Appendices for article: The role of additive and multiplicative noise in filtering complex dynamical systems

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Appendix A. Singular perturbation expansion for linear problem

In this Appendix, we apply singular perturbation theory to obtain one order higher than the usual averaging limit. We recall the linear model (2.1)–(2.2)

$$dx = (a_{11}x + a_{12}y)\,dt + \sigma_x dW_x, \tag{A 1}$$

$$dy = \frac{1}{\epsilon}(a_{21}x + a_{22}y)\,dt + \frac{\sigma_y}{\sqrt{\epsilon}}dW_y, \tag{A 2}$$

for a slow variable $x \in \mathbb{R}$ and fast variable $y \in \mathbb{R}$. Here, W_x, W_y are independent Wiener processes and the parameter ϵ characterizes the time scale gap. We assume throughout that $\sigma_x, \sigma_y \neq 0$ and that the eigenvalues of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ \frac{1}{\epsilon}a_{21} & \frac{1}{\epsilon}a_{22} \end{pmatrix}$$

are strictly negative, to assure the existence of a unique invariant joint density. Furthermore we require $\tilde{a} = a_{11} - a_{12}a_{22}^{-1}a_{21} < 0$ to assure that the leading order slow dynamics supports an invariant density (cf.(A 11)). The probability density function of the system (A 1)-(A 2) satisfies a Fokker-Planck equation,

$$\frac{\partial}{\partial t}\rho(x,y,t) = \frac{1}{\epsilon}\mathcal{L}_0^\star\rho(x,y,t) + \mathcal{L}_1^\star\rho(x,y,t), \tag{A 3}$$

where

$$\mathcal{L}_0^\star\varphi = -\frac{\partial}{\partial y}\left((a_{21}x + a_{22}y)\,\varphi\right) + \frac{1}{2}\sigma_y^2\frac{\partial^2}{\partial y^2}\varphi,$$

$$\mathcal{L}_1^\star\varphi = -\frac{\partial}{\partial x}\left((a_{11}x + a_{12}y)\,\varphi\right) + \frac{1}{2}\sigma_x^2\frac{\partial^2}{\partial x^2}\varphi. \tag{A 4}$$

The probability density function is expanded according to

$$\rho(x,y,t) = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots. \tag{A 5}$$

Substituting the series (A 5) into the Fokker-Planck equation (A 3), and collecting orders of ϵ, we obtain at lowest order, $\mathcal{O}(1/\epsilon)$,

$$\mathcal{L}_0^\star\rho_0 = 0. \tag{A 6}$$

Since the fast dynamics is ergodic, there exists a unique solution to (A 6),

$$\rho_0(x,y,t) = \rho_\infty(y;x)\hat{\rho}(x,t),$$

with $\rho_\infty(y;x)$ given by,

$$\rho_\infty(y;x) \propto \exp\Big(\frac{a_{22}\tilde{y}^2}{\sigma_y^2}\Big), \qquad \tilde{y} \equiv y + \frac{a_{21}}{a_{22}}x, \tag{A 7}$$

At the next order, $\mathcal{O}(1)$, we obtain

$$\mathcal{L}_0^\star \rho_1 = \frac{\partial}{\partial t}\rho_0 - \mathcal{L}_1^\star \rho_0. \tag{A 8}$$

To assure boundedness of ρ_1 (and thereby of the asymptotic expansion (A 5)) a solvability condition has to be satisfied prescribed by the Fredholm alternative. Equation (A 8) is solvable only if the right-hand-side is in the space orthogonal to the (one-dimensional) null space of the adjoint $\mathcal{L}_0$ of $\mathcal{L}_0^\star$. The adjoint to the Fokker-Planck equation is called the backward-Kolmogorov equation and propagates expectation values; hence ergodicity of the fast dynamics implies that the infinitesimal generator $\mathcal{L}_0$ has constants as the only kernel modes. The solvability condition for (A 8) therefore reads as

$$\int dy \Big(\frac{\partial}{\partial t}\rho_0 - \mathcal{L}_1^\star \rho_0\Big) = 0, \tag{A 9}$$

which yields the evolution equation for $\hat{\rho}$

$$\frac{\partial}{\partial t}\hat{\rho} = -\frac{\partial}{\partial x}\left(\tilde{a}x\hat{\rho}\right) + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2}\hat{\rho}\,. \tag{A 10}$$

