_I = E_I$ $\quad t = \begin{pmatrix} a_{11} & & 0 \\ & a_{ii} & \\ 0 & & a_{nn} \end{pmatrix}$

$e_{i_j} \mapsto a_{i_j i_j} e_{i_j}$ $\quad a_{ii} \in \mathbb{C}^\times$.

Sup $\langle f_1, \dots, f_d \rangle = E$ is fixed by T.

$\forall t \in T$ $\quad t \bullet E = E$.

Better Sup $\langle f_1, \dots, f_d \rangle \ne E_I$ this mean in the RREF we get

$$d\left\{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_d \end{bmatrix}\right. \sim \left.\underbrace{\begin{bmatrix} \overbrace{1\ 0 \cdots 0}^{d} & \\ 0\ 1 \cdots 0 & \\ 0\ 0\ 1 \cdots 0 & b_{ij} \\ 0\ 0\ 0 \cdots 0\ 1 & \end{bmatrix}}_{n}\right\}$$

"Toy case" (WLOG assume first d coordinates are "leading" terms.

there is a $b_{ij} \ne 0$. (or else $E = E_I = E_{1,\dots,d}$)

WLOG sup $b_{ij} \ne 0$ occur at $i=1, j=n$.

Let $t_k = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \frac{k}{b_{1n}} \end{pmatrix}$ where $k \in \mathbb{C}$.

(in general)

Cannot happen that both t_1, t_2 satisfy $t_i \bullet \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ b_{1n} \end{pmatrix} \in E$. Then E is not T-fixed $\square$.

In general, take $t = (a_{ii})$ s.t. $a_{ii} = \begin{cases} 1 & i \ne j \\ 2 & i = j \end{cases}$

$B := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 \cdots & * \end{pmatrix} \right\}$ or $B :=$ upper Δ matrices of $GL_n(\mathbb{C})$.

B is a Borel subgroup of G (maximal connected solvable)

Prop (2): $X = \bigsqcup_{I (\text{multi})} BE_I$.

$\boxed{\text{Rmk Thm: [Lie–Kolchin] } G \text{ smooth connected group solvable } / k. \text{ If } k = \bar{k}, G \text{ is trigonalizable.}}$

Pf (prop (2)): Omitted $\square$ ~~($\mathbb{C}^4$)~~ (the proof is straightforward)

Let's see B-orbits. e.g. $\mathbb{C}^4$.

B-orbit of $\langle e_1, \dots, e_d \rangle$ $\quad I = \{1,2,3\}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $(d=3)$

$\rightsquigarrow \begin{matrix} \cancel{1000} \\ 0100 \\ 0010 \\ 0001 \end{matrix} \overset{B\text{-}}{\rightsquigarrow} \begin{matrix} 01{*}{*} \\ 001{*} \\ 0001 \end{matrix}$ REF $\rightsquigarrow$ RREF $\begin{matrix} 0100 \\ 0010 \\ 0001 \end{matrix}$

B-orbit has 1-element. $\dim = 0$.

$I = \{1,3,4\}$ $\quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\begin{matrix} 1000 \\ 0100 \\ 0001 \end{matrix} \overset{B\text{-}}{\rightsquigarrow} \begin{matrix} 10{*}0 \\ 01{*}0 \\ 0001 \end{matrix}$ $\quad \dim = 2$.

$I = \{2,3,4\}$

$\begin{matrix} 100{*} \\ 010{*} \\ 001{*} \end{matrix}$ $\quad \dim = 3$.

$I = \{1,2,4\}$

$\begin{matrix} 1000 \\ 0010 \\ 0001 \end{matrix} \rightsquigarrow \begin{matrix} 1{*}00 \\ 0010 \\ 0001 \end{matrix}$ $\quad \dim 1$.

$$X = \underset{\text{IIS}}{BE_{1,2,3}} \sqcup \underset{\text{IIS}}{BE_{1,2,4}} \sqcup \underset{\text{IIS}}{BE_{1,3,4}} \sqcup BE_{2,3,4}$$
$$= \mathbb{A}^0_{\mathbb{C}} \sqcup \mathbb{A}^1_{\mathbb{C}} \sqcup \mathbb{A}^2_{\mathbb{C}} \sqcup \mathbb{A}^3_{\mathbb{C}}$$

general $(d < n)$. $\quad = \mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^3$.

$\begin{matrix} 00100 \\ 00010 \\ 00001 \end{matrix}$ $\quad \mathbb{C}^0$.

$\begin{matrix} 01{*}00 \\ 00010 \\ 00001 \end{matrix}$ $\quad \mathbb{C}^1$

$\begin{matrix} 01{*}{*}0 \\ 001{*}0 \\ \quad\ 1 \end{matrix}$

$\boxed{n=5,\ d=3}$ $\mathbb{C}^5$ $d=3$:

$I=1,2,3$ C_1, dim $=0$:
001 00
000 10
000 01

$1,2,4$ C_2, dim $=1$:
01*00
00010
00001

[the analytic closure is the same as Zariski for varieties]

$1,2,5$ C_3, dim $=2$:
e_3: 1 * * 0 0
e_2: 0 0 0 1 0
e_1: 0 0 0 0 1

$1,3,4$ C_4, dim $=2$:
e_4: 0 1 0 * 0
e_3: 0 0 1 * 0
e_1: 0 0 0 0 1

$1,3,5$ C_5, dim $=3$:
1 * 0 * 0
0 0 1 * 0
0 0 0 0 1

$1,4,5$ C_6, dim $=4$:
1 0 * * 0
0 1 * * 0
0 0 0 0 1

$2,3,4$ C_7, dim $=3$:
0 1 0 0 *
0 0 1 0 *
0 0 0 1 *

$2,3,5$ C_8, dim $=4$ [see]:
1 * 0 0 *
0 0 1 0 *
0 0 0 1 *

$2,4,5$ C_9, dim $=5$ [see]:
1 0 * 0 *
0 1 * 0 *
0 0 0 1 *

$3,4,5$ C_{10}, dim $=6$:
1 0 0 * *
0 1 0 * *
0 0 1 * *

0 1 2 3 4 5 6

all open cells

$10=\frac{20}{2}=\frac{4\cdot 5}{2}=\frac{5!}{2!3!}=\binom{5}{2}=10.$

Def The Schubert cells in the grassmannian are the orbits $C_I := BE_I$, i.e. the B-orbits of X.

The closure of the Schubert cell C_I (for Zariski topology) is called the Schubert variety $X_I := \overline{C_I}$.

$B = T \ltimes U$.

$U \lhd B$ $\quad U := \left\{\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & \ddots & \vdots \\ 0 & & & 1\end{pmatrix}\right\}$ is a maximal unipotent subgroup of G

$C_I = U \cdot E_I$ (only need U-orbits)

$U_{E_I} :=$ isotropy group of E_I on U [is a subgroup since is an stabiliser] $= \left\{ x\in U \text{ s.t. } a_{ij}=0 \text{ if } \begin{array}{l} i\notin I \\ j\in I\end{array}\right\}$

$\mathbb{C}^3$

$U_{\langle e_1,e_2\rangle}$ or $U_{1,2} = \left\{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1\end{pmatrix}\right\} = U$ dim $=3$

$U_{2,3} = \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}\right\}$ dim $=0$

$U_{1,3} = \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1\end{pmatrix}\right\}$ dim $=1$

e.g. $\mathbb{C}^5$ $d=3$

$I=\{1,2,3\}$ $U_{1,2,3}=U$

$I=1\,2\,4$ $\quad U_{124} = \left\{\begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1\end{pmatrix}\right\}$ $\quad a_{3,4}=0$

$3<4$, $3\notin I$

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1\end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix} \sim \begin{matrix} 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1\end{matrix} \sim \begin{matrix} 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1\end{matrix}$$ ✓

let $U^I \subset U$ the complementary subset of U

def by $a_{ij} = \begin{cases} 0 & \text{if } i \in I \\ & \text{or } j \notin I. \end{cases}$

Claim: U^I is a subgroup. (of course is closed subscheme)

Proof Omitted $\square$.

The map $U^I \longrightarrow X$ is a locally

$g \longmapsto gE_I$.

closed embedding.

Def: A morphism $f: X \to Y$ of schemes is called affine morphism if the inverse image of every affine open set of Y is an affine open set of X.

An affine morphism $f: X \to Y$ is called a closed embedding if for every affine open subset $\operatorname{Spec} B \subset Y$ with $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, the map $B \longrightarrow A$ is surjective (i.e., of the form $B \longrightarrow B/I$)

⚠ Not confuse with unramified which is the analogous of the differential geometric concept of immersion.

A map is called a locally closed embedding, if there is an open cover $\{U_i\}$ of X ($f: X \to Y$) such that $f|_{U_i}$ is a closed embedding.

Understanding

$$U^I \xrightarrow{f} X = Gr(d,n)$$
$$g \longmapsto [gE_I].$$

e.g. $I = \{2,4\}$, $d=2$, $n=4$ $X \hookrightarrow \mathbb{P}(\Lambda^2 \mathbb{C}^4)$

$1,2,\ 3,4$ $E_I = \langle e_2, e_4 \rangle$

$$U_{E_I} = \left\{ \begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \qquad U_{E_I} \cdot E_I = E_I.$$

$$U^I = \left\{ \begin{pmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \\ 0 & c \\ 0 & 1 \end{pmatrix}$$

Only $gE_I = E_I$ when $g \in U^I$ if $g = 1$.

$$\begin{bmatrix} 1 & c & 0 & b \\ 0 & 0 & 1 & a \end{bmatrix} \text{ RREF.}$$

Sup $gE_I = g'E_I$, $g, g' \in U^I$.

means $gg' \in U_I \cap U^I = \{id\}$.

f is injective on closed points.

p.d. f locally closed embedding. f is a map of varieties is just the restriction of the map $g: G \longrightarrow X = Gr(d,n)$, which is regular!!

$g \longmapsto g \cdot [e_{i_1} \wedge \cdots \wedge_{i_d}]$.

the action is rational

$$G \times X \xrightarrow{a} X, \quad (g,x) \longmapsto g \cdot x.$$

$a|_{G \times x_0}: G$

gp. action on scheme

$$\begin{tikzcd} G \times G \times X \arrow[d, "(m \times id)"'] \arrow[dr, "id \times a"] & \\ G \times X \arrow[dr, "a"'] \arrow[r, phantom, "\circlearrowleft"] & G \times X \arrow[d, "a"] \\ & X \end{tikzcd}$$

$g|_{U^I} = f$.

$$\operatorname{Im} f \subseteq U^I\text{-Orbit of } E_I \subseteq U\text{-orbit of } E_I =: C_I.$$
$$U\text{-orbit of } E_I = UE_I$$

$\operatorname{Im} f = U^I\text{-Orbit of } E_I$

$= U_I U^I\text{-Orbit of } E_I \overset{\text{thm}}{=} C_I$.

Thm $U_I U^I = U$.

Proof: Omitted

eg
$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d & 0 & e \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d & a & e+fa \\ 0 & 1 & b & fb \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for some reason this seem to work for me!

$$G = \operatorname{Spec}\left[\mathbb{C}\left[x_{11}, \ldots, x_{nn}, \frac{1}{\det x_{ij}}\right]\right] \xrightarrow{g} X = \operatorname{Proj}\left[\frac{\mathbb{C}[w_{ijk}]}{\text{plucker rels}}\right]$$

let $U_d \subset X$ def by $w_{i_1, \ldots, i_d} \neq 0$. ie. $w_{i_1, \ldots, i_d} = 1$.

then $U_d \cong \mathbb{A}^{d(n-d)}$ coordinates $(w_{i_1', \ldots, i_d'}, \ldots)$

all $(i_1', \ldots, i_d') \neq (i_1, \ldots, i_d)$

$g^{-1}(U_d) = ?$ is an aff. open s.t. blah.

So we have the usual cover of X

still do not know if they are open.

$$X = \bigcup_{I \text{ multi index}} \boxed{BE_I} = \bigcup_I UE_I.$$

Other cover is given by the projective cover

$$X = \bigcup_{I=\{i_1,\dots,i_d\}} U_{i_1,\dots,i_d} \qquad U_{i_1\dots i_d} = \{p_{i_1,\dots,i_d} = 1\}.$$

$$U^I \xrightarrow{f} X \hookrightarrow \mathbb{P}(\Lambda^d \mathbb{C}^n).$$

$$g \longmapsto [gE_I]$$

$$f^{-1}(U_{i_1,\dots,i_d}) = ?$$

e.g. $d=2$, $n=4$. $I=\{1,3\}$ $\quad a_{23}$

① $U^I = \left\{\begin{pmatrix}1&0&0&0\\0&1&*&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}$ $\quad I^c=\{2,4\}$

$E_I = \langle e_1, e_3\rangle = \text{rowsp}\begin{pmatrix}1&0&0\\0&0&1\\ &&\end{pmatrix}$ $\qquad U_I = \left\{\begin{pmatrix}1&*&*&*\\0&1&0&*\\0&0&1&*\\0&0&0&1\end{pmatrix}\right\}$

$E_I = [e_1 \wedge e_3] \rightsquigarrow U_{13} = \{p_{13} \neq 0\}$

$= [p_{12}; p_{13}; p_{14}; p_{23}; p_{24}; p_{34}]$ $\quad [12, 13, 14, 23, 24, 34]$

$\binom{4}{2}-1=5$

$[e_1\wedge e_3] = [0;1:0:0:0:0]$

$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$ eq. of $X \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^4) \cong \mathbb{P}^5$

$$U^I \xrightarrow{f} \cancel{X}\ \mathbb{P}(\Lambda^2\mathbb{C}^4) \qquad d(n-d)=4 \quad +1\ \binom{n}{d}=6-1=5.$$

$$g \longmapsto [gE_I] = [ge_1 \wedge ge_3]$$

$$\begin{pmatrix}1&0&0&0\\0&1&a&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{bmatrix}0\\a\\1\\0\end{bmatrix}$$ for each a a different subspace!

$e_1,\ f_3 = ae_2 + e_3$

$\langle e_1, e_3\rangle$. $[e_1\wedge f_3] = [a:1:0:0:0:0]$

$e_1 \wedge f_3 = a(e_1\wedge e_2) + e_1\wedge e_3 =$

Not confuse with U_I

Of course f is defined over $U_{i_1,\dots,i_d} = \{p_{i_1,\dots,i_d} \neq 0\}$

since $\boxed{\begin{matrix}B\\ \cup\\ U \text{ is unipotent of } G \text{ max'l} \ldots B\end{matrix}}$

e.g. $d=2, n=4$, $I=\{2,4\}$, $I^c=\{1,3\}$

② $U^I = \left\{\begin{pmatrix}1&*&0&*\\0&1&0&0\\0&0&1&*\\0&0&0&1\end{pmatrix}\right\}$ $\quad 24 \quad \frac{12}{14}\ 34$

$E_I = \langle e_2, e_4\rangle$ $\qquad U_I = \left\{\begin{pmatrix}1&0&*&0\\0&1&*&*\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}$

$$\begin{pmatrix}1&a&0&b\\0&1&0&0\\0&0&1&c\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix} = \begin{pmatrix}a&b\\1&0\\0&c\\0&1\end{pmatrix} \quad \text{RREF}$$

$(a,b,c) \neq (a',b',c')$ different subspaces!

$$\underbrace{(ae_1+e_2)}_{g_1} \wedge \underbrace{(be_1+ce_3+e_4)}_{g_2}$$

$= ac(e_1\wedge e_3) + a\, e_1\wedge e_4$

$\quad -b(e_1\wedge e_2) + c\, e_2\wedge e_3 + e_2\wedge e_4$

$\boxed{\text{sup } b=1}$ $= [-b : ac : a : c : 1 : 0]$

$\boxed{\text{sup } b\neq 0}$ $\in U_{2,4}$ always, $\notin U_{3,4}$ never

$$f: U^I \longrightarrow U_{i_1,\dots,i_d} \subset X$$

$\cong$

$\mathbb{A}^5 = \text{Spec}\,\mathbb{C}[x_{12}, x_{13}, x_{14}, x_{23}, x_{34}]$ $\quad (-)$

$d=2$, $I\in\{1,2\}$

$(x_{12}x_{34} - x_{13} + x_{14}x_{23})$

$\cong \mathbb{A}^4 = \text{Spec}$

$(\mathbb{C}[x_{12}, x_{13}, x_{14}, x_{23}])$.

~~$f_1 = e_1 + a_{13}e_3 + a_{14}e_4$~~

~~$f_2 = e_2 + a_{23}e_3 + a_{24}e_4$~~

~~$\langle f_1, f_2\rangle = \langle ae_1 + e_2, be_1 + ce_3 + e_4\rangle$~~

$1 = p_{24} \neq 0$.

$f_2 = e_2 + a_{13}e_3 + a_{14}e_4 + a_{34}e_4$ $\quad \boxed{2413}$

$f_4 = e_4 + a_{12}e_2 + a_{13}e_3 + a_{23}e_3$

$(f_2, f_4) = \begin{pmatrix}0 & 0\\ 1 & a_{12}\\ a_{13} & a_{13}+a_{23}\\ a_{14}+a_{34} & 1\end{pmatrix}$

temporary new order $2<4<1<3$ $i\in\{2,4\}$ $r=4$. $\{4,1,3\}$

$$f_2 = e_2 + \sum_{k\geq 4} a_{2k} e_k$$
$$f_4 = e_4 + \sum_{k>4} a_{4k} e_k$$

coordinates $\mathbb{A}^4 = \begin{pmatrix} a_{21}, a_{23}, \\ a_{41}, a_{43} \end{pmatrix}$

$$f_2 = e_2 + a_{21} e_1 + \cancel{a_{23} e_3}$$
$$f_4 = e_4 + a_{41} e_1 + a_{43} e_3$$

$\boxed{\text{Chk}}$ $a_{ij} = (-1)^j P_{1,\dots,\hat{i},\dots,r,j}$, $i+1$ is not!

$a_{21} = (-1)^{i+1} P_{4,1} = (-1) a$, $\cancel{a_{23} = (-1)^{i+1} P_{4,3} = 0}$

$a_{41} = (-1)^{i+1} P_{2,1} = (-1)^{i+1}(-b)$, $a_{43} = (-1)^{i+1} P_{2,3} = (-1)^{i+1} c$.

New coordinates a, b, c.

dim $=3$ as expected. (same coord as the beginning)

$$f_2 = e_2 + a e_1$$
$$f_4 = e_4 + b e_1 + c e_3.$$

$1<2<3<4$ again!

$$\widetilde{\mathrm{Im}\, f} = \left\{ \begin{array}{l} x_{12}x_{34} - x_{13} + x_{14}x_{23} = 0 \\ x_{23} = 0 \end{array} \right\} \subseteq \mathbb{A}^5.$$

$$\widetilde{\mathrm{Im}\, f} = \left\{ \begin{array}{l} x_{12}x_{34} - x_{13} = 0 \\ x_{23} = 0. \end{array} \right\} \subseteq \mathbb{A}^5$$

$\dim(\widetilde{\mathrm{Im}\, f}) = 3! = \dim X - 1.$

$f : f^{-1}(U_{\{2,4\}}) \hookrightarrow B \subset X$.

