What is the universal enveloping algebra? Part II

§1. Overview.

There is a forgetful functor.

$$F : \{\mathbb{C}\text{-algebras}\} \longrightarrow \left\{\begin{matrix}\text{Lie} \\ \text{algebras}\end{matrix}\right\}$$

$$(A, \cdot) \longmapsto (A, [\,,\,])$$

where $[x,y] := x \cdot y - y \cdot x$.

e.g. For a vector space V over $\mathbb{C}$ we denote

$$F(\mathrm{End}(V)) =: \mathfrak{gl}(V) \quad \text{general linear Lie algebra of } V.$$

Last time: we defined

$$U : \left\{\begin{matrix}\text{Lie algebras} \\ \text{over } \mathbb{C}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{associative (unital)} \\ \text{algebras over } \mathbb{C}\end{matrix}\right\}$$

and a map $i : \mathfrak{g} \longrightarrow U(\mathfrak{g})$. We showed last time that U is left adjoint to F, i.e.,

$$\mathrm{Hom}_{Lie}(\mathfrak{g}, \mathrm{For}(A)) \underset{\text{universal property}}{\simeq} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(U(\mathfrak{g}), A)$$

$$\mathfrak{g} \rightsquigarrow \underset{\substack{\text{unital associative} \\ \text{algebra}}}{U(\mathfrak{g})} := T(\mathfrak{g}) \Big/ \langle x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g} \rangle$$

$$\mathrm{Rep}\,\mathfrak{g} \underset{\substack{\text{equivalence of} \\ \text{monoidal categories}}}{\simeq} \mathrm{Rep}\,U(\mathfrak{g}) \quad \leftarrow \text{Tensor structure is given by the Hopf algebra structure in } U(\mathfrak{g})$$

<u>Action over $V \otimes W$:</u> let V, W two $\mathfrak{g}$-modules.

The $\mathfrak{g}$-module structure on $V \otimes W$ is given by

$$g \cdot (v \otimes w) = (g \cdot v) \otimes w + v \otimes g \cdot w.$$

For an associative algebra A there is no natural A-mod structure on $V \otimes W$.
The key idea is for $A = U(\mathfrak{g})$ we have a comultiplication Δ.

$U(\mathfrak{g})$ is a Hopf algebra and the action on $V \otimes W$ is given by:

$$\begin{tikzcd}
U(\mathfrak{g}) \arrow[r, "\Delta"] \arrow[ddr, dashed] & U(\mathfrak{g}) \otimes U(\mathfrak{g}) \arrow[d] \\
& \mathrm{End}(V) \otimes \mathrm{End}(W) \arrow[d, phantom, "\cong" rotate=90] \\
& \mathrm{End}(V \otimes W)
\end{tikzcd}$$

We can obtain irreducible $\mathfrak{g}$-mod by considering Verma modules V_λ
for a weight λ.

<u>Philosophy</u>: Understand f.d. irreducible $\mathfrak{g}$-mod by using
∞-dimensional modules (the V_λ's) which are quotients of $U(\mathfrak{g})$.

<u>§ 2. Filtered algebras</u> let $\mathbb{N} = \{0, 1, 2, \dots\}$.

A k-algebra A is $\mathbb{N}$-graded if

$$A = \bigoplus_{i=0}^{\infty} A_i,$$

where the A_i's are vector subspaces and $A_n \cdot A_m \subseteq A_{n+m}$.

An algebra A over $\mathbb{k}$ is $\mathbb{N}$-filtered if there is an increasing sequence

$$\{0\} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_i \subseteq \cdots \subseteq A$$

of subspaces of A such that

$$A = \bigcup_{i=0}^{\infty} F_i$$

and $\forall n, m \in \mathbb{N}$,

$$F_n \cdot F_m \subseteq F_{n+m}.$$

<u>Example</u>: The algebra of differential operators $B := \mathbb{k}\{x, d/dx\} \cong \mathbb{k}\langle x, y\rangle / \langle yx - xy = 1\rangle$ of $A := \mathbb{k}[x]$. Here $B \subseteq \operatorname{End} A$.

$$\frac{d}{dx} x^i = i x^{i-1} \qquad d/dx : A_i \longrightarrow A_{i-1}, \quad x : A_{i-1} \longrightarrow A_i$$

$$x^i \longmapsto i x^{i-1} \qquad x^{i-1} \longmapsto x^i$$

$$x \cdot x^i = x^{i+1}$$

$$x\, d/dx : A_i \longrightarrow A_i, \quad d/dx\, x : A_i \longrightarrow A_i$$

$$x^i \longmapsto i x^i \qquad x^i \longmapsto (i+1) x^i$$

We have the relation

$$x\, d/dx - d/dx\, x = 1$$

Let $F_i = \left\{ \begin{array}{l} \text{linear combinations of elements} \\ \text{of the form } x^j (d/dx)^k \text{ s.t} \\ \quad j+k \leq i. \end{array} \right\}$, e.g. $x + d/dx + x^2 \notin F_2$.