The associated averaged Langevin equation

$$dX = \tilde{a}X\,dt + \sigma_x dW_x, \tag{A 11}$$

supports an invariant measure provided $\tilde{a} < 0$. The solution of the reduced Fokker-Planck equation (A 10) is

$$\hat{\rho}(x,t) \propto \exp\left(-\frac{x^2}{2\sigma(t)^2}\right), \tag{A 12}$$

where

$$\sigma^2(t) = -\frac{\sigma_x^2}{2\tilde{a}}(1 - e^{2\tilde{a}t}).$$

To capture diffusive effects of the fast variable we perform here the perturbation theory one order higher. The solvability condition (A 9) assures that the $\mathcal{O}(1)$ equation (A 8) can be solved for ρ_1. In order for $\rho_0 + \epsilon\rho_1$ to be a density we require the necessary condition on ρ_1

$$\int dx\, dy\, \rho_1 = 0. \tag{A 13}$$

After lengthy algebra, we find

$$\begin{aligned}\rho_1(x,y,t) = \Big(\Big([\frac{2a_{11}a_{21}}{\sigma_y^2 a_{22}} - \frac{2a_{12}a_{21}^2}{\sigma_y^2 a_{22}^2}]x\tilde{y} + [\frac{a_{12}a_{21}}{\sigma_y^2 a_{22}} - \frac{a_{21}^2\sigma_x^2}{a_{22}\sigma_y^4}](\tilde{y}^2 + \frac{\sigma_y^2}{2a_{22}})\Big)\hat{\rho}(x,t) \\ + [\frac{a_{12}}{a_{22}} - \frac{2a_{21}\sigma_x^2}{a_{22}\sigma_y^2}]\tilde{y}\partial_x\hat{\rho}(x,t) + R(x,t)\Big)\rho_\infty(y;x),\end{aligned} \tag{A 14}$$

where $R(x,t)$ lies in the kernel of $\mathcal{L}_0^\star$ and will be chosen to satisfy the condition in (A 13), which reduces to.

$$\int dx\, R(x,t) = 0. \tag{A 15}$$

The function R can be determined by requiring that the solution ρ_1 satisfies the solvability condition of the next order, $\mathcal{O}(\epsilon)$,

$$\mathcal{L}_0^\star \rho_2 = \frac{\partial}{\partial t}\rho_1 - \mathcal{L}_1^\star \rho_1, \tag{A 16}$$

that is,

$$\int dy\Big(\frac{\partial}{\partial t}\rho_1 - \mathcal{L}_1^\star \rho_1\Big) = 0. \tag{A 17}$$

Substituting (A 14) to (A 17), we obtain

$$\frac{\partial}{\partial t}R + \frac{\partial}{\partial x}\left(\tilde{a}xR\right) - \frac{1}{2}\sigma_x^2\frac{\partial^2}{\partial x^2}R = \frac{a_{12}a_{21}}{a_{22}^2}\tilde{a}\partial_x(x\hat{\rho}) + \Big(\frac{a_{12}^2\sigma_y^2}{2a_{22}^2} - \frac{a_{12}a_{21}\sigma_x^2}{a_{22}^2}\Big)\partial_{xx}\hat{\rho}, \tag{A 18}$$

the solutions of which can be found by a self-similarity ansatz, $R(x,t) = \sigma(t)^{-1}f(\xi)$, where $\xi \equiv x\sigma(t)^{-1}$. After some algebra, we find a solution that satisfies the condition in (A 15),

$$R(x,t) = \frac{a_{12}}{\sigma_x^2 a_{22}^2}\Big(-a_{21}\sigma_x^2 + \frac{a_{12}\sigma_y^2}{2} - a_{21}\tilde{a}\sigma^2(t)\Big)\Big(\frac{x^2}{\sigma^2(t)} - 1\Big)\hat{\rho}(x,t). \tag{A 19}$$

It is not too difficult to check that (A 14) with (A 19) satisfies the constraint in (A 13). Furthermore, we have $\mathbb{E}_{\rho_1}(x) = \int dx\, dy\, x\, \rho_1(x,y) = 0$, which implies that the $\mathcal{O}(\epsilon)$-correction does not contribute a correction to the mean of x when compared to the averaged system (A 11).