$U_{\{2,4\}} = \mathrm{Spec}\, A$ $\quad B = X \cap U_{\{2,4\}} \subset \mathbb{P}^5$.

B affine open.

$U_{\{2,4\}} \cap X$ affine open dense of X.

$B = \{x_{24} \neq 0\}$ in $\mathbb{P}^5$.

$f : \mathrm{Spec}\, A/_I \hookrightarrow \mathrm{Spec}\, A$

$\rightsquigarrow A \twoheadrightarrow A/_I$

locally. closed immersion $\square$.

$\mathrm{Im}\, f = \bigcup E_I = C_I \text{ "}\cong\text{" } U^I$.

C_I is a locally closed subvariety of X.
(i.e. intersection of closed & open, or equivalently it is open on its closure)

Not confuse with U_I!

$$C_I = \mathrm{Im}\, f = \underbrace{\widetilde{\mathrm{Im}\, f}}_{\text{closed}} \cap \underbrace{U_{\{2,4\}}}_{\text{open}} \quad //.$$

$$C_I = \left\{ \begin{array}{l} f_1 = 0 \\ f_2 = 0 \end{array} \right\} \cap \underbrace{\{x_{24} \neq 0\}}_{\text{open affine}} \subseteq \{x_{24} \neq 0\}$$

$$= \left\{ \begin{array}{l} f_1 = 0 \\ f_2 = 0 \end{array} \right\} \subset \mathbb{A}^5 = U_{\{2,4\}}.$$

$$|I| := \sum_{j=1}^{d} (i_j - j) = (2-1) + (4-2) = 3.$$

Then $C_I \cong \mathbb{A}^{|I|}$ and $X_I = \overline{C_I} \cong \mathbb{P}^{|I|}$.

<u>Prop</u> (i) C_I is the set of d-dim subspaces E s.t

$$\dim(E \cap \langle e_1, \dots, e_j \rangle) = \#\left\{ k \,\middle|\, \begin{array}{l} 1 \leq k \leq d \\ i_k \leq j \end{array} \right\} = n_j$$

for all $j = 1, \dots, n$.

<u>e.g.</u> ① $I = \{1,3\}$. $C_I = \text{row}\begin{bmatrix} 0 & 1 & \partial & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U^I E_I$.

$E \in C_I$.

$\dim E \cap \langle e_1 \rangle = \langle e_1 \rangle = 1$. $j=1$

$d=2$. $\{\#k \mid 1 \leq k \leq 2\} \leq 2$. $n_1 = 1$

$\dim(E \cap \langle e_1, e_2 \rangle) = 1$. $n_2 = 1$ $j=2$.

② $I = \{2,4\}$ $C_I = \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$

$n_1 = 0$, $\dim(\ ) = 0$.

$n_2 = 1$, $\dim(\ ) = 1$.

(ii) X_I is the set of d-dim s.e. E s.t.

$$\dim(E \cap \langle e_1, \dots, e_j \rangle) \geq \#\{k \mid \dots\} = n_j$$

$\forall j \in \{1, \dots, n\}$

eg $I=\{1,3\}$ $\quad C_I = \text{row}\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ dim 1

$\{1,3\}$

$X_I = ?$ $\quad n_1 = 1$

$n_2 = 1$

$n_j = \#\{k \mid 1 \le k \le d,\ i_k \le j\}$

$E \cap \langle e_1 \rangle \ge 1$. $\quad C_I \subset \overline{C_I}$.

$\{1,4\}$ $\begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ dim 2

dim $\cap = ?$ $\quad \{1,2\}$ $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ dim 0

$I=\{2,4\}$ row $\{2,4\}$ $\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$ dim 3

$n_1 = 0$

$n_2 = 1$

$\{2,3\}$ $\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$ dim 2

Claim: [Cordeiro]

$$X_I = \bigcup_{J \le I} C_J.$$

$J \le I$ iff $j_k \le i_k \ \forall k \in \{1, \dots, d\}$.

~~eg $I = \{1,3\}$~~

Embedding $\quad U^I \xhookrightarrow{f} \mathbb{P}(\Lambda^d \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{d}-1}\mathbb{C}$.

$g \longmapsto g \cdot [E_I]$

revisited! $\quad E_I = [e_{i_1} \wedge \dots \wedge e_{i_d}]$

$g = \begin{bmatrix} 1 & a_{12} & \cdots \\ & \ddots & \vdots \\ 0 & & 1 \end{bmatrix}$, $\quad U_{E_I} = \left\{ \begin{pmatrix} 1 & a_{ij} \\ & 1 \end{pmatrix} \middle|\ a_{ij}=0 \ \begin{smallmatrix} i \notin I \\ j \in I \end{smallmatrix} \right\}$

$U^I = \left\{ \left(\ddots \right) \middle|\ \begin{smallmatrix} a_{ij}=0 \text{ if} \\ i \in I \text{ or } j \notin I \end{smallmatrix} \right\}$

eg. $I = \{2,4\}$. $\quad I^c = \{1,3\}$.

$E_I = \langle e_2, e_4 \rangle$.

$U_I = \left\{ \begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$ $\quad U^I = \left\{ \begin{pmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$

$U^I \cong \mathbb{A}^3 \subset \mathbb{A}^{16} \supset G \subset \mathbb{A}^{17}$. (open) (closed)

$U^I \hookrightarrow \mathbb{P}(\Lambda^2 \mathbb{C}^4)$.

$g \longmapsto g[E_I]$. $\quad 3 = |I|$.

$[E_I] = [e_2 \wedge e_4]$

$g \in U^I = \begin{pmatrix} 1 & a & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\quad \operatorname{Im} f \cong \mathbb{A}^3$.

$g\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 1 \\ 0 \\ 0 \end{pmatrix} = ae_1 + e_2$

$g\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ c \\ 1 \end{pmatrix} = be_1 + ce_3 + e_4$

$g \cdot [E_I] = [\overset{w_1}{-b} : \overset{w_2}{ac} : \overset{w_3}{a} : \overset{w_4}{c} : \overset{w_5}{1} : \overset{w_6}{0}]$

morph of varieties.

In general f is in every coordinate a polynomial in the matrix entries of g, then f is regular.

is injective, the image satisfies Plücker

is locally closed embedding.

$A \longrightarrow A/I$

$\mathbb{C}[w_{12}, \dots] / \phi \qquad \mathbb{C}[\dots] / \text{affine Plücker} + \{ w_3 w_4 = w_2,\ w_6 = 0 \}$

$$|I| = \sum_{j=1}^{d} (i_j - j), \quad U^I \xhookrightarrow{L} X$$
$$\text{im } L = C_I \subseteq \mathbb{A}^{|I|} \quad \boxed{\text{since } U^I \cong \mathbb{A}^{|I|}}$$

Prop. (i) C_I is the set of d-dim subspaces E of $\mathbb{C}^n$ s.t.
$$\dim(E \cap \langle e_1, \ldots, e_j\rangle) = n_j \quad \forall j = 1, \ldots, n$$
(ii) $X_I = \overline{C_I}$ is the set of " " " of $\mathbb{C}^n$ s.t.
$$\dim(E \cap \langle e_1, \ldots, e_j\rangle) \geq n_j, \quad j = 1, \ldots, n.$$

eg $I = \{1,3\}$ where $n_j = \#\{k \mid 1 \leq k \leq d, i_k \leq j\}$

(1) row $\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix} = C_I.$

flip (as in Fulton)

let $E \in C_I$, $E = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ * \\ 1 \\ 0 \end{pmatrix} \right\rangle.$

$\dim(E \cap \langle e_1\rangle) = 1.$ $n_1 = 1$

$\dim(E \cap \langle e_1, e_2\rangle) = 1.$ $n_2 = 1.$

$\dim(E \cap \langle e_1, e_2, e_3\rangle) = 2$ $n_3 = 2$

$n_4 = 2.$

$E \cap \langle \longrightarrow \rangle = 2$

$I = \{2,4\}$

$E \in \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}$

$n_1 = 0$, $n_2 = 1$, $n_3 = 1$, $n_4 = 2$. faster to compute!

$\dim(E \cap \langle e_1\rangle) = 0.$

$\cap \langle e_1, e_2\rangle = 1$

$\cap \langle e_1, e_2, e_3\rangle = 1$

$\cap \mathbb{C}^4 = 2.$

24, 23, 14, 13, 12

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closures. (1)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix} = C' \sqcup C_0$$
$$C' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$E \in C_0$, $\dim_1 = 1 \geq n_1$, $\dim_2 = 2 \geq n_2 = 1$, $2 \geq n_3 = 2$, $2 \geq n_4 = 2.$ E

---

closures $I \subseteq C$ where I ( ) = s.t.

$$E \in \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix} = \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}$$
$$\cup \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}$$
$$\cup \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{bmatrix}$$

Back to (1) converse condition.

let E s.t. $C_I = \langle e_1, e_3\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}.$

(1) $E \cap \langle e_1\rangle = 1$ pf $E \in C_I$ 2-dim.

(2) $E \cap \langle e_1, e_2\rangle = 1$ $e_1 \in E.$

(3) $E \cap \langle e_1, e_2, e_3\rangle = 2$ $e_2 \notin E$

(4) $E \cap \langle e_1, e_2, e_3, e_4\rangle = 2$ $ae_1 + be_2 + ce_3 \in E.$

$e_4 \notin E$ since $E \cap \langle e_1, e_2, e_3\rangle = E.$

Then $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & b & c & 0 \end{bmatrix}$ but $c \neq 0.$ or else $e_2 \in E.$

$\therefore E \in \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 1 & 0 \end{bmatrix}$ //.

$E \cap \langle e_1\rangle = 0$ $e_1 \notin E.$

$E \cap \langle e_1, e_2\rangle = 1$ $[a, 1, 0, 0] \in E.$

$E \cap \langle e_1, e_2, e_3\rangle = 1$ $* * 1 \notin E.$

$* * * 1 \in E.$

$E \cap \mathbb{C}^4 = 2 = E.$ $E = \begin{bmatrix} * & 1 & 0 & 0 \\ * & * & * & 1 \end{bmatrix} = \begin{bmatrix} * & 1 & & \\ * & 0 & * & 1 \end{bmatrix}$ //

---

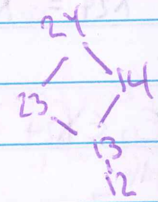
back to closures $I = \{1,3\}$

1	1	2
1	2	2
2	2	2
2	2	2
E_I	C_0	ϕ

$E_I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}$

$I = \{2,4\}$

0	1	0	1	1
1	1	1	1	2
1	1	2	2	2
2	2	2	2	2



$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}$ $\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}$

Gr: $X_I = \bigcup_{J \le I} C_J$ where $J \le I$ iff $j_k \le i_k$ $\forall k$.

eg $d=1$ $Gr(1,n) \xrightarrow{\sim} \mathbb{P}(\mathbb{C}^n) = \mathbb{P}^{n-1}$.
bijection

$$f_1 \longmapsto [f_1]$$

$I=\{i\}$. $\overline{C_I} = \bigcup_{J \le I} C_J$ $\quad J=\{j\}$ $\quad |J| = j-1$

Schubert varieties for $d=1$

$$X_0 \subset X_1 \subset \cdots \subset X_n, \quad X_j \cong \mathbb{P}^{j-1}.$$

$Gr(1,\mathbb{C}^4) = \mathbb{P}^3$, $\quad I=\{2\}$. $\quad [*100] = C_I$.

$C_I \cong \mathbb{A}^1$

$|I| = 1$.

$I=\{3\}$ $\quad C_I = [**10]$.

$|I| = 2 = 3-1$

eg $d=2$, $n=4$. Poset of Schubert varieties.

X_{34} = all X.

$X = Gr(2,4) \subset \mathbb{P}^5$

$\dim X = 4$.

34
|
24
/ \
14 23
\ /
13
|
12.

$X_{23} = C_{23} \cup C_{13} \cup C_{12}$

$X_{24} = X_{23} \cup C_{14} \cup C_{23}$

X_{24} is singular, why?

$\mathbb{P}^5$.

$$X = \{X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = 0\}$$

$X_{24} = X \cap T_{E_{12}}$.

X_{24} = all but $C_{34} = \begin{bmatrix} **10 \\ **01 \end{bmatrix}$.

$E_{12} \in X_{24}$

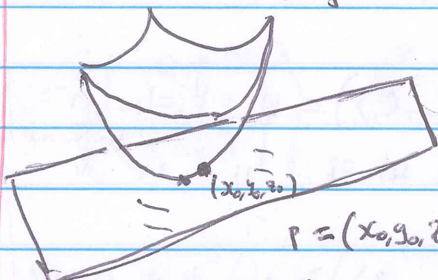
$= \begin{bmatrix} 1000 \\ 0100 \end{bmatrix}$ singular point

$[1:0:0:0:0:0]$ in Plücker coordinates

Tangent space revisited

Classical calculus

surface $S = (S, \mathcal{O}_S)$.



$P = (x_0,y_0,z_0) \in S$.

$m = m_P = \{f \in \mathcal{O}_S(U) \mid f(P) = 0\}$. $\quad U \subset S$ affine open

m/m^2 =: cotangent space of S.

$(m/m^2)^*$ =: tangent space of S.

$m/m^2 = V$ is a vector space over k where $k = R/m$ and R s.t. $V = Spec(R)$.

Def The Zariski cotangent space of a local ring (A,m) is defined to be m/m^2.

Lemma: $I \triangleleft A$, $S \subset A$ multiplicative set. $\pi: A \to A/I$ canonical projection. $T := \pi(S)$ then

$$\frac{S^{-1}A}{S^{-1}I} \xrightarrow{\sim} T^{-1}(A/I)$$

$$\frac{a}{s} \bmod S^{-1}I \longmapsto \frac{\pi(a)}{\pi(s)}$$

is a ring isomorphism. In particular, if $\mathfrak{p} \triangleleft A$ prime then $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong Frac(A/\mathfrak{p})$. → field (always)

Proof: Omitted (See at the back)

Cor: $m \triangleleft A$ maximal, $A_m/mA_m \cong A/m$.
A_m is a local ring with mA_m unique maximal ideal.

m/m^2 is an A/m vector space.

$(a+m)\cdot(b+m^2) = ab + m^2$ well defined since m ideal.

A module. $\mathfrak{m} \triangleleft A$ max'l. $1 \in A$.

$\downarrow$ $\mathfrak{m}/\mathfrak{m}^2$ is a $A/\mathfrak{m}$-v.s in obvious way (see prev. page)

$$\mathfrak{m} \xrightarrow{\varphi} \mathfrak{m} \otimes_A (A/\mathfrak{m})$$ extension of scalars

$$\mu \longmapsto \mu \otimes_A (1+\mathfrak{m})$$

φ is an additive map (or better a left A-module map).

let $\mu \in \mathfrak{m}^2$ then $\mu = \mu_1 \cdot \mu_2$.

$$\varphi(\mu) = \mu \otimes_A (1+\mathfrak{m})$$
$$= \mu_1 \otimes_A (\mu_2 + \mathfrak{m})$$
$$= \mu_1 \otimes_A 0 = 0.$$

$\mathfrak{m}^2 \subset \ker\varphi \leftarrow$ is a $\mathfrak{m}$-module of course

φ is surjective since if I have

$$\sum \mu_i \otimes_A (a_i + \mathfrak{m}) \in \mathfrak{m} \otimes_A A/\mathfrak{m}$$

then a preimage can be $\sum \mu_i a_i \in \mathfrak{m}$.

Sup $\mu \otimes_A (1+\mathfrak{m}) = 0$. Ask Geordie if this is ok!

and suppose $\mu \notin \mathfrak{m}^2$.

~~Every element of $\mathfrak{m} \otimes_A (A/\mathfrak{m})$ is of the form~~

$$\cancel{\sum \mu_i \otimes_A (a_i + \mathfrak{m}) = \underbrace{\left(\sum \mu_i a_i\right)}_{\in \mathfrak{m}} \otimes (1+\mathfrak{m})}$$

WRONG.

~~Then every element is a simple tensor of the form $\mu \otimes_A (1+\mathfrak{m})$. then the unique way for $\mu \otimes_A (1+\mathfrak{m})$ is for $\mu \in \mathfrak{m}^2$ by universal property of tensor~~

(o)

then

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\varphi} \mathfrak{m} \otimes_A (A/\mathfrak{m})$$

$$\mu + \mathfrak{m}^2 \longmapsto \mu \otimes_A (1+\mathfrak{m})$$

is an A-module iso, also iso of right $A/\mathfrak{m}$-modules since

$(\mu + \mathfrak{m}^2)\cdot(a+\mathfrak{m}) = \mu a + \mathfrak{m}$ (well def action)

and $\varphi(\mu a + \mathfrak{m}^2) = \mu a \otimes_A (1+\mathfrak{m}) = \mu \otimes_A (a+\mathfrak{m})$.

then

$$\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m} \otimes_A (A/\mathfrak{m})$$ as right $A/\mathfrak{m}$-vector space.

$$\mathfrak{m} \otimes_A (A/\mathfrak{m}) \cong \mathfrak{m}_\mathfrak{m} \otimes_{A_\mathfrak{m}} (A_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}). \quad \otimes$$

$$\mu \otimes (1+\mathfrak{m}) \longmapsto \frac{\mu}{1} \otimes_{A_\mathfrak{m}} 1 + \mathfrak{m}_\mathfrak{m}$$

Rmk: R comm ring, $S \subset R$ mult. then for M an R-module

$$S^{-1}M := M \otimes_R S^{-1}R.$$

Cor [we not using this] R ring, M, N R-mod

$$S^{-1}M \otimes_{S^{-1}R} S^{-1}N \cong S^{-1}(M \otimes_R N).$$

Note: (localising is exact then $S^{-1}(M/N) \cong S^{-1}M / S^{-1}N$.

Using Rmk above we get

$\mathfrak{m}_\mathfrak{m} = \mathfrak{m} \otimes_A A_\mathfrak{m}$ "=" $\mathfrak{m} A_\mathfrak{m}$ then

$$\mathfrak{m}_\mathfrak{m} \otimes_{A_\mathfrak{m}} (A_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}) \cong \mathfrak{m} \otimes_A \left(A_\mathfrak{m} \otimes_{A_\mathfrak{m}} A_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}\right)$$

$$\cong \mathfrak{m} \otimes_A A_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} \cong \mathfrak{m} \otimes_A A/\mathfrak{m}. \text{ proving } \otimes$$

as $A/\mathfrak{m}$-v.s.