Not graded!

Let A be an $\mathbb{N}$-filtered algebra, the associated graded algebra $\operatorname{gr}(A)$ is defined as the vector space

$$\operatorname{gr}(A) = \bigoplus_{n=0}^{\infty} G_n = F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus F_3/F_2 \oplus \cdots$$

where, $G_0 = F_0$ and $G_n = F_n / F_{n-1}$ for $n > 0$. The multiplication is induced by the multiplication in A. This is a graded algebra.

Any $\mathbb{N}$-graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$ admits a filtration $F_n = \bigoplus_{i=0}^{n} A_n$.

Prop A graded algebra is naturally filtered

Proof: $A = \bigoplus_{i=0}^{\infty} A_i$, $F_m = \bigoplus_{i=0}^{m} A_i$, $\bigcup F_m = A$

We have the following diagram

$$\begin{tikzcd} \left\{ \begin{array}{c} \text{graded} \\ \text{algebras} \end{array} \right\} \arrow[r, bend left, "i"] & \left\{ \begin{array}{c} \text{filtered algebras} \\ \text{over } k \end{array} \right\} \arrow[l, bend left, "gr"] \end{tikzcd}$$

Fact: If A is graded then $gr(i(A)) = A$.

Example: The symmetric algebra $\mathrm{Sym}(V)$ of a vector space V with basis $\{x_1, \ldots, x_n\}$ is the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ it is graded

$$\mathrm{Sym}(V) = \bigoplus_{i_1, \ldots, i_n} \mathbb{C}\{x_1^{i_1} \cdots x_n^{i_n}\}.$$

$T(V)$ is also graded

$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - [x,y] \rangle$ is filtered but not graded.

Can show

$$gr(U(\mathfrak{g})) = \mathrm{Sym}(\mathfrak{g}) \leftarrow \text{commutative algebra}$$

$$\neq U(\mathfrak{g}) \quad \therefore \quad U(\mathfrak{g}) \text{ is not graded.}$$

However $U(\mathfrak{g}) \simeq \mathrm{Sym}(\mathfrak{g})$ as vector spaces (they have the same PBW-basis!).

$$U(\mathfrak{g}) = \bigoplus \mathbb{C}\{e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n}\}.$$

If $\mathfrak{g}$ is abelian $U(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}) = gr(U(\mathfrak{g})) \Rightarrow U(\mathfrak{g})$ is graded.

§3. Verma modules.

Let $\mathfrak{g}$ be a semi-simple f.d. Lie algebra over $\mathbb{C}$ (e.g. $\mathfrak{sl}_n(\mathbb{C})$)

Let

$$\begin{array}{lll} \mathfrak{g} & & \rightsquigarrow U(\mathfrak{g}) \\ \cup & & \quad \cup \\ \mathfrak{b} & \rightsquigarrow \text{Borel subalgebra} & \rightsquigarrow U(\mathfrak{b}) \\ \cup & & \quad \cup \\ \mathfrak{h} & \rightsquigarrow \text{Cartan subalgebra} & \rightsquigarrow U(\mathfrak{h}) \end{array}$$

- Let $\lambda \in \mathfrak{h}^*$ $\quad \lambda : \mathfrak{h} \longrightarrow \mathbb{C}$

- $\mathbb{C}_\lambda$ is the 1-dimensional $\mathfrak{h}$-module

$$h \in \mathfrak{h}, \quad z \in \mathbb{C}, \quad h \cdot z = \lambda(h) z.$$

$\rightsquigarrow$ We extend $\mathbb{C}_\lambda$ to a $\mathfrak{b}$-module, $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$.

$$n \cdot z = 0 \quad \text{for } n \in \mathfrak{n}^+$$

$$h \cdot z = \lambda(h) z \quad \forall h \in \mathfrak{h}.$$

$\Rightarrow$ $\mathbb{C}_\lambda$ is a $U(\mathfrak{b})$-module.

- The $U(\mathfrak{b})$-module structure of $\mathbb{C}_\lambda$ can be also be defined directly
  - let $\{x_1, \dots, x_r\}$ be a basis of $\mathfrak{n}^+$
  - let $\{h_1, \dots, h_p\}$ be a basis of $\mathfrak{h}$

We can pick $\{x_1^{i_1} \cdots x_r^{i_r} h_1^{j_1} \cdots h_p^{j_p}\}$ as a PBW-basis for $U(\mathfrak{b})$

and define the action on $\mathbb{C}_\lambda$ as

$$(x_1^{i_1} \cdots x_r^{i_r} h_1^{j_1} \cdots h_p^{j_p}) \cdot z = \lambda(h_1)^{j_1} \lambda(h_2)^{j_2} \cdots \lambda(h_p)^{j_p} z$$

The **Verma module** associated with λ is the $U(\mathfrak{g})$-module:

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \quad \left(= U(\mathfrak{n}^-) \text{ as vector space} \right)$$

For $x \in U(\mathfrak{g})$ the action is $x \cdot (y \otimes z) := xy \otimes z$.

