On a normal form for one-dimensional excitable media

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(joint work with Lorenz Kramer)

We present a generic normal form for one-dimensional excitable media. The normal form is constructed around the well-known generic saddle-node bifurcation present in excitable media for isolated pulses. In the case of wave trains or pulses in a ring of length L with velocity c_0, this saddle-node will be disturbed and depends on the wavelength. The interaction with the preceding pulse (or more accurately with its inhibitor) modifies the bifurcation behaviour. The normal form reads

$$\partial_t X = -\mu - gX^2 - \beta(\gamma + X(t-\tau) + \gamma_1 X(t)) \, , \tag{1}$$

where $\beta = \beta_0 \exp(-\kappa\tau)$. Here X measures for example the deviation of the maximal amplitude of the pulse with respect to the saddle-node of the isolated pulse. The delay time $\tau = L/c_0$ is the temporal distance of two consecutive pulses, and κ is the decay rate of the inhibitor. The last term models the influence of the inhibitor of the preceding pulse.

Besides the saddle-node of the isolated pulse ($\beta \to 0$) and the shifted saddle-node of a wave train of finite wave length L, Equation (1) exhibits three new bifurcation scenarios. In particular, a Hopf bifurcation which may coalesce in a Bogdanov-Takens point with the saddle-node, and an inhomogeneous pitchfork bifurcation in which every second pulse dies. These bifurcations have so far not been observed and we can verify the predictions of our normal form in a modified Barkley model [1] and the Fitzhugh-Nagumo model [2].

We determine the parameters of the normal form by fitting to numerical data obtained by simulating a particular excitable medium, the Barkley model. We test the predictions against numerical simulations of partial differential equation models of excitable media. The quantitative agreement and the predictions are striking.

Moreover, we presented a non-perturbative approach to study bifurcations in excitable media [3]. It is based on the observation that close to the saddle-node the pulse shape is approximately a bell-shaped function. Employing a test function approximation that optimises the two free parameters of a bell-shaped function, i.e. its amplitude and its width, we find the actual bifurcation point and determine the pulse shape for close-to-critical pulses at excitabilities near the bifurcation. This method which makes explicit use of the bell-shaped character at the bifurcation point has also been successfully applied to other reaction diffusion systems such as bistable and autocatalytic systems. It was also successful in describing solution behaviour of reaction diffusion systems far away from the bifurcation [4, 5]. We show that this method is successful in describing retracting fingers in two-dimensional excitable media. Our method may be used to determine the parameters of the normal form (1) directly from the PDE.

References

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