Finally

$$\mathfrak{m}_\mathfrak{m} \otimes_{A_\mathfrak{m}} (A_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}) \cong \mathfrak{m}_\mathfrak{m}/(\mathfrak{m}_\mathfrak{m})^2$$

$$\frac{\mu}{s} \otimes_{A_\mathfrak{m}} (1+\mathfrak{m}_\mathfrak{m}) \longmapsto \frac{\mu}{s} + \mathfrak{m}_\mathfrak{m}^2.$$

Since the map

$$\mathfrak{m}_\mathfrak{m} \longrightarrow \mathfrak{m}_\mathfrak{m} \otimes_{A_\mathfrak{m}} (A_\mathfrak{m}/\mathfrak{m}_\mathfrak{m})$$ is surjective

and has kernel $(\mathfrak{m}_\mathfrak{m})^2$ (same argument as before).

Then we have the isomorphism of $A/\mathfrak{m}$ v.s.

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\varphi} \mathfrak{m}_\mathfrak{m}/(\mathfrak{m}_\mathfrak{m})^2$$

$\frac{\mu}{s} + (\mathfrak{m}_\mathfrak{m})^2 \mapsto \mu s^{-1} + \mathfrak{m}^2$ is the inverse map.

$$\mu + \mathfrak{m}^2 \longmapsto \mu + (\mathfrak{m}_\mathfrak{m})^2$$

eg: $A = k[x]$, $\mathfrak{m} = (x)$

$$\frac{ax + \mathfrak{m}^2}{1+x} \in \mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 \longmapsto (ax)(1+x)^{-1} + \mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2.$$

as $(1+x)^{-1} = (1-x) \mod (x)^2$.

then $\frac{ax}{(1+x)} = ax(1-x)$ in $(x)/(x)^2$.

(•) $m \xrightarrow[\text{map } \varphi]{\text{A-module}} m \otimes_A A/m$

$\mu \longmapsto \mu \otimes (1+m)$

$m^2 \subset \ker \varphi$.

let $\mu \in \ker \varphi$ then $\mu \otimes (1+m)$

$= 0 = \mu, \otimes (0+m)$

for all $\mu, \in m$.

for example $0 = \mu \otimes (0+m)$

$= \mu \otimes (\mu' + m)$ for some μ'

$= \mu\mu' \otimes (1+m)$

let A ring $I \subset A$ ideal, $N \in A$-mod

$0 \to I \to A \to A/I \to 0$ exact

$(-) \otimes_A N$ is right-exact.

then the following is exact:

$I \otimes_A N \to A \otimes_A N \to A/I \otimes_A N \to 0.$

$\wr\wr$

$IN \to N \to A/I \otimes_A N \to 0.$

$IN = \ker(N \to A/I \otimes_A N)$.

then in particular $I^2 = \ker(I \to A/I \otimes_A I)$

Right exactness of tensor functor R, ring, R-Mod:

$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ / $M\otimes\_$

$\boxed{M\otimes\_ \text{ is right exact}}$ (Showing)

ψ surjective, $\operatorname{Im}\varphi = \ker\psi$

p.d. is exact the following: M right R-module

$M \otimes_R A \xrightarrow{\varphi_*} M \otimes_R B \xrightarrow{\psi_*} M \otimes_R C \to 0$

let $m \otimes c \in M \otimes_R C$ $\exists b \in B$ st $\psi(b) = c$

$\psi_*(m \otimes b) = m \otimes c$; ψ_* is surjective

$\psi_*\varphi_*(m \otimes a) = m \otimes \psi\varphi(a) = m \otimes 0 = 0$.

then $\operatorname{im}\varphi_* \subset \ker\psi_*$

$\ker\psi_* \subset \operatorname{Im}\varphi_*$ (hard one!)

~~ψ_*~~ $\psi_*(\sum_i m_i \otimes b_i) = 0$.

$\sum m_i \otimes \psi(b_i) = 0$.

Lemma:

$\operatorname{Hom}(M \otimes_R N, P) \cong \operatorname{Hom}(M, \operatorname{Hom}_R(N,P))$

Proof:

$\psi : M \otimes_R N \to P \longleftrightarrow f : M \to \operatorname{Hom}(N,P)$

$m \mapsto \varphi_m : N \to P$

$\psi(\sum m_i \otimes_R n_i) \in P$.

$n \mapsto \varphi_m(n)$

$\varphi_m(n) \in P$

$\varphi_m(rn) = r\varphi_m(n)$

$r \in R$.

define f as

$f(m)(n) = \psi(m \otimes_R n)$. //

Details Omitted //

Lemma: $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact

then $0 \to \operatorname{Hom}_R(C,M) \xrightarrow{g_*} \operatorname{Hom}_R(B,M) \xrightarrow{f_*} \operatorname{Hom}(A,M)$

is exact. [functor $\operatorname{Hom}(-,M)$ is left-exact]

Proof:

~~let $h \in \operatorname{Hom}(B,M)$~~ ~~$h : B \to M$~~

~~$f_* h \in \operatorname{Hom}(A,M)$~~ ~~$f_* h = h \circ f$~~

~~let $h \in \operatorname{Hom}(A,M)$~~

① g_* is surjective.

$C \overset{h_1}{\underset{h_2}{\rightrightarrows}} M$, $B \xrightarrow{g} C$

$h_1 \circ g = h_2 \circ g \underset{A}{\Rightarrow} h_1 = h_2$

g surjective

g (epi)

② $\operatorname{Im} g_* \subset \ker f_*$ obvious

③ since $g \circ f = 0$ $h_1 \circ g \circ f = 0$.

p.d. $\ker f_* \subset \operatorname{Im} g_*$

Sup $h : B \xrightarrow{h} M$, $\exists h'$: $C \to M$, $h \circ f = 0$.

$A \xrightarrow{f} B$, $A \xrightarrow{0} M$

p.d. $\exists h' : C \to M$

s.t. $h = h' \circ g$.

Define h' s.t. $h'(c) = g(b_1) = g(b_2)$ st.

$b_1, b_2 \in g^{-1}(c)$. p.d. well defined.

$g(b_1 - b_2) = 0$, $b_1 - b_2 \in \ker g \subset \ker h$ //

$\underset{\operatorname{Im} f}{\shortparallel}$ this uses $h f = 0$ and $\underset{\ker g}{\operatorname{Im} f}$

$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ | $\mathrm{Hom}(-,M)$ left-exact

$0 \to \mathrm{Hom}(C,M) \longrightarrow \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M)$

Let P be an arbitrary R-module

$0 \to \mathrm{Hom}(C,P) \to \mathrm{Hom}(B,P) \to \mathrm{Hom}(A,P)$.

Apply $\mathrm{Hom}_R(M,-)$ [left exact]

$0 \to \mathrm{Hom}(M,\mathrm{Hom}(C,P)) \to \mathrm{Hom}(M,\mathrm{Hom}(B,P)) \to \mathrm{Hom}(M,\mathrm{Hom}(A,P))$.

$0 \to \mathrm{Hom}(M\otimes C,P) \to \mathrm{Hom}(M\otimes B,P) \to \mathrm{Hom}(M\otimes A,P)$

is exact $\forall P$. The functor $\mathrm{Hom}(-,P)$ has a magic property.

Lemma: If

$\mathrm{Hom}_R(C,P) \xrightarrow{\psi^*} \mathrm{Hom}_R(B,P) \xrightarrow{\varphi^*} \mathrm{Hom}_R(A,P)$

is exact $\forall P$ then $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is exact.

Proof: $P = C$

$\varphi^* \psi^*(id_C)) = \psi\circ\varphi = 0$. $\mathrm{Im}\,\varphi \subset \ker\psi$

$P = \mathrm{coker}\,\varphi$, h, $B \xrightarrow{h} \mathrm{coker}\,\varphi$

$h \in \mathrm{Hom}_R(B,P)$

$$\begin{tikzcd} & \mathrm{coker}\,\varphi \\ B \arrow[ur] & \\ A \arrow[u, "\varphi"] \arrow[uur, "0=\varphi^*(h)"'] & \end{tikzcd}$$

$h \in \ker\varphi^* = \mathrm{Im}\,\psi^*$

then $\exists h'$ st

$$\begin{tikzcd} & C \arrow[dr, "h'"] & \\ B \arrow[ur, "\psi"] \arrow[rr, "h"] & & \mathrm{coker}\,\varphi \end{tikzcd}$$

$h = \psi^* h' = h'\circ\psi$

$\Rightarrow \ker\psi \subset \ker h$

$\ker h = \mathrm{im}\,\varphi$

$\ker h = \ker(\mathrm{coker}\,\varphi) = \mathrm{im}\,\varphi$ $\square$

$\left(\mathrm{coker}\,\varphi = B/\mathrm{Im}\,\varphi\right)$

$G/P \hookrightarrow \mathbb{P}(V)$, $G = GL_n(\mathbb{C})$

$P = \mathrm{Stab}(\langle e_1, \dots, e_d\rangle)$

$G \curvearrowright G/P = X$ $\quad V = \Lambda^d \mathbb{C}^n$.

$G \curvearrowright X$

$G \curvearrowright V \Rightarrow G \curvearrowright \mathbb{P}(V)$

how is this action?

A: Is a representation of G!

Let $g \in G$. $f_g : V \longrightarrow V$

$\underbrace{e_{i_1}\wedge\dots\wedge e_{i_d}}_{\text{basis element of } V} \longmapsto \underbrace{ge_{i_1}\wedge\dots\wedge ge_{i_d}}_{\text{other basis element}}$

Does g permute the basis of $\Lambda^d\mathbb{C}^n$?

A: Yes, since f_g is invertible

$f_g \circ f_{g^{-1}} = id_V$

Then $g \in GL(V)$ in particular

$g \in \mathrm{End}_R(V)$. $\quad (\mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1)$

e.g. $T = (\mathbb{C}^\times)^n$, $V = \mathbb{C}^n$, $\mathbb{P}(\mathbb{C}^n) = \mathbb{P}^{n-1}$

$\boxed{n=2}$

$T = (\mathbb{C}^\times)^2$, $V = \mathbb{C}^2$

$t = (z_1, z_2)$, $(z_1,z_2)\cdot v$

$(z_1,z_2)\cdot(v_1,v_2) := (z_1v_1, z_2v_2)$.

$V = V_{(1,0)} \oplus V_{(0,1)}$

$T \curvearrowright \mathbb{C}^2$. $\quad (z_1,z_2) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$v \longmapsto \begin{bmatrix} z_1v_1 \\ z_2v_2 \end{bmatrix}$

$G \curvearrowright \mathbb{C}^2$

$\mathrm{Stab}\langle\begin{pmatrix}1\\0\end{pmatrix}\rangle = ?$. $\quad \mathcal{O}_{(1,0)} = V_{(1,0)} - (0,0)$

$\supset T$.

$\begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix}$.

$GL_2 \curvearrowright \mathbb{C}^2$ obvious representation (identity) is irreducible and faithful.

"Understanding" weights.

Let $\mathfrak{g}$ be a complex s.s. Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ (nilpotent subalgebra that is self-normalizing). Let V be a rep of $\mathfrak{g}/\mathbb{C}$ and let $\lambda \in \mathfrak{h}^*$. Then the weight space of V with weight λ is the subspace V_λ given by

$V_\lambda := \{v \in V \mid \forall h \in \mathfrak{h},\ h\cdot v = \lambda(h)v\}$

A weight is an element of

$\mathfrak{X} := \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{h}, \mathbb{C}) \subseteq \mathfrak{h}^*$.

(it suffices with a basis of $\mathfrak{h}$)

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ sl_2

$\boxed{\begin{array}{l} [h,e] = 2e \\ [h,f] = -2f \\ [e,f] = h \end{array}}$

e.1 $\boxed{sl_2} := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$ $a,b,c \in \mathbb{C}$

$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = ah + be + cf.$

$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \middle| - \right\} \subseteq sl_2.$

$\mathrm{Hom}_{\text{Lie-alg}}(\mathfrak{h}, \mathbb{C}) = \mathfrak{h}^*.$

If V is a rep of sl_2 $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$

let V be the standard rep of sl_2.

$\rho : sl_2 \longrightarrow \mathrm{End}(V)$ $V = \mathbb{C}^2$

$sl_2 \circlearrowright \mathbb{C}^2$ in obvious way

$\mathfrak{h} = \mathrm{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

$\subseteq sl_2.$

$\lambda \in \mathfrak{h}^*$, $\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$ is det by the image of h say $\lambda(h) = \lambda \in \mathbb{C}$. (abuse of notation here. ⚠)

$\mathfrak{h}^* \cong \mathbb{C}$. Want to check $\mathbb{C}^2 = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$

$V_\lambda = \{ v \in \mathbb{C}^2 \mid \underbrace{\forall h' \in \mathfrak{h},\ h' \cdot v = \lambda(h') \cdot v}_{\text{need check only for } h \text{ in } sl_2} \}.$

$\lambda(h) = \lambda.$

$h' = \mu h$, $\lambda(h') = \lambda(\mu h) = \mu \cdot \lambda(h) = \mu \cdot \lambda.$

let $v \in V_\lambda$ s.t. $h v = \lambda(h) v$

$h \cdot v = \lambda v.$

$\Rightarrow \mu h v = \mu \lambda v$

$h' v = \lambda(h') v.$

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $h - \lambda \mathrm{Id} = \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}$

$\det(h - \lambda \mathrm{Id}) = (1-\lambda)(1+\lambda) = 0 \Rightarrow \lambda = \pm 1.$

~~V_1~~ $\varpi : \mathfrak{h} \longrightarrow \mathbb{C}.$ $V_1 = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$h \mapsto 1$ $V_{-1} = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$

if $\lambda \neq \pm\varpi$

then $V_\lambda = 0$. $\mathbb{C}^2 = V_1 \oplus V_{-1}.$

e.1 $sl_2(\mathbb{C})$

$V = S^2(\mathbb{C}^2) = \mathbb{C}\{e_1^2, e_1 e_2, e_2^2\}$

$\dim V = 3.$ $\rho : sl_2(\mathbb{C}) \longrightarrow \mathrm{End}(V)$

MISLEADING CALCULUS

$g \in sl_2$ $g(e_1^2) = g(e_1) g(e_1)$?

$g(e_1 e_2) = g(e_1) g(e_2)$?

$(g+g') \cdot e_1^2 = (g+g') e_1 \, (g+g') e_1$

$= (g e_1 + g' e_1)^2$

$= (a_1 e_1 + a_2 e_2 + a_1' e_1 + a_2' e_2)^2$

$= (a_1 a_1' + a_1^2 + a_1'^2) e_1^2$

$+ (\quad\quad) e_2^2$

$+ (a_1 a_2 + a_1 a_2' + a_1' a_2) e_1 e_2.$

$g = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $g' = \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix}$

$(g+g') \cdot e_1^2$ $g \cdot e_1 = \begin{pmatrix} a \\ c \end{pmatrix}$

$(g+g') e_1^2 = \begin{pmatrix} a+a' \\ c+c' \end{pmatrix} \cdot \begin{pmatrix} a+a' \\ c+c' \end{pmatrix}$

$= [(a+a') e_1 + (c+c') e_2] \, [\quad\quad]$

~~$=$ taste~~ $a \leftrightarrow a_1$, $a' \leftrightarrow a_1'$

$c \leftrightarrow a_2$, $c' \leftrightarrow a_2'$

$= (a a' + a^2 + a'^2) e_1^2$

$+ (c c' + c^2 + c'^2) e_2^2$

$+ (a c + a c' + a' c) e_1 e_2.$

$g e_1^2 + g' e_1^2 = \begin{pmatrix} a \\ c \end{pmatrix} \cdot \begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} a' \\ c' \end{pmatrix} \cdot \begin{pmatrix} a' \\ c' \end{pmatrix}$

$= (a e_1 + c e_2)^2 + (a' e_1 + c' e_2)^2$

$= (a^2 + a'^2) e_1^2 + (c^2 + c'^2) e_2^2$

$+ (2ac + 2a'c') e_1 e_2.$

Lemma $v \in V_\lambda$, $e v \in V_{\lambda+2}$

$f v \in V_{\lambda-2}$

Pf

~~$h(ev)$~~

$(2e) \cdot v = [h, e] \cdot v = h(e v) - e h v = 2(e v)$, where $ehv = e\lambda v$

$\Rightarrow h(ev) = (\lambda+2)(ev)$ ▷

e.g. $\mathfrak{gl}_2 \supset \mathfrak{h} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \,\middle|\, \lambda, \mu \in \mathbb{C} \right\} \cong \mathbb{C}^2$

$\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{h_1, h_2\}$, $h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = a$

$h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = b$.

v.f. of h_1 & h_2. In $V = \mathbb{C}^2$ std rep

λ	0	1
h_1	$\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$	$\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$
h_2	$\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$	$\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$

(☆)

$\lambda \in \mathrm{Hom}_{Lie}(\mathfrak{h}, \mathbb{C}) \underset{h\,comm}{\cong} \mathfrak{h}^*$.

but $\mathfrak{h}^* \cong \mathbb{C}^2$

$\lambda \in \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ is det by

$\lambda(h_1) = x$ $\lambda = (x, y) \in \mathbb{C}^2$.

$\lambda(h_2) = y$.

$V_\lambda = \left\{ v \in V \,\middle|\, \begin{matrix} a \cdot v = xv \\ b \cdot v = yv \end{matrix} \right\}$

by (☆) we have

$V_{(0,1)} = \left\{ v \in V \,\middle|\, \begin{matrix} h_1 v = 0 \\ h_2 v = v \end{matrix} \right\} = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$.

$V_{(1,0)} = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$, $V_{(1,1)} = 0$.

$V_{(0,0)} = 0$.

$V_{(x,y)} = 0$ if $(x,y) \notin \{(1,0), (0,1)\}$.

$\mathbb{C}^2 = V_{(1,0)} \oplus V_{(0,1)}$

Comment about tensor product of Lie alg reps.

As we noticed in ⊕ if we try to define $V \otimes W$ for two representations V and W of a Lie algebra $\mathfrak{g}$. The definition

$g \cdot x \otimes y := (gx) \otimes (g \cdot y)$ does not work.

The trick is "differentiate" that definition

$g \cdot (x \otimes y) := g \cdot x \otimes y + x \otimes g \cdot y$

e.g. $\mathfrak{gl}_2(\mathbb{C})$, $\mathbb{C}^2 \otimes \mathbb{C}^2$ rep. $V = \mathbb{C}^2 \otimes \mathbb{C}^2$

$V = \mathbb{C}^2 \otimes \mathbb{C}^2$ $\rho : \mathfrak{gl}_2(\mathbb{C}) \to \mathrm{End}(V)$

$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $h_1(v_1 \otimes v_2)$

$h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $a h_1(e_1 \otimes e_1) = 2a(e_1 \otimes e_1)$

$h_1(e_1 \otimes e_1) = 2 e_1 \otimes e_1$, $h_1(e_1 \otimes e_2) = e_1 \otimes e_2$.