Prop. If $\lambda \in \mathbb{N}$, $V(\lambda)$ quotiented by its maximal $U(\mathfrak{g})$-submodule is the irreducible $\mathfrak{g}$-module of highest weight λ.

Example: $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, where $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} =: f$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: h$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =: e$.

$$\begin{aligned} \mathfrak{g} &= \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e \\ &\cup \\ \mathfrak{b} &= \mathbb{C}h \oplus \mathbb{C}e \\ &\cup \\ \mathfrak{h} &= \mathbb{C}h \end{aligned}$$

$$[h,f] = -2f \qquad [h,e] = 2e \qquad [e,f] = h$$

$$U(\mathfrak{h}) = \mathbb{C}[h]$$

A PBW-basis of $U(\mathfrak{sl}_2)$ is $\{f^i h^j e^k\}_{i,j,k=0}^{\infty}$.

Let $\lambda \in \mathfrak{h}^*$. $\quad 1 \otimes 1 \in V(\lambda)$

$$\begin{aligned}
f \cdot 1\otimes 1 &= f \otimes 1 & f \cdot f \otimes 1 &= f^2 \otimes 1 \\
h \cdot 1 \otimes 1 &= \lambda \cdot 1 \otimes 1 & h \cdot f \otimes 1 &= (fh - 2f) \otimes 1 = (\lambda - 2)\, f \otimes 1 \\
e \cdot 1 \otimes 1 &= 0 & e \cdot f \otimes 1 &= (fe + h) \otimes 1 = \lambda \cdot 1 \otimes 1
\end{aligned}$$

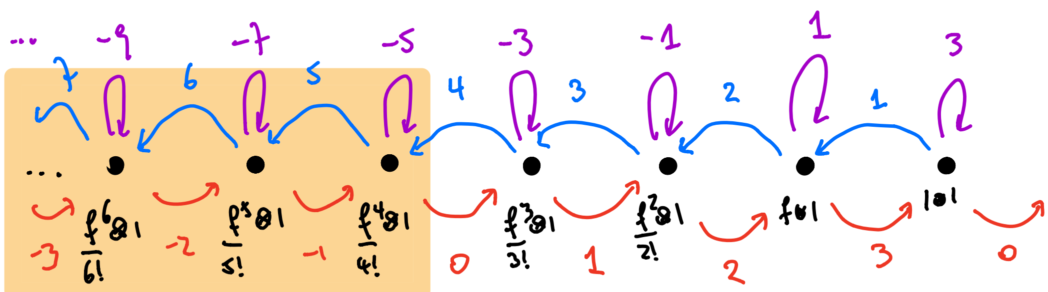
$\{f^k \otimes 1\}$ is a basis for $V(\lambda)$. We choose $\left\{ \frac{f^k}{k!} \otimes 1 \right\}$ as a basis

$$f \cdot \frac{f^k}{k!} \otimes 1 = (k+1) \frac{f^{k+1}}{(k+1)!} \otimes 1,$$

$$h \cdot \frac{f^k}{k!} \otimes 1 = (3-2k) \frac{f^k}{k!} \otimes 1,$$

$$e \cdot \frac{f^k}{k!} \otimes 1 = \begin{cases} (4-k) \dfrac{f^{k-1}}{(k-1)!} \otimes 1 & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases}$$

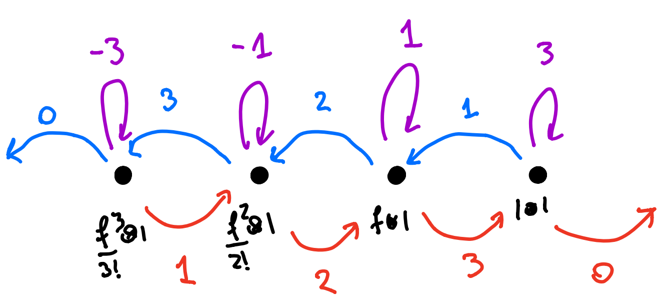
For example, if $\lambda(h) = 3$, we have $V(3)$ as $U(\mathfrak{g})$-module is:



this is the unique non-trivial proper submodule of $V(3)$

The quotient of $V(3)$ with its proper submodule is

$L(3) =$



This is the $\mathfrak{sl}_2$-module of highest weight 3.

Thanks for attending!

(Next time: Weyl's character formula!)