Appendix B. Convergence of solutions for the linear system

We formulate the following convergence result.

Theorem 0.1. *Consider the linear multi-scale system* (A 1)-(A 2)*. Assume that the matrix A has negative eigenvalues and that $\tilde{a} = a_{11} - a_{12}a_{22}^{-1}a_{21} < 0$. Let x^ϵ be the solution of* (A 1)-(A 2) *and $\tilde{X}$ be the solution of the reduced equation,*

$$d\tilde{X} = \tilde{a}\tilde{X}\, dt + \sigma_x dW_x - \sqrt{\epsilon}\sigma_y\frac{a_{12}}{a_{22}}dW_y, \tag{B 1}$$

corresponding to the same realizations W_x, W_y and the same initial condition $x^\epsilon(0) = \tilde{X}(0)$. Then the error $e(t) = x^\epsilon(t) - \tilde{X}(t)$ is bounded for finite time T by

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2\Big) \le c\epsilon^2. \tag{B 2}$$

Proof. We follow the general line of proof as outlined in Pavliotis and Stuart (2008), and extend the results of Zhang (2011) to the next order. Consider the slow drift term $f(x,y,t) = a_{11}x + a_{12}y$ and the averaged slow vector field $F(x) = \tilde{a}x$ where we recall $\tilde{a} = a_{11} - a_{12}a_{22}^{-1}a_{21} < 0$. Introducing the infinitesimal generators $\mathcal{L}_0$ and $\mathcal{L}_1$ as the formal L^2-adjoints of the operators (A 4), consider the following Poisson equation,

$$\mathcal{L}_0\phi(x,y,t) = f(x,y) - F(x) = a_{11}x + a_{12}y - \tilde{a}x, \tag{B 3}$$

for smooth functions ϕ satisfying $\langle\phi\rangle_{\rho_\infty} \equiv \int \phi(x,y,t)\rho_\infty(y;x)\, dy = 0$. The existence of solutions to the Poisson equation is assured by a Fredholm alternative since $\langle f - F\rangle_{\rho_\infty} = 0$. Using Itô's formula we write

$$\frac{d\phi}{dt} = \frac{1}{\epsilon}\mathcal{L}_0\phi + \mathcal{L}_1\phi + \sigma_x\partial_x\phi\dot{W}_x + \frac{\sigma_y}{\sqrt{\epsilon}}\partial_y\phi\dot{W}_y. \tag{B 4}$$

Combining (B 3) and (B 4), we deduce that

$$f(x,y) = F(x) - \sqrt{\epsilon}\sigma_y \partial_y \phi \dot{W}_y + \epsilon\Big(\frac{d\phi}{dt} - \mathcal{L}_1\phi - \sigma_x \partial_x \phi \dot{W}_x\Big) \tag{B 5}$$

and substituting this equality into the slow equation (A 1), we obtain

$$dx = F(x)\,dt + \sigma_x dW_x - \sqrt{\epsilon}\sigma_y dW_y + \epsilon\Big(\frac{d\phi}{dt} - \mathcal{L}_1\phi - \sigma_x \partial_x \phi \dot{W}_x\Big)dt. \tag{B 6}$$

In the case of our linear system (A 1)-(A 2) we find $\phi(y) = \frac{a_{12}}{a_{22}}y$ as a solution of the Poisson equation (B 3) and find

$$dx^\epsilon = \tilde{a}x^\epsilon\,dt + \sigma_x dW_x - \sqrt{\epsilon}\sigma_y \frac{a_{12}}{a_{22}} dW_y + \epsilon\frac{d\phi}{dt}dt. \tag{B 7}$$

Note that this equation is simply a rewrite of (A 1) where we replace the fast variable y using (A 2). We have

$$\int_0^t \frac{d\phi}{ds}ds = \phi(t) - \phi(0) = -\frac{a_{12}}{a_{22}}(y^\epsilon(t) - y^\epsilon(0)).$$