$h_1 e_1 = e_1$ $h_2(e_1 \otimes e_1) = 0$. $h_2(e_1 \otimes e_2) = e_1 \otimes e_2$.

$h_1 e_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(e_1 \otimes e_1) = 2a(e_1 \otimes e_1)$

---

e.g. $T \subset GL_2$, $S^2(\mathbb{C}^2) = \mathbb{C}\{e_1^2, e_1e_2, e_2^2\} = V$

$\begin{pmatrix} a & \\ & b \end{pmatrix} e_1^2 = ?$ tensor Action of Lie groups

$g \cdot (x \otimes y) = gx \otimes g \cdot y$.

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} e_1^2 = a^2 e_1^2$.

$\rho : T \to \mathrm{End}(V)$

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} e_1 e_2 = ab\, e_1 e_1$ $V \cong \mathbb{C}^3$

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} e_2^2 = b^2 e_2^2$ $T \subset G \curvearrowright V$

$\lambda \in X(T)$

$V_\lambda = \{ v \in V \mid \forall t \in T,\ tv = \lambda(t) v \}$.

$X(T) = \mathrm{Hom}(T, \mathbb{C}^\times) =$

$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto a^n b^m \,\middle|\, n, m \in \mathbb{Z} \right\} \simeq \mathbb{Z}^2$.

e_1^2 weight $(2,0)$. $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(v \otimes w) = \lambda(v \otimes w)$.

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(v_1 e_1 + v_2 e_2) \otimes (w_1 e_1 + w_2 e_2)$

$= \begin{pmatrix} v_1 a \\ v_2 b \end{pmatrix} \otimes \begin{pmatrix} w_1 a \\ w_2 b \end{pmatrix} = [v_1 a e_1 + v_2 b e_2] \otimes [w_1 a e_1 + w_2 b e_2]$

$(v_1 e_1 + v_2 e_2) \otimes (w_1 e_1 + w_2 e_2) = v \otimes w$.

$= v_1 w_1 e_1^2 + (v_1 w_2 + v_2 w_1) e_1 e_2 + v_2 w_2 e_2$.

The other is $(v_1 a e_1 + v_2 b e_2) \otimes (w_1 a e_1 + w_2 b e_2)$

$= a^2 v_1 w_1 e_1^2 + ab(v_1 w_2 + v_2 w_1) e_1 e_2 + b^2 v_2 w_2 e_2^2$

$t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\forall t$, $\boxed{t(v \otimes w) = \lambda(t)\, v \otimes w}$.

$\lambda \in X(T)$,

$\lambda_{(n,m)} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a^n b^m$.

$\lambda_{(n,m)} \longleftrightarrow (n,m)$

$t(v \otimes w) = a^n b^m\, v \otimes w$.

possible eigenvectors of t in $V \cong \mathbb{C}^3$ (still do not use $\lambda \in X(T)$)

$t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in T$.

first

$t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ forces to the e.v. to satisfy

(☆) $v_1 w_2 + v_2 w_1 = 0$ and $v_2 w_2 = 0$.

sup $v_2 \neq 0$, $\underset{(☆)}{\Rightarrow}$ $w_2 = 0$. $\Rightarrow v \otimes w = \begin{pmatrix} a \\ 0 \end{pmatrix} \otimes \begin{pmatrix} b \\ 0 \end{pmatrix}$

$= ab\, e_1^2$

$= c e_1^2$.

$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $v \otimes w = \lambda\, v \otimes w$.

$h_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$. $V_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$

$V_0 = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$.

$h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$h_2 e_2 = e_2$ $V_1 = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$

$V_0 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ does not belong to T!!

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a \neq 0$, $b \neq 0$ $\in T$.

$t = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $V_a = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$, $V_1 = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$

$t_2 = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ $V_b = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$, $V_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$.

$T = \mathbb{C}^\times \times \mathbb{C}^\times$.

t_1 $v \otimes w$

$t_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 a \\ v_2 \end{pmatrix}$, $t_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 b \end{pmatrix}$

$t_1 \cdot (v \otimes w) = \begin{pmatrix} a v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} a w_1 \\ w_2 \end{pmatrix}$

$= a^2 v_1 w_1 e_1^2 + a(v_1 w_2 + w_1 v_2) e_1 e_2$

$+ v_2 w_2 e_2^2$

$t_1(v \otimes w) = \lambda(v \otimes w) = \lambda v_1 w_1 e_1^2 + \lambda(\cdots) + \lambda v_2 w_2 e_2^2$

If $\lambda = 0$.

$v_2 w_2 = 0$, sup $v_2 = 0 \underset{a \neq 0}{\Rightarrow}$ $v_1 w_2 = 0$.

$w_1 \neq 0 \Rightarrow w_2 = 0 = w_1 = 0$

I got the 0 vector not an eigenvector

then $v_1 = 0$.

then $\lambda \neq 0$.

$\lambda v_2 w_2$

Better way to compute e.v. $V \cong \mathbb{C}^3$.

$t_1 = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $t_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^2 \end{bmatrix}$

$t_{(a,1)}$

$t_1(e_1 \otimes e_1) = a^2 e_1^2$.

$t_1(e_1 \otimes e_2) = a e_1 e_2$

$t_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\, e_1 e_2$.

$t_1(e_2 \otimes e_2) = e_2 \otimes e_2$

$|t_1 - \lambda Id| = 0 = (a^2 - \lambda)(a - \lambda)(1 - \lambda) = 0$.

$\lambda = \begin{cases} a^2 \\ a \\ 1 \end{cases}$ $a \neq 0$.

$V_{a^2} = \langle e_1 \otimes e_1 \rangle$

$V_a = \langle e_1 \otimes e_2 \rangle$

$V_1 = \langle e_2^2 \rangle$

For several $t \in T$

$$t = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & b^2 \end{bmatrix} \qquad \lambda = \begin{cases} a^2 \\ ab \\ b^2 \end{cases}$$

a, b, ab, a^2, b^2 all $\neq 0$.

$$V_{a^2} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad V_{ab} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad V_{b^2} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

unique possible eigenvectors

Common e.vectors, for all $t \in T$.

$$\underset{\lambda}{\overset{\Downarrow}{X(T)}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto a^n b^m \right\}_{(n,m)\in\mathbb{Z}^2} \cong \mathbb{Z}^2$$

Let $\lambda : T \longrightarrow \mathbb{C}^\times$ fixed.

$\lambda = \lambda(n,m)$

e.g.) $\lambda = (1,1)$ fixed

$$V_\lambda = \{ \tilde{v} \in \mathbb{C}^3 \mid \forall t \in T,\ t \cdot \tilde{v} = \lambda(t) \cdot \tilde{v} \}$$

should verify

$$t_a \cdot \tilde{v} = \lambda(t) \cdot \tilde{v} = a \tilde{v}$$

$t_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \overline{t_a} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda(t_a) = a^1 \cdot 1^1 = a$

$t_a \cdot \tilde{v} = a \tilde{v}$.

$\Rightarrow \tilde{v} \in V_a$ as espace for t_a

$\Rightarrow \tilde{v} \in \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ (if $a \neq 1$) or

$\tilde{v} \in \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle$ if $a = 1$. no restrictions!

Since all a occur, $\tilde{v} \in \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$

this is compatible with $t_b = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$

$\Rightarrow V_{(1,1)} = \mathbb{C}\{e_1 e_2\}$

$\lambda = (1,0)$

$$V_{(1,0)} = \{ \tilde{v} \in \mathbb{C}^3 \mid \forall t \in T,\ t \cdot \tilde{v} = \lambda(t) \cdot \tilde{v} \}$$

$t = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $\quad a \neq 1$

e.g. $t_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad \overline{t_0} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda(t_0) = a^1 b^0 = 2$.

need $\tilde{v}$ s.t. $\overline{t_0}\, \tilde{v} = 2 \tilde{v}$

$\Rightarrow \tilde{v} \in \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ all ok!

try with $t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\quad \overline{t_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\lambda(t_1) = 1 \Rightarrow \tilde{v} \in \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle \cap \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$

$\Rightarrow \tilde{v} = 0$.

$\Rightarrow V_{(1,0)} = 0$.

$\lambda = (2,0)$

$t = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \overline{t} = \begin{pmatrix} a^2 & & \\ & a & \\ & & 1 \end{pmatrix}$ $\quad a^2 \neq a$ if $a \neq 1$.

$\lambda(t) = a^2$ need $\tilde{v}$ s.t.

$\overline{t}\, \tilde{v} = a^2 \tilde{v} \Rightarrow \tilde{v} \in \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle$.

$t = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ $\qquad \overline{t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{pmatrix}$

need $\tilde{v}$ s.t.

$\overline{t}\, \tilde{v} = \tilde{v} \Rightarrow \tilde{v} \in \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle$

$\lambda(t) = 1$ $\quad$ valid for all $b \in \mathbb{C}$!

then $V_{(2,0)} = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle$. etc.

$\begin{pmatrix} a & \\ & b \end{pmatrix} e_1^2 = a^2 e_1^2$ $\quad$ weigth $(2,0)$

$\begin{pmatrix} a & \\ & b \end{pmatrix} e_1 e_2 = ab\, e_1 e_2$ $\quad$ " $(1,1)$

$\begin{pmatrix} a & \\ & b \end{pmatrix} e_2^2 = b^2 e_2^2$ $\quad$ " $(0,2)$

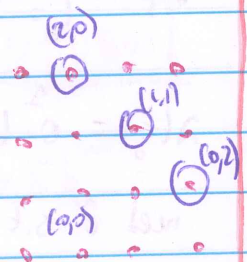
So, $GL_2 \curvearrowright^{V:} \mathbb{C}\{e_1^2, e_1 e_2, e_2^2\} \cong \mathbb{C}^3$

$\overset{\cong}{S^2(\mathbb{C}^2)}$

$V|_{T \subset GL_2}$ is a rep of T.

$X(T)$ = 1-dim reps of T.

$$V|_T = \bigoplus_{\lambda \in X(T)} V_\lambda .$$



$$\mathbb{C}^3 = V_{(1,1)} \oplus V_{(2,0)} \oplus V_{(0,2)} .$$

$$T = \begin{pmatrix} * & & & \\ & * & & \\ & & \ddots & \\ & & & * \end{pmatrix} \subseteq GL_n \curvearrowright \Lambda^2(\mathbb{C}^n)$$

weight spaces are

$\mathbb{C} e_1 \wedge e_2$ $(1,1,0,0,\text{—})$

$\mathbb{C} e_1 \wedge e_3$ $(1,0,1,0,\text{——})$.

$\vdots$

Thm if V f.d rep of s.s Lie alg $\mathfrak{g} \supset \mathfrak{h}$

then

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda .$$

eg $\mathfrak{sl}_2$, $\mathbb{C}^2 = V$ $\mathfrak{h}^* = \mathbb{C}\varpi_1$

$V_{\varpi_1} \cong \mathbb{C}$ $\mathbb{C}^2 \cong V_{\varpi} \oplus V_{\varpi - 2}$.

$V_{\varpi_1 - 2} \cong \mathbb{C}$

$V_\lambda = 0$ for all others.

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $V_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ $\mathbb{C}^2 = V_1 \oplus V_{-1}$.

$V_{-1} = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$.

$\varpi_1 = 1$ fundamental weight.

$\mathfrak{h} = \mathrm{span}_{\mathbb{C}} h \cong \mathbb{C}$, $\mathfrak{h}^* \cong \mathbb{C}$.

⊛ Thm $T \subseteq G \curvearrowright V$, then $[v] \in \mathbb{P}(V)$ is T-fixed iff $v \in \bigcup_{\lambda \in X(T)} V_\lambda \setminus \{0\}$.

Proof:

$(\Leftarrow)$ let V a G-rep.

$$V|_T = \bigoplus_{\lambda \in X(T)} V_\lambda$$

let $\lambda \in X(T)$ s.t $V_\lambda \neq 0$.

let $v \in V_\lambda$ $v \neq 0$.

$\forall t \in T$ $t \cdot v = \lambda(t) v$. $\lambda(t) \in \mathbb{C}^\times$

let $t \in T$

$t \cdot [v] = [t \cdot v] = [\lambda(t) v] = [v]$

then $[v]$ is T-fixed □

$(\Rightarrow)$ Suppose v has non-zero projections in at least two distinct V_λ.

i.e. $\exists \lambda, \mu$ s.t $\pi_\lambda(v) = v_\lambda \neq 0$

$\pi_\mu(v) = v_\mu \neq 0$.

to prove $[v]$ is not T-fixed.

$\lambda, \mu \in X(T)$ $\lambda \neq \mu$.

$\exists t \in T$ s.t $\lambda(t) \neq \mu(t)$.

$t \cdot [v_\lambda + v_\mu + \cdots]$

$= [\lambda(t) v_\lambda + \mu(t) v_\mu + \cdots] \neq [v]$ □

"The" Thm

$T = \mathbb{G}_m^r$ torus over k

Rep $T \cong \mathfrak{X}$-graded k-modules

where

$\mathfrak{X} = \mathrm{Hom}_{\text{alg gps}}(T, \mathbb{G}_m)$ denotes the

"$X(T)$" character lattice of T

"The" Cor $G \curvearrowright V$

$$V|_T = \bigoplus_{\lambda \in \mathfrak{X}} V_\lambda .$$

This gives us a better proof of the fact that in $Gr(d,n) =: X$

(where $G := GL_n(\mathbb{C}) \curvearrowright X$ since acts in $V := \Lambda^d \mathbb{C}^n$) $V = \Lambda^d \mathbb{C}^n$.

we have

$$\left\{ \begin{matrix} T\text{-fixed points} \\ \text{in } X \end{matrix} \right\} = \left\{ \begin{matrix} E_I \ ; \\ I \text{ multi-index} \\ i_1 < i_2 < \dots < i_d \end{matrix} \right\}$$

$$T \curvearrowright \mathbb{P}(V), \quad T \curvearrowright V \cong \mathbb{C}^{\binom{n}{d}}.$$

what are the weight spaces?

$$\lambda \in X(T) = \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^n.$$

$$= \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \\ & & & a_n \end{pmatrix} \longmapsto a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} \right\}$$

$$V_\lambda = \{ v \in V \mid \forall t \in T, t \cdot v = \lambda(t) v \}.$$

$$t \cdot v = \lambda v$$

$$t = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \quad t \cdot (e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_d})$$
$$= (te_{i_1} \wedge te_{i_2} \wedge \dots \wedge te_{i_d})$$
$$= \left(\prod a_{i_j}\right)(e_{i_1} \wedge \dots \wedge e_{i_d})$$

$$t e_{i_j} = a_{i_j} e_{i_j}$$

$$V = \bigoplus_{\substack{I \\ \text{multi-index} \\ i_1 < i_2 < \dots < i_d}} V_I$$

$$t : V \longrightarrow V.$$
$$e_I \longmapsto \left(\prod_{i \in I} a_i\right) e_I.$$

$$t = \begin{bmatrix} \prod_{i \in I_1} a_i & 0 & \cdots \\ & \prod_{i \in I_2} a_i & \\ & & \ddots \end{bmatrix}$$ t is a diagonal matrix.

e.g. $d = 2$, $n = 4$.

$$t = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \qquad \binom{n}{2} = 6.$$

$t(e_1 \wedge e_2) = ab\, e_1 \wedge e_2$. weight $(1,1,0,0)$.

$$t = \begin{pmatrix} ab & 0 & 0 & 0 & 0 & 0 \\ 0 & ac & 0 & 0 & 0 & 0 \\ 0 & 0 & ad & 0 & 0 & 0 \\ 0 & 0 & 0 & bc & 0 & 0 \\ 0 & 0 & 0 & 0 & bd & 0 \\ 0 & 0 & 0 & 0 & 0 & cd \end{pmatrix}$$

$[12, 13, 14, 23, 24, 34]$ Plücker coordinates!

$\updownarrow$

weights

$[1,1,0,0] \rightsquigarrow ab$

$[1,0,1,0]$ ac

$[1,0,0,1]$ ad

$[0,1,1,0]$ bc

$[0,1,0,1]$ bd

$[0,0,1,1]$. cd.

V_I where $I = \{i_1, \dots, i_d\}$ is the same as the weight space V_λ where $\lambda : T \longrightarrow \mathbb{C}^\times$ given by

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \longmapsto \prod_{i \in I} a_i$$

<u>Thm</u> $T \subseteq G \curvearrowright V$ $V = \bigoplus_{\lambda \in \mathfrak{X}} V_\lambda$. Then:

$[v] \in \mathbb{P}(V)$ is a 1-dim orbit G-module $\iff v \in V_\chi \oplus V_{\chi'}$

<u>Proof</u>: $(\Leftarrow)$ let $v \in V_\chi \oplus V_{\chi'}$ w/ no trivial projections)

$$t \cdot [v] = [v_\chi \cdot \chi(t) + v_{\chi'} \cdot \chi'(t)]$$
$$= \left[v_\chi + \frac{\chi'(t)}{\chi(t)} v_{\chi'} \right]$$

We have a map $\mu: T \to \mathbb{C}^\times$, $z \mapsto z_1^{n_1} z_2^{n_2} \cdots$, $n \in \mathbb{Z}$.

$$\varphi: T \longrightarrow \mathbb{P}^1$$

$$t \longmapsto [\mu(t) ; \chi(t)]$$

If $\mu = \chi$ the image is one point $[1:1]$

If $\mu \neq \chi$ we can consider

$$\frac{\mu}{\chi} : T \longrightarrow \mathbb{C}^\times$$

$$t \longmapsto z_1^{n_1 - n_1'} z_2^{n_2 - n_2'} z_3^{n_3 - n_3'} \cdots$$

$\frac{\mu}{\chi} = \lambda : T \longrightarrow \mathbb{C}^\times$ $\lambda \in X(T)$.

$\mu = \chi \Rightarrow \lambda$ is the map sending all to 1

$$z \longmapsto z_1^0 z_2^0 \cdots z_n^0$$

weight $(0, \dots, 0)$

$\mu \neq \chi$ means λ is not the $(0, \dots, 0)$ wt.

$\mathbb{G}_m = \mathbb{C}^\times \xrightarrow{\overline{\varphi}} \mathbb{C}^\times$ $\quad \mathbb{C}^\times = \operatorname{Spec} k[t,t^{-1}]$.

$k[t,t^{-1}] \xrightarrow{\varphi} k[t,t^{-1}]$ $\quad \alpha t^i \in k[t,t^{-1}]^\times$

$t \longmapsto \alpha t^i$

$$k[t,t^{-1}]^\times = \left\{ \sum_{i \geq -k}^{m} a_i t^i \right\}^\times = \{ a_i t^i \}.$$

$\mathbb{C}^\times = \operatorname{Spec} k[t,t^{-1}] \longrightarrow \operatorname{Spec} k[t,t^{-1}] = \mathbb{C}^\times$

$\ni m = (t - \alpha) \longmapsto \varphi^{-1}(m) = \varphi^{-1}(t-\alpha)$, $\alpha \neq 0$

comultiplication of $\mathbb{C}^\times = \mathbb{G}_m \to \Delta$

$$k[t,t^{-1}] \xrightarrow{\Delta} k[t,t^{-1}] \otimes k[t,t^{-1}]$$

$$t \longmapsto t \otimes t.$$

Lemma $\operatorname{Hom}_{\text{alg gp}}(k^\times, k^\times) \cong \mathbb{Z}$, $k = \overline{k}$.
$\{ z \mapsto z^n \mid n \in \mathbb{Z} \}$

Pf $k^\times = \operatorname{Spec}(k[t,t^{-1}])$

$f \in \operatorname{Hom}_{\text{alg gp}}(k^\times, k^\times) \rightsquigarrow \overline{f}$

$k^\times = \operatorname{Spec}(k[t,t^{-1}])$.

$\overline{f} : k[t,t^{-1}]$

Lemma: X alg scheme $/k$

$$\operatorname{Hom}(X, \mathbb{G}_m) \cong \operatorname{Hom}_{k\text{-alg}}(k[t,t^{-1}], \mathcal{O}_X(X)) \cong \mathcal{O}_X(X)^\times$$

The morphism $G \to \mathbb{G}_m$ corresponding to $f \in \mathcal{O}_G(G)^\times$ is a homomorphism iff

$$\Delta_G(f) = f \otimes f.$$

Proof / Remarks / Facts:

Theorem: G, H affine group schemes then:

$$\operatorname{Hom}_{\text{gp sch}}(G,H) \cong \operatorname{Hom}_{\text{Hopf}}(k[H], k[G])$$

Definition: A Hopf algebra homomorphism is a bi-algebra homomorphism.

Thm Let $\phi : H \to K$ a bialgebra map, H, K Hopf algebras, then

$$\phi S_H = S_K \phi = \phi^{-1}$$

Def: A group scheme homomorphism from G to H is just a map $G \to H$ st

$$\begin{tikzcd} G \times_k G \arrow[r, "\varphi \times_k \varphi"] \arrow[d, "m_G"'] & H \times_k H \arrow[d, "m_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

commutes.

⚠ Remark: No need conditions on involution nor identity. (This is very strange!)

$$(*) \quad \begin{tikzcd} k[G] \otimes_k k[G] & k[H] \otimes_k k[H] \arrow[l] \\ k[G] \arrow[u, "\Delta_G"] & k[H] \arrow[l, "\varphi^*"] \arrow[u, "\Delta_H"'] \end{tikzcd}$$

$$G = \operatorname{Spec} \mathcal{O}_G(G) \xrightarrow{\ \varphi\ } \mathbb{G}_m = \operatorname{Spec}(k[t,t^{-1}])$$

$$k[t,t^{-1}] \xrightarrow{\ \varphi^*\ } \mathcal{O}_G(G)$$
$$t \longmapsto f \in \mathcal{O}_G(G)^\times$$

we know : $\varphi^*(t) = f \in \mathcal{O}_G(G)^\times$

Let us see condition on f given by the diagram (★)

$$\begin{array}{ccc}
k[G]\otimes_k k[G] & \xleftarrow{\ \varphi^*\otimes\varphi^*\ } & k[H]\otimes_k k[H] \\
\Delta_G \uparrow & & \uparrow \Delta_H \\
k[G] & \longleftarrow & k[H] = k[t,t^{-1}]
\end{array}$$

$$\begin{array}{ccc}
\Delta_G(f) & \longleftarrow & t\otimes t \\
\uparrow & & \uparrow \\
f & \longleftarrow & t
\end{array}$$

$\varphi^*\otimes\varphi^*(t\otimes t) = \varphi^*(t)\otimes\varphi^*(t) = f\otimes f.$

Then
$$f\otimes f = \Delta_G(f)$$

In particular, if $G = k^\times = \mathbb{G}_m$.

$$f \in k[t,t^{-1}]^\times = \{\lambda t^i \mid \lambda \in k^\times = k-\{0\},\ i\in\mathbb{Z}\}$$

$$f\otimes f = \Delta_G(f)$$

$$(\lambda t^i)\otimes(\lambda t^i) = \Delta_G(\lambda t^i)$$

$$\Delta_G(\lambda t^i) = \lambda\Delta_G(t^i) = \lambda\,[\Delta_G(t)]^i$$

Since Δ is an algebra map.

$$\lambda^2(t^i\otimes t^i) = \lambda(\Delta_G(t))^i = \lambda(t\otimes t)^i$$

$\Delta_G : A \longrightarrow A\otimes A$ is hom of algebras

What means to $A\otimes A$ be an algebra?

Define it for $A\otimes B$, where A,B algebras

$$\begin{tikzcd}
(A\otimes B)\otimes(A\otimes B) \arrow[r, "1\otimes\tau\otimes 1"] \arrow[dr, dashed, "m_{A\otimes B}"'] & (A\otimes A)\otimes(B\otimes B) \arrow[d, "m_A\otimes m_B"] \\
 & A\otimes B
\end{tikzcd}$$

Then $(a\otimes b)\cdot(c\otimes d) = ac\otimes bd$

In particular $(a\otimes b)^n = a^n\otimes b^n$.

$$\lambda^2(t^i\otimes t^i) = \lambda\, t^i\otimes t^i$$

$\Rightarrow \lambda = \lambda^2 \quad \lambda\neq 0 \Rightarrow \lambda = 1.$

Then $f = t^i \quad \forall i\in\mathbb{Z}.$

$\Rightarrow \operatorname{Hom}_{alg\,gp}(k^\times, k^\times) \cong \mathbb{Z}$

$\Rightarrow \operatorname{Hom}_{alg\,gp}(T, k^\times) \cong \mathbb{Z}^r, \quad T = (k^\times)^r.$

$$z \longmapsto z_1^{n_1}z_2^{n_2}\cdots z_r^{n_r}$$
$$T \longrightarrow k^\times$$

$$k[t,t^{-1}] \longrightarrow k[T]$$
$$t \longmapsto f\in k[T]^\times,$$

$$k[t,t^{-1}] \longrightarrow k[t,t^{-1}]\otimes k[t,t^{-1}]\otimes\cdots \cong k[t_1,t_1^{-1},t_2,t_2^{-1},\ldots,t_r,t_r^{-1}]$$

$$k[T]^\times = \{\lambda t_1^{n_1}t_2^{n_2}\cdots t_r^{n_r}\} \ni f$$

$\Delta_T(f) = f\otimes f$. multiindex: $t_\alpha^{n_\alpha}$. $n_\alpha\in\mathbb{Z}^r$.

$$\Delta_T(\lambda t_1^{n_1}\cdots t_r^{n_r}) = \lambda^2\, t_\alpha^{n_\alpha}\otimes t_\alpha^{n_\alpha}$$
$$= \lambda\,[t_\alpha^{n_\alpha}\otimes t_\alpha^{n_\alpha}] \Rightarrow \lambda = 1$$

$$f = t_1^{n_1}\cdots t_r^{n_r}.$$

$$\boxed{\begin{array}{c} k[t,t^{-1}] \longrightarrow k[t,t^{-1}] \\ t \longmapsto t^n \end{array}}$$

$$k^\times \longrightarrow k^\times, \quad z \longmapsto z^n$$

$$\operatorname{Spec} k[t,t^{-1}] \longrightarrow \operatorname{Spec} k[t,t^{-1}]$$
$$\cup\!\!\!\!\!\shortmid$$
$$(t-\alpha) \lhd k[t,t^{-1}] \longmapsto (t-\alpha^n)$$

$$\varphi^{-1}((t-\alpha)) = \{p(t)\in k[t,t^{-1}] : p(t^n)\in(t-\alpha)\}$$

$$p(t^n)\in(t-\alpha) \iff p(\alpha^n) = 0 \iff (t-\alpha^n)\mid p(t).$$

$$\iff p(t)\in(t-\alpha^n) \qquad = (t-\alpha^n) \qquad \square$$

Thm $T \subset G \curvearrowright V = \bigoplus_{\lambda \in X(T)} V_\lambda$.

$[v]$ 1-dim orbit $\iff v \in V_\lambda \oplus V_\mu$ (non trivially)

Proof: ($\Leftarrow$) O.K.

($\Rightarrow$) $\boxed{\text{Thm: Different characters are linearly independent}}$

Lemma:

$V = \bigoplus_{\lambda \in X(T)} V_\lambda$ $\quad [\lambda_1(t) v_\lambda + \lambda_2(t) v_\mu + v_\beta]$ $(t \in T)$

has dimension 2.

Proof

First start with this lemma

Lemma:

S set of vectors $S \subset V$, $\lambda S \subset S$, $0 \notin S$.

suppose $\dim S = n+1$ then

$\dim([s] : s \in S) = n$.

Proof omitted

Consider $S = \{\lambda_1(t) v_\lambda + \lambda_2(t) v_\mu + \lambda_3(t) v_\beta\}$ $t \in T$.

where $v_\lambda, v_\mu, v_\beta$ are all L.I.

Claim $\dim S = 3$. Obviously $\dim S \leq 3$

Suppose $\dim S \leq 2$. Then we would have

~~Since v_β, v_λ and v_μ are all L.I.~~

$\Rightarrow \lambda_3$

$\lambda_1(t) v_\lambda + \lambda_2(t) v_\mu + \lambda_3(t) v_\beta \overset{!}{=} a(t) f_1 + b(t) f_2$

$f_1, f_2 \in \text{span}\{v_\lambda, v_\mu, v_\beta\}$

$f_1 = \mu_1 v_\lambda + \mu_2 v_\mu + \mu_3 v_\beta$

$f_2 = \mu_1' v_\lambda + \mu_2' v_\mu + \mu_3' v_\beta$.

then we would have

For each $t \in T$

$\lambda_1(t) = a(t) \mu_1 + b(t) \mu_1'$

$\lambda_2(t) = a(t) \mu_2 + b(t) \mu_2'$

$\lambda_3(t) = a(t) \mu_3 + b(t) \mu_3'$

WLOG

$\gamma_1 \lambda_1(t) + \gamma_2 \lambda_2(t) = \gamma_3 b(t)$

$b(t) = \bar\gamma_1 \lambda_1(t) + \bar\gamma_2 \lambda_2(t) = b(t)$

$\Rightarrow \lambda_3(t)$ is linear combination $\left(\begin{matrix}t \text{ indepen-}\\ \text{dent}\end{matrix}\right)$ of $\lambda_2(t), \lambda_1(t)$ for each t

$\Rightarrow$ all $\bar\gamma$ are 0 since $\lambda_1, \lambda_2, \lambda_3$ are LI.

but $\bar\gamma = 0 \Rightarrow \lambda_1, \lambda_2$ are LD ✗

then $\dim S = 3$

then $\dim [S] = 2$.

then orbit of $[v]$ dim 1

$\Rightarrow v \in V_\lambda \oplus V_\mu$ □

$\text{End}(k^n \otimes_k A)^\times = \text{End}(A^{\oplus n})^\times$

$= GL_n(A)$

$A \mapsto GL_n(A)$ the functor for GL_n

Why $k^n \otimes_k A \cong A^{\oplus n}$?

e.g. $n=2$, A f.g. k-algebra

$k^2 \otimes A \longrightarrow A \oplus A$.

$\binom{\mu}{\lambda} \otimes a \longmapsto \mu a \oplus \lambda a = \binom{\mu a}{\lambda a}$

General GL_V is the functor

$A \longmapsto \text{End}(V \otimes_k A)^\times = GL(V \otimes_k A)$

Zariski tangent space:

vector space over the residue field A/m

(Zariski tangent space)

(A,m) local ring, m/m^2 cotangent space$^\vee$ of A

If X is a scheme the Zariski cotangent space $T^\vee_{X,p}$ at a point p is the cotangent space of the local ring

$$\mathcal{O}_{X,p} \left(= \lim_{U \ni p} \mathcal{O}(U) = \left\{ \langle \tfrac{f}{g}, U \rangle : g|_U \neq 0 \right\}\right)$$

(If $p \in U = \operatorname{Spec} A$, then $\mathcal{O}_{X,p} \cong A_p$)

Some motivation and ideas:

A derivation at a point p of a manifold X is a $\mathbb{R}$-linear operation taking functions $f : U_p \to \mathbb{R}$, $U_p \ni p$ open "near" p ($f \in \mathcal{O}_p$) and output elements $f'(p) \in \mathbb{R}$, s.t. they satisfy Leibnitz rule:

$$(fg)' = f'g + g'f \quad \left(\text{let } m \text{ maximal of } \mathcal{O}_p\right)$$

So a derivation is the same as a map $m \longrightarrow \mathbb{R}$ (to extend to $\mathcal{O}_{X,p}$ use the map $\mathcal{O}_{X,p} \longrightarrow m$

$$f \longmapsto f - f(p).$$

$m \triangleleft \mathcal{O}_{X,p}$, $m = \left\{ \frac{f}{g} \,\middle|\, \begin{matrix} f \in p \\ g \notin p \end{matrix} \right\}$

$$= \left\{ \frac{f}{g} \,\middle|\, \begin{matrix} f(p) = 0 \\ g(p) \neq 0 \end{matrix} \right\}.$$

but m^2 maps to 0.

i.e. $f(p) = g(p) = 0$.

$$(fg)'(p) = 0 = f'(p)g(p) + f(p)g'(p).$$

So we have a map

$$m/m^2 \longrightarrow \mathbb{R}. \text{ i.e.}$$

an element of $(m/m^2)^\vee$.

Exercise: $f \in (m/m^2)^\vee$ satisfy the Leibnitz rule (gives a derivation)

(i.e. this process is reversible)

Solution:

$\varphi : m/m^2 \longrightarrow \mathbb{R}$. $\mathbb{R}$-linear

$$f + m^2 \longmapsto \varphi(f) \in \mathbb{R}.$$

$$\hat{\varphi} : \mathcal{O}_{X,p} = A_p \longrightarrow m$$

$$f \longmapsto f - f(p).$$

$$\tilde{\varphi} : \mathcal{O}_{X,p} = A_p \longrightarrow \mathbb{R}.$$

$$f \longmapsto \varphi(f - f(p))$$

$$\tilde{\varphi}(f) := f'(p).$$

$$(fg)'(p) = \tilde{\varphi}(\underbrace{fg - f(p)g(p)}_{\in m}).$$

$\overset{?}{\parallel}$

$$f(p)g'(p) + f'(p)g(p)$$

$$= f(p)\tilde{\varphi}(g - g(p)) + \tilde{\varphi}(f - f(p))g(p)$$

$$= \tilde{\varphi}((f - f(p))g(p) + (g - g(p))f(p)).$$

To prove

$$fg - f(p)g(p) - (f - f(p))g(p) - (g - g(p))f(p) \in m^2.$$

$$= fg - f(p)g(p) - f \cdot g(p) + f(p)g(p) - g \cdot f(p) + f(g)g(p)$$

$$= fg - f \cdot g(p) - g \cdot f(p) + f(p)g(p)$$

$$= (f - f(p))(g - g(p)) \in m^2 \;/\!/.$$

A vague motivation

Let f a function on $\mathbb{A}^n$. Near 0 it is approximated by a linear function on the tangent space. e.g. $x^2 + xy + x + y \sim x + y$ $(x,y) \to 0$.

Hence m/m^2 is the cotangent space (linear functions) on the tangent

If A is a ring, $m \triangleleft A$ maximal, $f \in m$ then $\operatorname{Spec}[A/(f)]$ tangent space at m is the tangent space at m in $\operatorname{Spec} A$ cut out by the equation $f \bmod m^2$.

e.g. Spec $\mathbb{Z} = X$. $\quad \mathfrak{p} \lhd \mathbb{Z} \quad \mathfrak{p} = (p)$
p prime or 0.

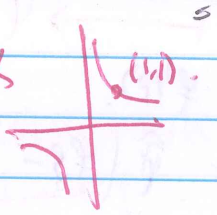
$$\mathcal{O}_{X,\mathfrak{p}} = \mathbb{Z}_{\mathfrak{p}} = \left\{ \frac{n}{m} \,\middle|\, p \nmid m \right\}$$

residue fields

$$\frac{\mathbb{Z}_{\mathfrak{p}}}{p\mathbb{Z}_{\mathfrak{p}}} \underset{\text{Thm}}{\cong} \mathrm{Frac}\left(\mathbb{Z}/p\mathbb{Z}\right) = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p.$$

if $\mathfrak{p} = (0)$: $\quad \frac{\mathbb{Z}_{\mathfrak{p}}}{\mathfrak{p}\mathbb{Z}_{\mathfrak{p}}} = \mathbb{Q}/(0)\mathbb{Q} = \mathbb{Q} = \mathrm{Frac}\left(\mathbb{Z}/0\mathbb{Z}\right) = \mathbb{Q}$.

e.g. $k[x,y]/(xy-1) =: X$



Compute $T_{(1,1)}X$.

Remark The ring $\mathcal{O}_{X,\mathfrak{p}}$, $\mathfrak{p} \in X$, is local and $m_{\mathfrak{p}} \lhd \mathcal{O}_{X,\mathfrak{p}}$ unique maximal ideal.

$\frac{\mathcal{O}_{X,\mathfrak{p}}}{m_{\mathfrak{p}}}$ residue field denoted by $k(\mathfrak{p})$

~~Claim $m := m_{\mathfrak{p}}$ is a $k(\mathfrak{p})$ vector space~~

~~define $k(\mathfrak{p})$ action on $m_{\mathfrak{p}}$~~

~~$\cdot : k(\mathfrak{p}) \times m \longrightarrow m$~~

~~$((a+m), f) \longmapsto af$~~

Claim m/m^2 is a $k(\mathfrak{p})$ vector space

$$\cdot : k(\mathfrak{p}) \times m/m^2 \longrightarrow m/m^2$$
$$(a+m, f+m^2) \longmapsto (af + m^2)$$

well defined:

$a' = a \bmod m \qquad af = a'f' \bmod m^2$?