Integrating (B 7) and (B 1), we obtain the solution of the full system

$$x^\epsilon(t) = x^\epsilon(0) + \tilde{a}\int_0^t x^\epsilon(s)ds + \sigma_x \int_0^t dW_x - \sqrt{\epsilon}\sigma_y\frac{a_{12}}{a_{22}}\int_0^t dW_y - \epsilon\frac{a_{12}}{a_{22}}(y^\epsilon(t) - y^\epsilon(0)) \tag{B 8}$$

and of the reduced system

$$\tilde{X}(t) = \tilde{X}(0) + \tilde{a}\int_0^t \tilde{X}(s)ds + \sigma_x\int_0^t dW_x - \sqrt{\epsilon}\sigma_y\frac{a_{12}}{a_{22}}\int_0^t dW_y, \tag{B 9}$$

respectively. Defining the error $e(t) = x^\epsilon(t) - \tilde{X}(t)$ with $e(0) = 0$, we have

$$e(t) = \tilde{a}\int_0^t e(s)ds - \epsilon\frac{a_{12}}{a_{22}}(y^\epsilon(t) - y^\epsilon(0)),$$

such that upon using the triangle inequality we obtain the bound

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2\Big) \le 2\Big(\int_0^T \mathbb{E}\Big(\sup_{0\le t\le T}|e(s)|^2\Big)ds + \epsilon^2\mathbb{E}\Big(\sup_{0\le t\le T}|y^\epsilon(t) - y^\epsilon(0)|^2\Big)\Big). \tag{B 10}$$

The upper bound in (B 2) is obtained upon using the Gronwall lemma and by applying the result in Zhang (2011),

$$\mathbb{E}\Big(\sup_{0\le t\le T}|y^\epsilon(t)|^2\Big) \le \mathcal{O}(\log(T\epsilon^{-1})). \tag{B 11}$$

□

We remark that extensions to the multi-dimensional case can be readily made provided there are suitable non-degeneracy conditions for the noise.

Appendix C. First and second order statistics of the reduced model with additive and multiplicative noises

In this Appendix we compute the first and second order statistics of the reduced stochastic model in (D 1) with initial condition $U(t_0) = U_0$. Define

$$J(s,t) = \int_s^t \hat{\lambda} ds' + \sqrt{\epsilon}\frac{\sigma_\gamma}{d_\gamma} dW_\gamma(s') = \hat{\lambda}(t-s) + \sqrt{\epsilon}\frac{\sigma_\gamma}{d_\gamma}(W_\gamma(t) - W_\gamma(s)) \equiv \hat{\lambda}(t-s) + J_W(s,t),$$

where

$$\begin{aligned} J_W(s,t) &= \sqrt{\epsilon}\frac{\sigma_\gamma}{d_\gamma}(W_\gamma(t) - W_\gamma(s)) \\ \langle J_W(s,t)\rangle &= 0 \\ Var(J_W(s,t)) = \langle J_W(s,t)^2\rangle &= \epsilon\frac{\sigma_\gamma^2}{d_\gamma^2}(t-s). \end{aligned}$$

The explicit solution for (D 1) can be written as follows,

$$\begin{aligned} U(t) &= e^{-J(t_0,t)}U_0 + \int_{t_0}^t (\hat{b} + f(s))e^{-J(s,t)}ds + \sigma_u \int_{t_0}^t e^{-J(s,t)}dW_u(s) + \sqrt{\epsilon}\frac{\sigma_b}{\lambda_b}\int_{t_0}^t e^{-J(s,t)}dW_b(s) \\ &= A + B + C + D. \end{aligned}$$

Therefore, we can compute the mean, the variance and the covariance

$$\begin{aligned} \langle U(t)\rangle &= \langle A\rangle + \langle B\rangle, \qquad \text{(C 1)} \\ Var(U(t)) &= \langle |U(t)|^2\rangle - |\langle U(t)\rangle|^2 \\ Cov(U(t), U(t)^*) &= \langle U(t)^2\rangle - \langle U(t)\rangle^2, \end{aligned}$$

where

$$\langle |U(t)|^2\rangle = \langle |A|^2\rangle + \langle |B|^2\rangle + \langle |C|^2\rangle + \langle |D|^2\rangle + 2Re[\langle A^* B\rangle] \qquad \text{(C 2)}$$

$$\langle U(t)^2\rangle = \langle A^2\rangle + \langle B^2\rangle + 2\langle AB\rangle. \qquad \text{(C 3)}$$