$f' = f \bmod m^2$

$$af - a'f' = af - (a'-a+a)f'$$
$$= \underbrace{a(f-f')}_{\in m^2} - \underbrace{(a'-a)}_{\in m}\underbrace{f'}_{\in m} \in m^2 \;\square.$$

e.g. (Cont.) $\quad m_{\mathfrak{p}} = \{ g \in k[X] \mid g(1,1) = 0 \}$
$= (x-1, y-1) \bmod (xy-1)$

$\mathfrak{p} = (1,1)$

$$\mathcal{O}_{X,\mathfrak{p}} = \left\{ \frac{f}{g} \,\middle|\, f, g \in \frac{k[x,y]}{(xy-1)},\ g(1,1) \neq 0 \right\}$$

$k[X]_{m_{\mathfrak{p}}} = A_{m_{\mathfrak{p}}} \qquad A = k[X]$

$$k(\mathfrak{p}) = \frac{A_{m_{\mathfrak{p}}}}{m_{\mathfrak{p}} A_{m_{\mathfrak{p}}}} \cong A/(x-1, y-1)_{\bmod (xy-1)} \cong k.$$

$$m = m_{\mathfrak{p}} A_{m_{\mathfrak{p}}} = \left\{ \frac{f}{g} \,\middle|\, \begin{array}{l} f(1,1) = 0 \\ g(1,1) \neq 0 \end{array} \right\} \lhd \mathcal{O}_{X,\mathfrak{p}}$$

$$\frac{m}{m^2} = \left\{ \frac{f}{g} + m^2 \,\middle|\, \begin{array}{l} f(1,1) = 0 \\ g(1,1) \neq 0 \end{array} \right\}$$

$f \in \mathcal{O}_{X,\mathfrak{p}} \rightsquigarrow f - f(\mathfrak{p}) \in m$.

find a basis for $\frac{m}{m^2}$ $\left(= \mathrm{span}_k(x)\right)$.

$m = (x-1, y-1)$

$m^2 = \left((x-1)^2, (y-1)^2, (x-1)(y-1)\right)$

$m^2 \subseteq m \qquad x(x-1) \in m$

$x(x-1) = x-1 \bmod m^2$.

$y(y-1) = y-1 \bmod m^2$

$xy = (x-1) + (y-1) + 1 \bmod m^2$.

but $xy = 1$ in A.

$$\left.\begin{array}{l} (x-1)^2 = 0 \Rightarrow x^2 = 2x-1 \\ (y-1)^2 = 0 \Rightarrow y^2 = 2y-1 \end{array}\right\} (*)$$

$(x-1)(y-1) = 0 \Rightarrow xy - x - y + 1 = 0 \Rightarrow x + y = 2$.

by $(*)$ everything reduces to $ax + by + c$ but

$x + y = 2 \Rightarrow m/m^2 = \{ \lambda x + c + m^2 \}$
$= \{ \lambda(x-1) + m^2 \}$.

$v \in T_{X,\mathfrak{p}} = \left(\frac{m}{m^2}\right)^{\vee}$

$\lambda x + c \in m \Rightarrow$ condition over c

$$v : \frac{m}{m^2} \longrightarrow k$$
$$\lambda(x-1) + m^2 \longmapsto \text{—}$$
$$(x-1) \longmapsto \lambda(x-1) \in k$$

Giving a function $f: \frac{m}{m^2} \to k$

where $\frac{m}{m^2} = \mathrm{Span}_k(x-1)$ is the same

as a number in k. $f(x-1) = -f(y-1)$

$x+y-2 = 0 = (x-1)+(y-1)$ // Y

$\left(\frac{m}{m^2}\right)^\vee \xrightarrow{\sim} \{(x,y)\in \mathbb{A}^2_k \mid x+y+2=0\}$

$f \longmapsto (f(x-1)+1, f(y-1)+1)$

$\lambda \in \left(\frac{m}{m^2}\right)^\vee$ $\lambda = \lambda(x-1)$ $\lambda \in k$.

$\left(\frac{m}{m^2}\right)^\vee \xrightarrow{\sim} \{x+y+2=0\}$.

$\lambda \longmapsto (\underbrace{\lambda+1}_{a}, \underbrace{-\lambda+1}_{b}) \in Y$

$a+b=2$.

$3 \longmapsto (4,-2)$

$1 \longmapsto (2,0)$

$-1 \longmapsto (0,2)$

$0 \longmapsto (1,1)$ etc...

The tangent space of $\mathrm{Spec}\, A/(f)$ at m is the same as the intersection of the tangent space of $\mathrm{Spec}\, A$ at m and the hyperplane $f \bmod m^2 = 0$

e.g. $X = \mathrm{Spec}\, \frac{k[x,y]}{(xy-1)}$ $Y = \mathbb{A}^2_k$.

$T_{(1,1)} Y = k^2$

$T_{(1,1)} X = k^2 \cap (xy-1 \bmod m^2)$

$m^2 = ((x-1)^2, (x-1)(y-1), (y-1)^2)$

$xy-1 \equiv (x-1)(y-1) + x+y-2 \bmod m^2$

$\equiv x+y-2 \bmod m^2$.

$T_{(1,1)} X = k^2 \cap \{x+y=2\}$

$= \{x+y=2\}$ //.

Back to M. Brion notes $n,d \in \mathbb{N}$ dim.

$U^I \longrightarrow X$ locally closed embedding.

$g \longmapsto [gE_I]$

$C_I := BE_I = UE_I = U^I E_I$. $I = (i_1, \dots, i_d)$

$U^I \cong \mathbb{A}^{|I|}_{\mathbb{C}} = \mathbb{C}^{|I|}$. affine space

$|I| = \sum_{k=1}^{d} i_k - k$ $\overline{C_I} =: X_I$ Schubert variety

Prop. (i) C_I is the set of d-dim $E \subset \mathbb{C}^n$ s.t

$\dim(E \cap \langle e_1, \dots, e_j \rangle) = n_j$, $j = 1 \dots n$.

(ii) X_I is the set of d-dim $E \subset \mathbb{C}^n$ s.t

$\dim(E \cap \langle e_1, \dots, e_j \rangle) \geq n_j$.

Thus
$$X_I = \bigcup_{J \leq I} C_J.$$

where $J \leq I$ if $j_k \leq i_k$ $\forall k \in \{1, \dots, d\}$

and $n_j := \#\{k \mid 1 \leq k \leq d, i_k \leq j\}$

e.g. $I = \{1,3\}$ $n=4$, $d=2$.

$E_I = \langle e_1, e_3 \rangle$ $BE_I = ?$ $E \in BE_I$

$\begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \end{bmatrix} \underset{\substack{B\text{-action} \\ (\text{or } U^I \text{action})}}{\rightsquigarrow} \begin{bmatrix} 1&0&0&0 \\ *&*&1&0 \end{bmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&*&1&0 \end{bmatrix}$

1-dim'l cell.

$|I| = \sum i_k - k = \cancel{1-1} + 3-2 = 1$.

$\dim(E \cap \langle e_1 \rangle) = 1$ (1) $n_1 = 1$

$\dim(E \cap \langle e_1, e_2 \rangle) = 1$ (2) $n_2 = 1$

$\dim(E \cap \langle e_1, e_2, e_3 \rangle) = 2$ (3) $n_3 = 2$

$\dim$ ———— $= 2$ (4) $n_4 = 2$.

conversely if (1) then $e_1 \in E$

if 2 $e_2 \notin E$ if (3)

if (4) $e_4 \notin E$.

$\dim(E \cap \langle e_1, e_2, e_3 \rangle) = 2$. but $e_2 \notin E$.
$e_1 \in E$.

$\Rightarrow \dim E \cap \langle e_2, e_3 \rangle = 1$

$\Rightarrow \lambda e_2 + e_3 \in E$ for some λ

$\Rightarrow E = \begin{pmatrix} 1&0&0&0 \\ 0&*&1&0 \end{pmatrix}$ //.

$I = \{1,3\}$.

$E_I = \begin{pmatrix} 1000 \\ 0010 \end{pmatrix} \xrightarrow{BE_I} \begin{pmatrix} 1000 \\ 0*10 \end{pmatrix}$

$BE_I = C_I$, $\overline{C_I} = X_I = ?$ $\quad E_{1,2}$

$\begin{pmatrix} 1000 \\ 0h10 \end{pmatrix} = \begin{pmatrix} 1000 \\ 01\frac{1}{h}0 \end{pmatrix} \xrightarrow[h \to \infty]{} \begin{pmatrix} 1000 \\ 0100 \end{pmatrix}$

$C_I =$

$\dim E \cap \langle e_1 \rangle = 1 \quad \geq 1 \quad n_1 = 1$

$\dim E \cap \langle e_1, e_2 \rangle = 2 \Rightarrow e_2 \in E. \quad n_2 = 1$

$\dim E \cap \langle e_1, e_2, e_3 \rangle = 2 \quad n_3 = 2$

$E \cap \langle e_1, e_2, e_3, e_4 \rangle = 2. \quad n_4 = 2$

Generally $\begin{pmatrix} 1000 \\ 0100 \end{pmatrix}$ satisfies $\cap I$

eg	X		dim
	4	34	4
		24	3 X_{24}
		14 \ 23	2
		13	1
		12	0

$\dim X = Gr(2,4) = 12 - 2 = d(n-d) = 4$

$\dim C_{12} = 1$, $\dim C_{13} = 1$.

$\dim C_{34} = 2+2 = 4 = |I|$

$\dim C_{24} = 1+2 = 3 = |I|$

$\dim C_{14} = 0+2 = \dim C_{23} = 1+1$

$\dim C_{13} = 1$.

X_{24} is a 3 dim variety.

$X_{24} \subset X = \mathbb{P}(\wedge^2 \mathbb{C}^4) = \mathbb{P}^5$.
closed $= \mathrm{Proj}\, \mathbb{C}[x_0, \ldots, x_5]$.

$X = \mathrm{Proj}\, \dfrac{\mathbb{C}[x_0, \ldots, x_5]}{(x_0x_5 - x_1x_4 + x_2x_3)}$

$x_0 = P_{12}$
$x_1 = P_{13}$
$x_2 = P_{14}$
$x_3 = P_{23}$
$x_4 = P_{24}$
$x_5 = P_{34}$

$\begin{pmatrix} 0100 \\ 0001 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} *100 \\ *0*1 \end{pmatrix}$

$e_2 \wedge e_4 \quad e_1 \wedge e_4 + e_2 \wedge e_4$.

$x_{34} = 0$. $\quad e_1 \wedge e_4 \neq 0$. etc.

$p = E_{12} \in X \hookrightarrow \mathbb{P}^5$.

$T_pX = ?$ $\quad X = \mathrm{Proj}\, \mathbb{C}[x_0, \ldots, x_5] / \text{Plucker}$

E_{12} lives in the open $x_0 \neq 0$.

$U = \mathrm{Spec}\, \dfrac{\mathbb{C}[x_1, \ldots, x_5]}{\underbrace{\{x_5 - x_1x_4 + x_2x_3 = 0\}}_{(f)}}$ $\quad \dim = 4 \quad \cap \mathbb{A}^5$

$T_pX = T_p\mathbb{A}_4 \cap \{f \bmod m^2 = 0\}$

$m^2 = ?$

$p = e_1 \wedge e_2 = [1:0:0:0:0:0] \in \mathbb{P}^5$.

$p \in \mathbb{A}^5$ is $(0,0,0,0,0)$

$T_p\mathbb{A}^5 = \mathbb{R}^5 = \mathbb{C}^5$

$m = (x_1, x_2, x_3, x_4, x_5)$

$m^2 = \underline{\qquad}$

write $f = x_5 \bmod m^2$.

then $(T_{E_{12}}X)^\vee = \{x_5 = 0\}$.

$X_{24} = X \cap T_{E_{12}}X$. , $\dim X_{24} = 3$

satisfies $x_1x_4 = x_2x_3$
$x_5 = 0$

$m/m^2 \subseteq \mathrm{Span}(x_1, \ldots, x_5)$

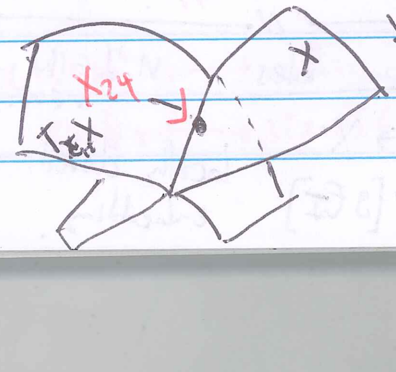
$X = X_{34}$ 4-dim. variety.

$p \in X_{34}$ $\quad p$ non-singular in X_{34}
$\dim T_pX = 4$

$X = X_{34} \quad \dim = 4$

$E_{12} \in X \quad T_{E_{12}}X \quad \dim = 4$

$\dim X \cap T_pX = 3$
$\downarrow \quad \downarrow$
$4 \cap 4$



$E_{12} \in X$

$p = E_{12} \in U = \operatorname{Spec} \underline{\mathbb{C}[x_1 \ldots x_5]}_{(\text{Plucker})}$, $\dim = 4$

$T_pX = \mathbb{A}^5 \cap \{x_5 = 0\} \subset \mathbb{A}^5$.

$\dim T_pX = 4 = \dim X$

p not singular.

let $Z = X \cap T_pX$, $\dim Z = 3$

$Z \supset U_Z = \operatorname{Spec} \frac{\mathbb{C}[x_1 \ldots x_5]}{(\text{Plucker},\ x_5)}$ $\dim = 3$

in an open

$T_pZ = ?$ $U_Z = \operatorname{Spec} A/(x_5)$.

by (A) $\operatorname{Spec} A = U \subset_{\text{open}} X$.

then $T_pZ = T_pX \cap \{x_5 = 0\}$

since $x_5 = x_5 \bmod m_p^2$.

then $T_pZ = T_pX$ then $\dim T_pZ = 4$

but $\dim Z = 3$

then p is singular. let $q \neq p$.

e.g. $q = E_{13} \in X$. let's see if E_{13} is singular in X or Z.

$q = E_{13}$, $q = e_1 \wedge e_3 = [0:1:0:0:0:0] \in \mathbb{P}^5$

$U_q = \operatorname{Spec} \underline{\mathbb{C}[y_1, \ldots, y_5]}_{\text{Plucker}} \subset \mathbb{A}^5$ $\dim = 4$

$x_1 \neq 0$.

p_{13} $U_q = \operatorname{Spec} \frac{\mathbb{C}[y_1, \ldots y_5]}{(y_1y_5 - y_4 + y_2y_3)}$

$x_0 = y_1$
$x_2 = y_2$
$x_3 = y_3$
$x_4 = y_4$
$x_5 = y_5$.

$q = [0;0;0;0;0] \in \mathbb{A}^5$.

$T_qX = \mathbb{A}^5 \cap (\text{Plu} \bmod m^2)$

Plucker mod $m^2 = -y_4$, $-y_4 = 0$.
$y_4 = 0$.

~~$T_qX = \{ \ldots \mathbb{A}^5_{\mathbb{C}}, \{y_4 = 0\}$ $\dim = 4$~~

$T_qX = \mathbb{A}^5 \cap \{y_4 = 0\}$ $\dim = 4$

q is not singular.

let $Z = X \cap T_pX$

$Z \supset_{\text{open}} U_Z = \operatorname{Spec} \frac{\mathbb{C}[x_1, \ldots, x_5]}{(\text{Plucker},\ x_5)}$, $\dim = 3$

$\cap$

$\mathbb{A}^5$.

Lets compute

T_qZ, $T_qU_Z = ?$

$U_Z = \operatorname{Spec}\left(A/(x_5)\right)$

$\operatorname{Spec} A = X$

$T_qX = \mathbb{A}^5 \cap \{y_4 = 0\}$

$T_qZ = T_qX \cap \{x_5 \bmod m_q^2 = 0\}$

$m_q =$, well q is $(1,0,0,0,0)$ in U_Z

p is $(0,0,0,0,0)$ in U_Z.

$m_q = (x_1 - 1, x_2, x_3, x_4, x_5)$

$m_p = (x_1, x_2, x_3, x_4, x_5)$.

$x_5 \bmod m_q^2 = x_5$.

$T_qZ = \mathbb{A}^5 \cap \{y_4 = 0\} \cap \{x_5 = 0\}$.

$\dim = 3$. non-singular!

$\dim Z = 3$

One can be convinced that E_{12} is the unique singular point of Z.

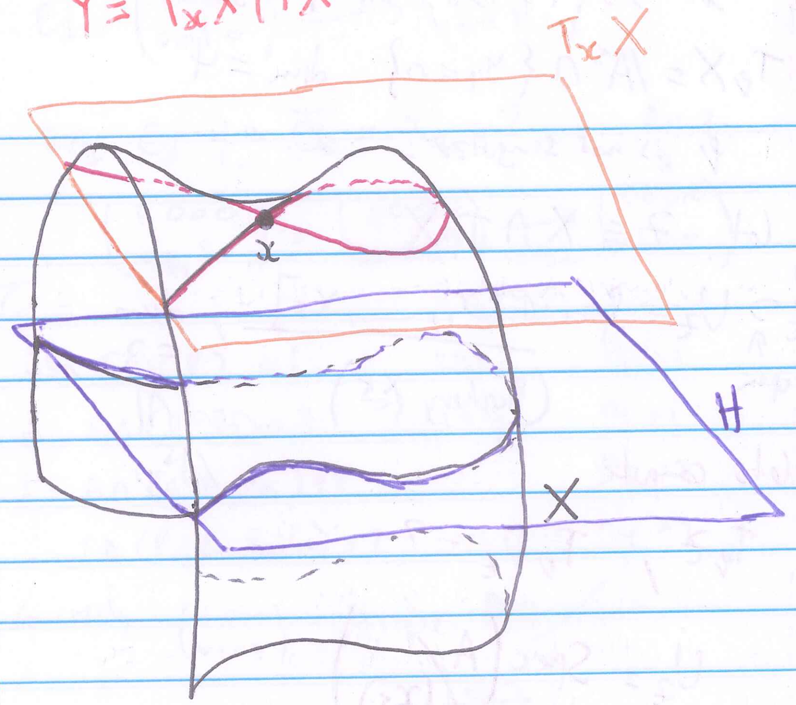
$\operatorname{Sing} Z$ $\dim = 1$

$\operatorname{Sing} Z$ $\operatorname{codim} = 2 \Rightarrow Z$ normal

The following is a picture where a subvariety of a non-singular variety is the intersection of the variety and T_pX for some $p \in X$

p not singular point in X but singular point in $T_pX \cap X$

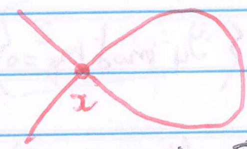
$Y = T_x X \cap X$



$H \cap X$ regular. x is singular point

$Y = T_xX \cap X$ singular. of Y.

Y



1-dim "cone"

Thm 8.15 (Hartshorne) [Criteria for singularity]
Let X be an irreducible separated scheme of finite type over $k = \bar{k}$.
Then $\Omega_{X/k}$ is locally free of rank $n = \dim X$ iff X is non singular $/k$.

Kähler Differentials

A ring, B an A-algebra, M be a B-module

Def An A-derivation of B into M is a map $d : B \to M$ s.t.

(1) d is additive
(2) $d(bb') = b\,db' + b'\,db$
(3) $da = 0 \quad \forall a \in A$.

Def The module of relative differential forms of B over A is the B-module $\Omega_{B/A}$, together with an A-derivation $d : B \to \Omega_{B/A}$ s.t. is universal: If M is other B-module and $d' : B \to M$ derivation then $\exists!\, f : \Omega_{B/A} \to M$ B-module homomorphism

s.t. $d' = f \circ d$, i.e.:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, dashed, "\exists! f"] \\ & M \end{tikzcd}$$

Sheaf of Differentials

Let $X \xrightarrow{f} Y$ morph of schemes,
$\Delta : X \longrightarrow X \times_Y X$ diagonal.

$\Delta(X)$ locally closed subscheme

i.e. $\Delta(X) \subseteq W$
closed open

see next page

Let J be the ideal sheaf of $\Delta(X)$ in W then

see next page for f^*

$$\Omega_{X/Y} := \Delta^*(J/J^2)$$

Thm 8.13 [Hartshorne]
A ring $Y = \operatorname{Spec} A$, $X = \mathbb{P}^n_A$ then

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{\oplus(n+1)} \to \mathcal{O}_X \to 0$$

is exact.

Rmk $U = \operatorname{Spec} A$ open in Y
$V = \operatorname{Spec} B$ open in X
$f(V) \subseteq U$ then $V \times_U V$ open affine subset of $X \times_Y X$

$V \times_U V \cong \operatorname{Spec}(B \otimes_A B)$

Prop: $\Omega_{V/U} \cong (\Omega_{B/A})^{\sim}$

Let X/k non singular. Two constructions:
Tangent sheaf: $\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$
it is a locally free sheaf of rank $n = \dim X$.

Canonical sheaf: $\omega_X := \Lambda^n(\Omega_{X/k})$

where $n = \dim X$

If X projective and non singular

Geometric genus of X is

$$p_g = \dim_k \Gamma(X, \omega_X).$$

Under suitable conditions p_g coincides with the arithmetic genus

$P_a = (-1)^n (P_X(0) - 1)$, $n = \dim X$, P_X Hilbert polynomial

Recall $f: X \to Y$ has the data of $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$ mor of sheaves on Y

$\underbrace{f^*\mathcal{G}}_{\text{called inverse image}} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is an $\mathcal{O}_X$-module

for any $\mathcal{G}$ sheaf of $\mathcal{O}_Y$-modules.

$f^{-1}\mathcal{G}$ is $f^{-1}\mathcal{O}_Y$ module and

$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$

then $f^\#$ corresponds to a map from $f^{-1}(\mathcal{O}_Y)$ to $\mathcal{O}_X$, then $\mathcal{O}_X$ is also a $f^{-1}(\mathcal{O}_Y)$ module.

Also, define $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for two $\mathcal{O}_X$ modules as the sheafification of

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

which is an $\mathcal{O}_X$-module.

If $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a group.

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

is a natural iso of groups for every $\mathcal{F}$ $\mathcal{O}_X$-module and $\mathcal{G}$ $\mathcal{O}_Y$-module.

Def $Y \subset X$ closed subscheme of X

$i: Y \hookrightarrow X$ inclusion morphism

The ideal sheaf of Y, denoted $\mathcal{J}_Y$ is the kernel of $i^\#: \mathcal{O}_X \to i_* \mathcal{O}_Y$.

Review of Divisors

let C non singular projective curve in $\mathbb{P}^2_k$

$L \cap C$ is a finite set of C.

C of degree d $\#\, C \cap L = d$ (counting multiplicity)

$L \cap C = \sum n_i P_i$ divisor.

Weil Divisors (X noetherian integral separated regular)

Def A prime divisor on X is a closed integral subscheme of codimension 1

A divisor is

$$D = \underbrace{\sum n_i Y_i}_{\text{finite sum}} \quad Y_i \text{ prime divisor}, \; n_i \in \mathbb{Z}.$$

D is effective if all $n_i \geq 0$.

let Y be a prime divisor, let η its generic point

$\mathcal{O}_{\eta, X}$ is a discrete valuation ring with valuation v_Y, let $f \in K^*$ a non-zero rational function on X. $v_Y(f) \in \mathbb{Z}$.

$v_Y(f) > 0$ say f has a zero along Y of order $v_Y(f)$

$v_Y(f) < 0$ " " " " pole along Y " " $-v_Y(f)$

For $f \in K^*$ $v_Y(f) = 0$ except for finitely many Y prime divisors.

$f \in K^*$ defines a divisor

$$(f) = \sum v_Y(f) \cdot Y \rightsquigarrow \text{principal divisor}$$

eg. $(f/g) = (f) - (g)$

$$\mathrm{Im}\begin{pmatrix} K^* \longrightarrow \mathrm{Div}\, X \\ f \longmapsto (f) \end{pmatrix} =: \mathrm{Principal}(X)$$

$D \sim D'$ iff $D - D'$ is principal

we say D and D' are equivalent.

$$\mathrm{Div}X / \sim \; =: \text{Divisor class group of } X =: \mathrm{Cl}\, X$$

eg.

$\mathrm{Cl}\, \mathbb{A}^n = 0$.

$\mathbb{P}^n_k$ $H := \{x_0 = 0\}$

$D = \sum n_i Y_i$, $\deg D = \sum n_i \deg Y_i$

$\deg Y_i$ is the degree of Y_i as a hypersurface.

Prop $X = \mathbb{P}^n$. $H := \{x_0 = 0\}$.

(a) D divisor of degree d, then
$$D \sim dH$$
(b) $\deg((f)) = 0 \quad \forall f \in K^\times$

(c) $\deg : Cl\,X \xrightarrow{\sim} \mathbb{Z}$ is isomorphism.
$$H \longmapsto 1.$$

Cartier divisors

Extends the definition to any scheme.

Cartier divisor $\rightsquigarrow$ locally looks like the divisor of a rational function.

Let X scheme. For each open $U = \operatorname{Spec} A$

$A \supset S = \{\text{non-zero divisors}\}$

$$K(U) := S^{-1}A = \left\{ \frac{a}{b} \;\middle|\; \begin{array}{l} a \in A \\ b \text{ non zero divisor} \end{array} \right\}$$

$K(U)$ is called total quotient ring of A

U open (not necessarily affine)

$S(U) := \{\text{non-zero divisors in } \mathcal{O}_x \ \forall x \in U\}$
$\subseteq \Gamma(U, \mathcal{O}_X)$

$U \mapsto S(U)^{-1}\,\Gamma(U, \mathcal{O}_X)$ is a presheaf.

Let $\mathcal{K}$ be the associated sheaf called sheaf of total quotient rings of $\mathcal{O}_X$.

The notion of $\mathcal{K}$ replaces the notion of function field K for an integral scheme.

$\mathcal{K}^*$ invertible set of $\mathcal{K}$. $\mathcal{O}_X^*$ the same for $\mathcal{O}_X$

Def A Cartier divisor on a scheme X is a global section of $\mathcal{K}^*/\mathcal{O}_X^*$.

Can be described by the data of open covering $\{U_i\}$ of X, $f_i \in \Gamma(U_i, \mathcal{K}^*)$ s.t. $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$

A Cartier div is principal if is in
$$\operatorname{Im}\left(\Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*) \right)$$
let X normal scheme

Def D is locally principal if X can be covered by open sets U s.t. $D|_U$ is principal.

Prop
$$\text{Cartier divisors} = \text{locally principal Weil divisors}$$

Invertible sheaves:

Prop $\mathcal{L}$ invertible sheaf (i.e. locally free $\mathcal{O}_X$ module of $rk = 1$) then
$\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$ where $\mathcal{L}^{-1} := \mathcal{L}^\vee$
$$\mathcal{L}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$$

Proof [A lot of work, DO IT, remember the definition of $\mathcal{H}om_{\mathcal{O}_X}$].

$\operatorname{Pic} X =$ iso classes of inv. sheaves $(= H^1(X, \mathcal{O}_X^*))$

let D Cartier represented by $\{(U_i, f_i)\}$

let $\mathcal{L}(D)$ be the invertible sheaf s.t.
$$\mathcal{L}(D)|_{U_i} \cong \frac{1}{f_i} \mathcal{O}_X|_{U_i}$$
$$\mathcal{O}_{U_i} \xrightarrow[\sim]{\text{Hom}} \mathcal{L}(D)|_{U_i}$$
$$1 \longmapsto \frac{1}{f_i}$$

A Cartier divisor is effective if can be represented by $\{(U_i, f_i)\}$ s.t. all $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. The associated subscheme of codimension 1 is the closed subscheme def by the sheaf of ideals $\mathcal{I}$ which is locally generated by f_i.

Rmk
$$\left\{ \begin{array}{c} \text{effective} \\ \text{Cartier} \\ \text{divisors} \end{array} \right\} \xleftrightarrow[1-1]{\sim} \left\{ \begin{array}{c} \text{locally principal} \\ \text{closed subschemes} \\ \text{of } Y \end{array} \right\}$$

Prop: let D be an effective Cartier divisor on X, let Y be the associated subscheme. Then $\mathcal{J}_Y \cong \mathcal{L}(-D)$ where $\mathcal{J}_Y$ is the sheaf of ideals generated locally by f_i.

[$D = \{(U_i, f_i)\}$ s.t. $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$, and $\mathcal{J}_Y$ is the ideal of Y]

Back to M. Brion notes (again)

For d, n arbitrary $d \le n$.

$Gr(d,n) \hookrightarrow \mathbb{P}(\Lambda^d \mathbb{C}^n)$.

e.g. $X_{1,2,\dots,d} = C_{1,\dots,d} = E_{1,2,\dots,d}$

$X_{n-d+1, n-d+2, \dots, n} = Gr(d,n) = X$.

$X_{n-d, n-d+2, \dots, n} = ?$

$I' = \{n-d, n-d+2, \dots, n\}$

e.g.

$$\left[\begin{array}{c|ccc} 0 & \cdots & 0 & 1 \\ 0 & \cdots 0 & 1 & 0 \\ 0 & \cdots 0 & 1 & 00 \\ \vdots & & \vdots & \\ 0 \cdots & & & \\ 0 \cdots 0 & 001 & \cdots & 0 \\ 0 \cdots 0 & 01 & \cdots & 0 \\ 0 \cdots 010 & & \cdots & 0 \end{array}\right]$$

e_{n-d}, d

Better

$$\left[\begin{array}{ccc|c} & & & 0 \quad\quad 1 \\ & 0 & & 0 \quad\quad 10 \\ & & & 001 \cdots \\ & & & 010 \cdots 0 \\ 0 & \cdots & 00 & 100 \cdots 0 \end{array}\right] = \left[\begin{array}{c|c} 0_{n-d} & I_d \end{array}\right]$$

$n-d$, d, $I = \{n-d+1, \dots, n\}$

$I' \supseteq$

$$\left[\begin{array}{cc|c} 0 & \begin{matrix}0\\0\\0\\0\end{matrix} & \begin{matrix} 0 \quad 010 \\ 0010 \cdots 0 \\ 010 \cdots 0 \end{matrix} \\ 0 \cdots 0 & 1 & 0 \cdots 0 \end{array}\right]$$

$n-d$

$$\sim \left[\begin{array}{cc|c} * & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix} * & I_{d-1} \\ \vdots & \\ * & 10 \end{matrix} \\ (***) & 1 & 00 \cdots 0 \end{array}\right] = C_{I'}$$

$\dim C_{I'} = (n-d)d - 1$.

e.g. $e_4\ e_2$

$$\begin{bmatrix} 0001 \\ 0100 \end{bmatrix} \sim \begin{bmatrix} *0*1 \\ *100 \end{bmatrix}$$

$$\begin{bmatrix} 0001 \\ 0010 \end{bmatrix} \sim \begin{bmatrix} **01 \\ **10 \end{bmatrix}$$

$$|I'| = \sum_{k=1}^{d} (i'_k - k) = n-d-1 + n-d + n-d + \cdots \le d(n-d)$$

$$|I| = \sum_{k=1}^{d} (i_k - k) = n-d + n-d \cdots \le d(n-d)$$

O.K!

$\operatorname{codim} C_{I'} = 1$.

$X - C_{I'} = C_I$, $X_I = \bigcup_{J \le I} C_J$.

But C_I is all E d-dim s.t.

$E \cap \langle e_1 \rangle$

$n_1 = 0$, $n_{n-d} = 0$

$n_2 = 0$, $n_{n-d+i} = i$.

$\vdots$

$n_{n-1} = d-1$.

$n_n = d$

E st

$E \cap \langle e_1, \dots, e_{n-d} \rangle = 0$.

$X_{I'} = X - C_I = \{E \mid E \cap \langle e_1, \dots, e_{n-d} \rangle \ne 0\}$

~~$X_{I'} \subset \mathbb{P}(\Lambda^d \mathbb{C}^n)$~~

~~$\{[a_{1,2,\dots,d} : a_{1,\dots,i,\dots} : \dots]\}$~~

~~st $a_{1,\dots,n-d} \ne 0$ i.e. the first coord~~

~~$X_{I'} \subset \mathbb{P}(\Lambda^d \mathbb{C}^n)$ (x_{i_j}) the set of $[1 : x : x \cdots x]$ first coord non 0.~~

~~$X_{I'} \cong \mathbb{A}^{\binom{n}{d}-1}$~~

FALSE!

$X_{I'} = X - C_I = \{E \mid E \cap \langle e_1, \dots, e_{n-d} \rangle \ne 0\}$

e.g $n=4$, $d=2$. $I' = \{2,4\}$

$X_{I'}$, $C_{I'} =$

$$C_{I'} = \begin{bmatrix} *100 \\ *0*1 \end{bmatrix}$$

$\dim = 3$.

$$C_I = \begin{bmatrix} **10 \\ **01 \end{bmatrix}$$

$I = \{3,4\}$.

$$\begin{array}{ccc} & 34 & & 4 \\ & 24 & & 3 \\ 14 & & 23 & 2 \\ & 13 & & 1 \\ & 12 & & 0 \end{array}$$

eg. (Cont)

$$E = \left\langle \begin{pmatrix} 2\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 2\\0\\0\\1 \end{pmatrix} \right\rangle \qquad v_1, v_2$$

$$E \in \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}$$

$[E] = [v_1 \wedge v_2]$

$[E] = [v_1 \wedge v_2]$

$$E \in \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}$$

$v_1 \wedge v_2 = (2e_1 + e_2) \wedge (2e_1 + e_4)$

$= 2e_1 \wedge e_4 - 2e_1 \wedge e_2 + e_2 \wedge e_4$

$[-2 : 0 : 2 : 0 : 1 : 0]$ look this!

$E \cap \langle e_1, e_2 \rangle \neq 0$.

$$E = \left\langle \begin{pmatrix} 3\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix} \right\rangle$$

$(3e_1 + e_2 + e_4) \wedge (e_1 + e_3)$

$= 3e_1 \wedge e_3 + -e_1 \wedge e_2 + e_2 \wedge e_3 - e_1 \wedge e_4 - e_3 \wedge e_4$

$= [-1 : 3 : -1 : 1 : 0 : -1]$

for something in $\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}$ is impossible

to generate $e_3 \wedge e_4$.

same for other lower cells. eg

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}$$

$P_{34} = 0 \iff E \cap \langle e_1, e_2 \rangle \neq 0$.

$\iff E \in X_{2,4}$.

$E \in X_{I'} = X - C_{I_{max}} \iff E \in \underbrace{U_{n-d+1, \dots, n} \cap X}_{U_{max} \cap X}$

$U_{max} = \{ [\cdots : a] \mid a \neq 0 \} \subset \mathbb{P}(\wedge^d \mathbb{C}^n)$ open affine

$\cong \mathbb{A}^{\binom{n}{d}-1}$

$X_{I'} = U_{max} \cap X \cong \mathbb{A}^{d(n-d)-1} = \mathbb{A}^{|I'|} = \mathbb{A}^{|I_{max}|-1}$

$X = X_{I'} \sqcup C_{I_{max}}$

$\cong X_{I'} \sqcup \mathbb{A}^{d(n-d)}$

$X = \underbrace{(H_{max} \cap X)}_{D} \sqcup \mathbb{C}^{d(n-d)}$

Any divisor D' on X is equivalent to a unique integer multiple of D.

Let Y be a scheme and $\mathcal{A}$ a quasi-coherent sheaf [A quasicoherent sheaf is a sheaf of $\mathcal{O}_X$-modules which is locally $\tilde{M}$ for M an A-module and $U = \operatorname{Spec} A$ open) of $\mathcal{O}_X$-algebras (i.e. q.c. sheaf of $\mathcal{O}_X$-modules and a sheaf of rings at the same time). and $f: X \to Y$

There is a unique scheme X^Y s.t. $f: X \to Y$, $V \subset Y$ open affine then $f^{-1}(V) \cong \operatorname{Spec} \mathcal{A}(V)$ f is compatible with restrictions $U \hookrightarrow V$ on Y the map $f^{-1}(V) \hookleftarrow f^{-1}(U)$ correspond to the restriction homs $\mathcal{A}(V) \longrightarrow \mathcal{A}(U)$.

$X =: \operatorname{Spec} \mathcal{A}$

<u>Vector Bundles</u>

Let Y be a scheme. A (geometric) vector bundle of rank n over Y is a scheme X and morph $f: X \to Y$ + data of open cover of Y $\{U_i\}$ and isomorphisms

$\psi_i : f^{-1}(U_i) \xrightarrow{\sim} \mathbb{A}^n_{U_i}$

where $\mathbb{A}^n_{U_i} = \operatorname{Spec} B[x_1, \dots, x_n]$

$\cong \operatorname{Spec} B \times \mathbb{A}^n_R = U_i \times \mathbb{A}^n_R$

for affine $U_i = \operatorname{Spec} B$. s.t. for any open affine $V = \operatorname{Spec} A \subseteq U_i \cap U_j$ the automorphism $\psi = \psi_j \circ \psi_i^{-1}$:

$\psi_j \circ \psi_i^{-1} : \mathbb{A}^n_V \longrightarrow \mathbb{A}^n_V$ is given by

* ($\mathbb{A}^n_V = \operatorname{Spec} A[x_1, \dots, x_n]$)

a "linear" automorphism θ of $A[x_1, \dots, x_n]$

i.e. $\theta(x_i) = \sum_j a_{ij} x_j$ $a_{ij} \in A$, $\theta(a) = a$ $\forall a \in A$.