Now we are going to compute each term in the right hand side of (C 1), (C 2), and (C 3). In all of these computations, we will use the fact that

$$\begin{aligned} \langle ze^{bx}\rangle &= \langle z\rangle e^{b\langle x\rangle + \frac{b^2}{2}Var(x)} \\ \langle zwe^{bx}\rangle &= \Big(\langle z\rangle\langle w\rangle + cov(z, w^*)\Big)e^{b\langle x\rangle + \frac{b^2}{2}Var(x)}, \end{aligned}$$

for any constant b, assuming that the real valued Gaussian random variable x is independent of both complex valued Gaussian random variable z and w.

We find that

$$\begin{aligned} \langle A\rangle &= e^{-\hat{\lambda}(t-t_0)}\langle U_0 e^{-J_W(t_0,t)}\rangle = e^{-\hat{\lambda}(t-t_0)}\langle U_0\rangle\langle e^{-J_W(t_0,t)}\rangle \\ &= \langle U_0\rangle e^{-\hat{\lambda}(t-t_0) + \frac{1}{2}Var(J_W(s,t))} = \langle U_0\rangle e^{\left(-\hat{\lambda} + \frac{\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}\right)(t-t_0)} \\ \langle B\rangle &= \int_{t_0}^t (\hat{b} + f(s))e^{-\hat{\lambda}(t-t_0)}\langle e^{-J_W(s,t)}\rangle ds = \int_{t_0}^t (\hat{b} + f(s))e^{\left(-\hat{\lambda} + \frac{\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}\right)(t-s)}ds \\ \langle |A|^2\rangle &= e^{-2\hat{\gamma}(t-t_0)}\langle |U_0|^2 e^{-2J_W(s,t)}\rangle = \Big(|\langle U_0\rangle|^2 + Var(U_0)\Big)e^{2\left(-\hat{\gamma} + \epsilon\frac{\sigma_\gamma^2}{d_\gamma^2}\right)(t-t_0)} \end{aligned}$$

$$\begin{aligned}\langle |B|^2\rangle &= \int_{t_0}^t\int_{t_0}^t \Big\langle \Big((\hat b+f(s))e^{-J(s,t)}\Big)\Big((\hat b+f(r))e^{-J(r,t)}\Big)^*\Big\rangle\, ds\, dr,\\ &= \int_{t_0}^t\int_{t_0}^t \big(|\hat b|^2+\hat b f(r)^*+\hat b^* f(s)+f(s)f(r)^*\big)e^{-2\hat\gamma t+\hat\lambda s+\hat\lambda^* r}\langle e^{-J_W(s,t)-J_W(r,t)}\rangle\, ds\, dr,\\ &= \int_{t_0}^t\int_{t_0}^t \big(|\hat b|^2+\hat b f(r)^*+\hat b^* f(s)+f(s)f(r)^*\big)e^{-2\hat\gamma t+\hat\lambda s+\hat\lambda^* r}e^{\frac{\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}\left[2t-r-s+2\min(t-r,t-s)\right]}\, ds\, dr,\end{aligned}$$

Using the Itô lemma, we deduce:

$$\begin{aligned}\langle |C|^2\rangle &= \sigma_u^2\int_{t_0}^t e^{-2\hat\gamma(t-s)}\langle e^{-2J_W(s,t)}\rangle ds = -\frac{\sigma_u^2}{2(-\hat\gamma+\epsilon\frac{\sigma_\gamma^2}{d_\gamma^2})}\Big(1-e^{2(-\hat\gamma+\epsilon\frac{\sigma_\gamma^2}{d_\gamma^2})(t-t_0)}\Big)\\ \langle |D|^2\rangle &= -\frac{\epsilon\sigma_b^2}{2|\lambda_b|^2(-\hat\gamma+\epsilon\frac{\sigma_\gamma^2}{d_\gamma^2})}\Big(1-e^{2(-\hat\gamma+\epsilon\frac{\sigma_\gamma^2}{d_\gamma^2})(t-t_0)}\Big)\\ \langle A^*B\rangle &= e^{-\hat\lambda^*(t-t_0)}\langle U_0^*\rangle\int_{t_0}^t(\hat b+f(s))e^{-\hat\lambda(t-t_0)}\langle e^{-J_W(t_0,t)-J_W(s,t)}\rangle ds,\end{aligned}\tag{C 4}$$