~~The condition of linear automorphism is weird but makes sense if A is f.g k-algebra~~

~~So if $A = k[a_k]$ then $a_{ij} = \sum_{k,m} \lambda_{ijk} a_k^m$~~

~~$\sum_j a_{ij} x_j = \sum_{j,k,m} a_k^m \otimes \lambda_{ijk} x_j$~~ ~~where k,m are multi-indices.~~

~~$= \sum_{k,m} a_k^m \otimes \sum_j \lambda_{ijk} x_j$~~

~~$= \sum_{j,k,m} a_k^m \otimes \underbrace{\lambda_{jkm}}$ x_j~~

~~we killed the dependence of j here.~~

NOT QUITE

~~**Better**:~~

~~only need to know where to map a_k for each k, then a_k^m is determined.~~

~~(this kill dependence of m)~~

Is the matrix $(a_{ij})^t|$ specialised on m!

Conclusion: The map θ can be determined by elements $\lambda_{ij} \in k$. The matrix $M = (\lambda_{ij})_{n\times n}$ determines the map of vector spaces $\mathbb{A}^n_k$. and the ~~condition~~ "linear" condition makes the specialisation a linear map!!

This fixes m

The condition $\boxed{\theta(a) = a}$ makes $\psi_j \circ \psi_i^{-1}$:

$$\operatorname{Spec} A \times \mathbb{A}^n_k \longrightarrow \operatorname{Spec} A \times \mathbb{A}^n_k$$

$$(m, v) \longmapsto (m, M(v))$$

M invertible $n\times n$ matrix with entries in k.

---

let $\mathcal{E}$ locally free sheaf on Y of $\mathrm{rk} = n$. $S(\mathcal{E})$ be the symmetric algebra take $X = \operatorname{Spec} S(\mathcal{E})$, $f : X \to Y$ such that $f^{-1}(V) \cong \operatorname{Spec}(S(\mathcal{E})(V))$ and $f^{-1}(U) \hookrightarrow f^{-1}(V)$ for $U \subset V$ is the restriction $S(\mathcal{E})(V) \longrightarrow S(\mathcal{E})(U)$. for V affine open

Choose U affine open of Y s.t. $\mathcal{E}|_U$ is free. choose a basis for $\mathcal{E}$ and let $\psi : f^{-1}(U) \longrightarrow \mathbb{A}^n_U$ be the map resulting by identifying $\mathcal{O}(U)[x_1, \dots, x_n]$ with $S(\mathcal{E}(U))$.

Then X is a vector bundle with map f covering $\{U\}$ and maps ψ. This is the geometric vector bundle associated to $\mathcal{E}$. we denote it by $\mathbb{V}(\mathcal{E})$.

Prop let X be $\mathbb{P}^n_k$ over a field k. let $D = \sum n_i Y_i$ Weil divisor, $\deg D := \sum n_i \deg Y_i$. where $\deg Y_i$ is the degree of the polynomial defining Y as a hypersurface. let $H = \{x_0 = 0\}$

Then:

(a) If D is a divisor of degree d, then $D \sim dH$.

(b) $f \in K^*$ then $\deg(f) = 0$.

(c) $\deg : \mathrm{Cl}\, X \longrightarrow \mathbb{Z}$ is iso of groups.

Proof let $S = k[x_0, \dots, x_n]$ be the homog. coordinate ring of X. If g is homogeneous of degree d, then $g = g_1^{n_1} \cdots g_r^{n_r}$ where g_i are irreducible polynomials. Then g_i defines hypersurface Y_i. [Remember factors of homogeneous polynomials are homogeneous too]

define $(g) = \sum n_i Y_i$, then $\deg(g) = d$.

let $f \in K^*$, $f = g/h$ for two homogeneous $\deg(f) = \deg(g) - \deg(h) = 0$. This proves (b)

let D of degree d, $D = D_1 - D_2$ for 2 effective divisors D_1, D_2. of degrees d_1, d_2.

$d_1 - d_2 = d$. $D_1 = (g_1)$, $D_2 = (g_2)$

This is possible since irred surface corresponds to a homogeneous prime ideal of height 1 in S which is principal. Taking powers and products one can produce all effective divisors by some (g) for homogeneous g.

Now take $f = \dfrac{g_1}{x_0^d g_2}$ is quotient of two homogeneous of degree d_1, (since $d + d_2 = d_1$)

$(f) = (g_1) - (x_0^d g_2) = 0$.

$(g_1) = (x_0^d g_2) = dH + (g_2)$

$dH = (g_1) - (g_2) = D$.

$\operatorname{Pic}(X) \cong \mathbb{Z}$ is freely generated by $[X_{I'}]$, $I' = X \cap H_{\max}$.

Equivalently, any line bundle on X is isom to a unique tensor power of the line bundle $L := \mathcal{O}_X(D)$, where $D := X_{I'}$, the pullback of $\mathcal{O}(1)$ via the Plücker embedding.

Let S be a graded ring and M a graded S-module ($S = \bigoplus_{d\geq 0} S_d$) ($M = \bigoplus_{d\geq 0} M_d$, $S_d \cdot M_i \subseteq M_{d+i}$). ($\operatorname{Proj} S = \{p \mid p \not\supseteq \bigoplus_{d>0} S_d =: S_+\}$)

The sheaf associated to M on $\overset{X}{\operatorname{Proj} S}$, denoted $\widetilde{M}$ is the unique sheaf on X s.t. for any homogeneous $f \in S_+$:

$$\widetilde{M}|_{D_+(f)} \cong \underbrace{(M_{(f)})^{\sim}}_{\text{this is the } \widetilde{M} \text{ for affine case!}} \quad (\star)$$

where we identify $(D_+(f), \mathcal{O}|_{D_+(f)})$ with $\operatorname{Spec} S_{(f)}$ as isomorphic locally ringed spaces.

$S_{(f)}$ (resp. $M_{(f)}$) is the subring of elements of degree 0 in the localised ring S_f (resp. M_f).

[Note $\{D_+(f)\}$ are a open covering of X]

$\mathcal{O}_X(n) := (S(n))^{\sim}$. $\mathcal{O}_X(1) :=$ twisting sheaf of Serre.

Let F sheaf of $\mathcal{O}_X$-modules on $\operatorname{Proj} S$.

$F(n) := F \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. $n \in \mathbb{Z}$.

Prop

$S = \bigoplus_{d\geq 0} S_d$. graded ring $X = \operatorname{Proj} S$.

Assume that S is generated by S_1 as an S_0-algebra.

(a) The sheaf $\mathcal{O}_X(n)$ is invertible.

proof (a)

We need to show $\mathcal{O}_X(n)$ is locally free of rk=1. Let $f \in S_1$. Consider $\mathcal{O}_X(n)|_{D_+(f)}$

i.e. $S(n)^{\sim}|_{D_+(f)} \cong S(n)_{(f)}^{\sim}$ on $\operatorname{Spec} S_{(f)}$

but $\left.\begin{array}{l} S(n)_{(f)} \cong S_{(f)} \\ f^n a \longmapsto a \end{array}\right\} \Rightarrow$ $S(n)_{(f)}$ is a $S_{(f)}$-module of rank 1

This in fact implies that $\mathcal{O}_X(n)|_{D_+(f)}$ is locally isomorphic to $\mathcal{O}_X$. (i.e. of rank 1)

Also $(\widetilde{M})_p = M_{(p)} := \left\{\begin{array}{l}\text{the module of} \\ \text{deg 0 in } M_p\end{array}\right\}$

And we had before $\mathcal{O}_p \cong S_{(p)}$

To understand $(\star)$ we can understand first the structure sheaf of $\operatorname{Proj} S =: X$. $\mathcal{O}_X$ is s.t.

(a) $\mathcal{O}_p = S_{(p)}$

(b) There is for every $f \in S_+$ an iso of $\overset{\text{affine}}{\text{schemes}}$

$$(D_+(f), \mathcal{O}_X|_{D_+(f)}) \cong \underbrace{\operatorname{Spec} S_{(f)}}_{\text{affine scheme is Spec of a ring.}}$$

In particular, the sheaves are isomorphic.

$$\mathcal{O}_X|_{D_+(f)} \cong \mathcal{O}_{S_{(f)}}$$

and the sets (or topological spaces) are the same

$$D_+(f) \cong \operatorname{Spec} S_{(f)}$$

$(\star)$ is the same as $(D(f), \widetilde{M}|_{D(f)}) \cong (\operatorname{Spec} S_{(f)}, \widetilde{M_{(f)}})$

To show this happen for other open sets of the form $D_+(f)$, $f \in S_+$ we use S gen by S_1 as S_0-algebra. So $D_+(f)$ with $f \in S_1$ cover X $\square$

(b) For any graded S-module M, $\widetilde{M}(n) \cong (M(n))^{\sim}$. In particular, $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$.

proof (b)

Follow from $(M \otimes_S N)^{\sim} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ where M, N graded S-modules when S is generated by S_1. This is true since for $f \in S_1$ $(M \otimes_S N)_{(f)} \cong M_{(f)} \otimes_{S_{(f)}} N_{(f)}$.

[localization property is valid for homogeneous case] $\square$

(c) Let T be another graded ring gen by T_1 as T_0-algebra, let $\varphi: S \to T$ degree preserving homomorphism. Consider

$Y = \mathrm{Proj}\, T$, $(X = \mathrm{Proj}\, S)$ $\varphi : S \longrightarrow T$

and $U = \{ P \in \mathrm{Proj}\, T \mid P \not\supseteq \varphi(S_+) \}$

U is open in Y and this defines a map $f : U \longrightarrow X$. Then

a) $f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)|_U$

b) $f_*(\mathcal{O}_Y(n)|_U) \cong (f_* \mathcal{O}_U)(n)$

Proof:

Note $\tilde{T}$ on X is just $f_*(\mathcal{O}_U)$ by construction of f. And the 2 eq. follow from the two more general facts

1) $\forall$ S-module M
$$f^*(\tilde{M}) \cong (M \otimes_S T)^{\sim}|_U$$

2) $\forall$ T-module N
$$f_*(\tilde{N}|_U) \cong ({}_S N)^{\sim} \quad \text{where}$$
${}_S N$ is N as a S-module (restriction to S)

Morphisms to $\mathbb{P}^n$

A ring $\mathbb{P}^n_A := \mathrm{Proj}\ A[x_0, \ldots, x_n]$

$x_0, \ldots, x_n \in \Gamma(\mathcal{O}_{\mathbb{P}^n}(1))$

$\mathcal{O}(1)$ is generated by x_i, i.e. they generate $\mathcal{O}(1)$ as a module over $\mathcal{O}_P$ $\forall P \in \mathbb{P}^n_A$.

Let X scheme over A and $\varphi : X \to \mathbb{P}^n_A$ an A-morphism. $\mathcal{L} = \varphi^*(\mathcal{O}(1))$ is an invertible sheaf and $s_0, \ldots, s_n$ where $s_i := \varphi^*(x_i) \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$. Furthermore, the converse is true: $\mathcal{L}$ and s_i determine φ.

Thm A ring X/A scheme.

(a) $\varphi : X \to \mathbb{P}^n_A$ A-mor, then $\varphi^*(x_i)$ generate the invertible sheaf $\varphi^* \mathcal{O}(1)$

(b) Given $\mathcal{L}$ invertible sheaf on X and $s_0, \ldots, s_n$ global sections generating $\mathcal{L}$.

Then there is a unique A-morph.
$\varphi : X \longrightarrow \mathbb{P}^n_A$ s.t. $\mathcal{L} \cong \varphi^* \mathcal{O}(1)$.
and $s_i = \varphi^*(x_i)$.

Proof (b)

Suppose $\mathcal{L}$ inv. sheaf and $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ global sections generating $\mathcal{L}$. For $i \in \{0, \ldots, n\}$ define
$$X_i := \{ P \in X \mid (s_i)_P \notin m_P \mathcal{L}_P \}$$

$\{X_i\}$ is an open cover of X.

Now we define a map $X_i \to U_i$ where $U_i = \{x_i \neq 0\} \subseteq \mathbb{P}^n_A$

$U_i \cong \mathrm{Spec}\, A[y_0, \ldots, y_n]$ $y_j = x_j / x_i$ and y_i is omitted.

Lets define a map
$$A[y_0, \ldots, y_n] \xrightarrow{\bar{\varphi}_i} \Gamma(X_i, \mathcal{O}_{X_i})$$
$$y_j \longmapsto s_j / s_i \quad A\text{-linear.}$$

this is well defined since $\forall P \in X_i$, $(s_i)_P \notin m_P \mathcal{L}_P$. Also s_j / s_i is an element of $\Gamma(X_i, \mathcal{O}_{X_i})$ since $\mathcal{L}$ is locally free of rank 1 trivialised over X_i. This induces a mor of schemes
$$X_i \xrightarrow{\varphi_i} U_i.$$

they clearly glue $\left(\frac{s_j}{s_i} \cdot \frac{s_i}{s_w} = \frac{s_j}{s_w} \text{ for example} \right)$

then we set
$$\varphi : X \longrightarrow \mathbb{P}^n_A$$

Since $\bar{\varphi}$ is A-linear φ is A morphism.

also $\varphi_i^*(y_j) = s_j / s_i = \varphi_i^*(x_j / x_i)$

$\Rightarrow \varphi^*(x_j) = s_j$ (Why?) $\to$ Not hard!

φ is clearly unique. $\square$

$\mathrm{Aut}(\mathbb{P}^n_k) = PGL(n, k)$. $(a_{ij}) \subset k^{(n+1)^2}$

Any $(n+1)^2$ matrix (invertible) gives an aut. of $k[x_0, \ldots, x_n]$ $x_i \longmapsto \sum_j a_{ij} x_j$.

$(\lambda a_{ij}) \cong (a_{ij})$.

Conversely, every Aut φ of $\mathbb{P}^n_k$ induces an automorphism of $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ so $\varphi^*(\mathcal{O}(1))$ must be a generator. Then it is $\mathcal{O}(1)$ or $\mathcal{O}(-1)$ but $\mathcal{O}(-1)$ has no global sections.

$\Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is the k-vector space generated by $x_0,\dots,x_n$ as a k-module (i.e. $\{x_i\}$ is a basis)

Since φ is an aut, s_i should be other basis of $\Gamma(\mathbb{P}^n, \mathcal{O}(1))$ i.e. $s_i = \sum a_{ij} x_j$ then (a_{ij}) as seen before as an automorphism of $\mathbb{P}^n$ should be φ (φ is uniquely det. by the s_i as we saw just before.

Def Let S be a graded ring $X = \mathrm{Proj}\, S$, $\mathcal{F}$ sheaf of $\mathcal{O}_X$-modules. The graded S-module associated to $\mathcal{F}$ is

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$$

Is a graded S-module. If $s \in S_d$, s determines naturally a section $s \in \Gamma(X, \mathcal{O}_X(d))$. For $t \in \Gamma(X, \mathcal{F}(n))$ we define $s.t$ in $\Gamma(X, \mathcal{F}(n+d))$ by taking $s \otimes t$ and using the natural map

$$\mathcal{F}(n) \otimes \mathcal{O}_X(d) \cong \mathcal{F}(n+d)$$

Thm A ring $S = A[x_0,\dots,x_r]$ $X = \mathrm{Proj}\, S =: \mathbb{P}^r_A$. Then $\Gamma_*(\mathcal{O}_X) \cong S$.

Def X scheme over Y, $\mathcal{L}$ invertible sheaf on X is very ample (relative to Y) if $\mathcal{L} \cong i^*\mathcal{O}(1)$ for an immersion $i : X \to \mathbb{P}^r_Y$ $\forall r$.

A morphism $f : X \to Y$ is an immersion if it produces an iso between X and an open subscheme of a closed subscheme of Y.

Prop [Criteria for being a closed immersion]

Let $\varphi : X \to \mathbb{P}^n_A$ mor. of schemes $/A$, with $\mathcal{L} := \varphi^*\mathcal{O}(1)$ and resp. sections $s_0,\dots,s_n$ as before.

Then φ is closed immersion iff:

(1) X_i is affine, and

(2) the maps $A[y_0,\dots,\hat{y}_i,\dots,y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$, $y_j \longmapsto s_j/s_i$ are surjective

Proof Identify X_i as an affine closed subscheme of U_i. Prove surjectivity of a map between affine spaces. Prove X closed subscheme of $\mathbb{P}^n_A$.

For $A = \mathrm{Spec}\, k$ we have more things (assume $k = \bar{k}$). Local criteria!

Thm $\varphi : X \to \mathbb{P}^n_k$.

$\mathcal{L}$ is above and s_i generating $\mathcal{L}$

[It means $\forall p \in X$ the germs of s_i on the stalk $\mathcal{L}_p$ generate $\mathcal{L}_p$ as an $\mathcal{O}_p$-module]

[More generally, X scheme $\mathcal{F}$ sheaf of $\mathcal{O}_X$-modules we say $\mathcal{F}$ is generated by global sections if there is a family $\{s_i\}$ of elements of $\Gamma(X, \mathcal{F})$ s.t. $(s_i)_p$ generate $\mathcal{F}_p$ as $\mathcal{O}_p$-module $\forall p \in X$]

Let $V \subseteq \mathcal{L}(X, \mathcal{F})$ be the subspace generated by s_i. Then φ is closed immersion iff:

(1) elements of V separate points i.e. $P, Q \in X$ $\exists s \in V$ s.t $s \in m_P \mathcal{L}_P$ but $s \notin m_Q \mathcal{L}_Q$ or vice versa, and

(2) elements of V "separate tangent vectors" i.e. $\forall p \in X$, $\{s \in V \mid s_p \in m_p \mathcal{L}_p\}$ spans the k-v.s. $m_p \mathcal{L}_p / m_p^2 \mathcal{L}_p$.

Def $\mathcal{L}$ inv. on X noetherian is ample if $\forall \mathcal{F}$ coherent on X $\exists n_0(\mathcal{F}) \in \mathbb{N}_{>0}$ s.t $\forall n \geq n_0$ $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections. ($\mathcal{L}^n := \mathcal{L}^{\otimes n}$)

Rmk "Ample" is absolute, depends only on X.
"Very ample" is relative to a morphism $X \longrightarrow Y$, where X is a scheme over Y.

ex X affine, every line bundle is ample.

Thm (Serre) X projective scheme over a noetherian ring A. Let $\mathcal{L}$ very ample invertible sheaf on X (we denote $\mathcal{L} := \mathcal{O}_X(1)$ and $\mathcal{L}^r := \mathcal{O}_X(r)$ as a convenient notation). Let $\mathcal{F}$ coherent $\mathcal{O}_X$-module. $\exists n_0$ s.t. $\forall n \geq n_0$, $\mathcal{F}(n)$ is generated by finite number of global sections. [We denoted here $\mathcal{F}(n)$ for $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$]

In other words, a very ample sheaf $\mathcal{L}$ on a projective scheme X over a noetherian ring is ample.

The converse is false, but if $\mathcal{L}$ ample then $\mathcal{L}^m$ is very ample for $m \in \mathbb{N}_{>0}$

"Ample is a 'stable' version of very ample"

Back to M. Brion notes