where

$$\begin{aligned}\langle e^{-J_W(t_0,t)-J_W(s,t)}\rangle &= e^{\frac12 Var(J(t_0,t))+\frac12 Var(J(s,t))+Cov(J(t_0,t)J(s,t))}\\ &= e^{\frac{\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}\left[(t-t_0)+(t-s)+2\min(t-t_0,t-s)\right]} = e^{\frac{\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}\left[(t-t_0)+3(t-s)\right]},\end{aligned}\tag{C 5}$$

since $t_0\le s\le t$. Substituting (C 5) into (C 4), we obtain

$$\langle A^*B\rangle = e^{\left(-2\hat\gamma+\frac{\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}\right)(t-t_0)}\langle U_0^*\rangle\Big[\int_{t_0}^t(\hat b+f(s))e^{\frac{3\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}(t-s)}\, ds\Big].\tag{C 6}$$

Following the same procedure, we obtain

$$\begin{aligned}\langle A^2\rangle &= e^{-2\hat\lambda(t-t_0)}\langle U_0^2 e^{-2J_W(s,t)}\rangle = \Big(\langle U_0\rangle^2+Cov(U_0,U_0^*)\Big)e^{2\left(-\hat\lambda+\epsilon\frac{\sigma_\gamma^2}{d_\gamma^2}\right)(t-t_0)}\\ \langle AB\rangle &= e^{-\hat\lambda(t-t_0)}\langle U_0\rangle\int_{t_0}^t(\hat b+f(s))e^{-\hat\lambda(t-t_0)}\langle e^{-J_W(t_0,t)-J_W(s,t)}\rangle ds\\ &= e^{\left(-2\hat\lambda+\frac{\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}\right)(t-t_0)}\langle U_0\rangle\Big[\int_{t_0}^t(\hat b+f(s))e^{\frac{3\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}(t-s)}\, ds\Big]\\ \langle B^2\rangle &= \int_{t_0}^t\int_{t_0}^t\Big\langle\Big((\hat b+f(s))e^{-J(s,t)}\Big)\Big((\hat b+f(r))e^{-J(r,t)}\Big)\Big\rangle\, ds\, dr\\ &= \int_{t_0}^t\int_{t_0}^t\big(\hat b^2+\hat b f(r)+\hat b f(s)+f(s)f(r)\big)e^{-2\hat\lambda t+\hat\lambda s+\hat\lambda r}\langle e^{-J_W(s,t)-J_W(r,t)}\rangle\, ds\, dr\\ &= \int_{t_0}^t\int_{t_0}^t\big(\hat b^2+\hat b f(r)+\hat b f(s)+f(s)f(r)\big)e^{-2\hat\gamma t+\hat\lambda s+\hat\lambda r}e^{\frac{\epsilon}{2}\frac{\sigma_\gamma^2}{d_\gamma^2}\left[2t-r-s+2\min(t-r,t-s)\right]}\, ds\, dr.\end{aligned}$$

Appendix D. Convergence of solutions for the nonlinear system

Theorem 0.2. *Consider the SPEKF model in (3.1). Assume that* $f(t)$ *is bounded,* $\Xi_n\equiv -n\hat\gamma+\epsilon\frac{n^2\sigma_\gamma^2}{2d_\gamma^2}<0$ *for* $1\le n\le 4$, *and* $\tilde\gamma$ *has sufficient decay of correlations. Let* u^ϵ *be a solution of (3.1) and* U *be a solution of*

$$\frac{dU}{dt} = -(\hat\lambda+\sqrt{\epsilon}\frac{\sigma_\gamma}{d_\gamma}\dot W_\gamma)U+\hat b+f(t)+\sigma_u\dot W_u+\sqrt{\epsilon}\frac{\sigma_b}{\lambda_b}\dot W_b,\tag{D 1}$$

where $\hat\lambda = \hat\gamma - i\omega$, *corresponding to the same realizations* W_u, W_b, W_γ *and the same initial condition* $u^\epsilon(0) = U(0)$. *Then there is a constant* $\tilde C(\epsilon, T)$ *such that the error* $e(t) = u^\epsilon(t) - U(t)$ *is bounded for finite time* T *by*

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2\Big) \le c\epsilon^2\Big(\tilde C(\epsilon,T) + C\frac{\sigma_u^2\sigma_\gamma^2}{d_\gamma^3}T\log(1+\epsilon^{-1}d_\gamma T)\Big)e^{c(1+\epsilon\frac{\sigma_\gamma^2}{d_\gamma^2})T}. \tag{D 2}$$

Proof. Following the proof of Theorem 0.1, we solve the following Poisson problem for ϕ,

$$\mathcal{L}_0\phi = F_1(u,\tilde b,\tilde\gamma) - \bar F_1(u) = -\tilde\gamma u + \tilde b, \tag{D 3}$$

$$\int\int \phi(u,\tilde b,\tilde\gamma)p_\infty(\tilde b,\tilde\gamma)\,d\tilde b\,d\tilde\gamma = 0, \tag{D 4}$$

where $F_1(u) = -(\tilde\gamma+\hat\lambda)u + \hat b + \tilde b + f(t)$ and $\bar F_1(u) = -\hat\lambda u + \hat b + f(t)$ are the drift terms of the slow dynamics in (3.1) and of the reduced model in (D 1), respectively. One can verify that

$$\phi(u,\tilde b,\tilde\gamma) = \frac{\tilde\gamma u}{d_\gamma} - \frac{\tilde b}{\lambda_b} \tag{D 5}$$

is a solution for the Poisson problem in (D 3)-(D 4). Applying the Itô-formula we obtain

$$\frac{d\phi}{dt} = \frac{1}{\epsilon}\mathcal{L}_0\phi + \mathcal{L}_1\phi + \frac{\tilde\gamma}{d_\gamma}\sigma_u\dot W_u - \frac{\sigma_b}{\sqrt{\epsilon}\lambda_b}\dot W_b + u\frac{\sigma_\gamma}{\sqrt{\epsilon}d_\gamma}\dot W_\gamma,$$

and subsequently, we can rewrite (D 3) as follows

$$F_1(u,\tilde b,\tilde\gamma) = \bar F_1(u) + \mathcal{L}_0\phi = \bar F_1(u) + \epsilon\big(\frac{d\phi}{dt} - \mathcal{L}_1\phi\big) - \epsilon\frac{\tilde\gamma}{d_\gamma}\sigma_u\dot W_u + \sqrt{\epsilon}\frac{\sigma_b}{\lambda_b}\dot W_b - \sqrt{\epsilon}u\frac{\sigma_\gamma}{d_\gamma}\dot W_\gamma.$$

Substituting into the slow equation in (3.1), we have

$$\frac{du^\epsilon}{dt} = \bar F_1(u^\epsilon) + \sigma_u\dot W_u + \sqrt{\epsilon}\Big(\frac{\sigma_b}{\lambda_b}\dot W_b - u^\epsilon\frac{\sigma_\gamma}{d_\gamma}\dot W_\gamma\Big) + \epsilon\Big(\big(\frac{d\phi^\epsilon}{dt} - \mathcal{L}_1\phi^\epsilon\big) - \frac{\tilde\gamma}{d_\gamma}\sigma_u\dot W_u\Big). \tag{D 6}$$

The solutions of the full model (D 6) is given by

$$\begin{aligned}u^\epsilon(t) &= u^\epsilon(0) + \int_0^t(-\hat\lambda u^\epsilon(s) + \hat b + f(s))\,ds + \sigma_u\int_0^t dW_u(s) + \sqrt{\epsilon}\frac{\sigma_b}{\lambda_b}\int_0^t dW_b(s) - \sqrt{\epsilon}\frac{\sigma_\gamma}{d_\gamma}\int_0^t u^\epsilon(s)dW_\gamma(s)\\ &\quad + \epsilon(\phi^\epsilon(t) - \phi^\epsilon(0)) - \epsilon\int_0^t \bar F_1(u^\epsilon(s))\frac{\tilde\gamma^\epsilon(s)}{d_\gamma}ds - \epsilon\frac{\sigma_u}{d_\gamma}\int_0^t\tilde\gamma(s)^\epsilon dW_u(s),\end{aligned}$$

and the solution of the approximate model (D 1) is given by

$$U(t) = U(0) + \int_0^t(-\hat\lambda U(s) + \hat b + f(s))\,ds + \sigma_u\int_0^t dW_u(s) + \sqrt{\epsilon}\frac{\sigma_b}{\lambda_b}\int_0^t dW_b(s) - \sqrt{\epsilon}\frac{\sigma_\gamma}{d_\gamma}\int_0^t U(s)dW_\gamma(s).$$

Defining $e(t) \equiv u^\epsilon(t) - U(t)$ and $e(0) = 0$, we obtain upon using the triangle inequality

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2\Big) \le 4\Big(&\int_0^T\mathbb{E}|e(s)|^2\,ds + \epsilon\frac{\sigma_\gamma^2}{d_\gamma^2}\mathbb{E}\Big|\int_0^T\int_0^T e(s)e(s')dW_\gamma(s)dW_\gamma(s')\Big|\\ &+ \epsilon^2\theta(T) + \epsilon^2\frac{\sigma_u^2}{d_\gamma^2}\mathbb{E}\Big|\int_0^T\int_0^T\gamma(s)\gamma(s')dW_u(s)dW_u(s')\Big|\Big),\end{aligned} \tag{D 7}$$

where

$$\theta(T) \equiv \mathbb{E}\Big(\sup_{0\le t\le T}\Big|\phi(t) - \phi(0) - \int_0^t\bar F_1(s)\frac{\tilde\gamma(s)}{d_\gamma}ds\Big|^2\Big). \tag{D 8}$$

By Itô isometry, (D 7) becomes

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e(t)|^2\Big) \le c\Big((1+\epsilon\frac{\sigma_\gamma^2}{d_\gamma^2})\int_0^T \mathbb{E}|e(s)|^2\,ds + \epsilon^2\theta(T) + \epsilon^2\frac{\sigma_u^2}{d_\gamma^2}\int_0^T \mathbb{E}|\gamma(s)|^2 ds\Big)$$
$$\le c\Big((1+\epsilon\frac{\sigma_\gamma^2}{d_\gamma^2})\int_0^T \mathbb{E}|e(s)|^2\,ds + \epsilon^2\theta(T) + C\epsilon^2\frac{\sigma_u^2\sigma_\gamma^2}{d_\gamma^3}T\log(1+\epsilon^{-1}d_\gamma T)\Big)\Big),$$

where we use Lemma 3.5 in Zhang (2011),

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\gamma(t)|^2\Big) \le C\frac{\sigma_\gamma^2}{d_\gamma}\log(1+\epsilon^{-1}d_\gamma T). \tag{D 9}$$

Provided $\tilde\gamma(t)$ decorrelates sufficiently fast and additionally $\Xi_n \equiv -n\hat\gamma + \epsilon n^2\sigma_\gamma^2/(2d_\gamma^2) < 0$ is satisfied, moments up to order-n are bounded (see Appendix D in Branicki and Majda 2013). Since θ in (D 8) is a function of as high as fourth order moments, our assumptions guarantee that there is a constant $\tilde C(\epsilon,T)$ such that $\theta(T)\le \tilde C(\epsilon,T)$. Subsequently, the main result in (D 2) is obtained upon employing the Gronwall lemma. □

The upper bound in (D 2) is not tight since we don't have an explicit expression for $\tilde C$ as a function of ϵ. Figure 5 suggests that (at least for the parameters chosen therein) that $\tilde C \approx \mathcal{O}(\epsilon^{-1})$.

References

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