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Parametric envelope solitons in coupled Korteweg-de Vries equations

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Abstract

We demonstrate that a system of linearly coupled Korteweg-de Vries equations, which *inter alia* is a general model of resonantly coupled internal waves in a stratified fluid, can give rise to broad envelope solitons produced by a double phase- and group-velocity resonance between the fundamental and second harmonics for certain wavenumbers. We derive asymptotic equations for the amplitudes of the two harmonics, which are identical to the second-harmonic-generation equations in a diffractive medium, that have recently attracted a lot of attention in nonlinear optics and give rise to the so-called parametric solitons. To check if the predicted solitons are close to exact solutions of the coupled Korteweg-de Vries equations, we perform direct numerical simulations, with initial conditions suggested by the above-mentioned parametric-soliton solution to the asymptotic equations. Since the latter is known only in a numerical form, we use for them a recently developed analytical variational approximation. As a result, we observe very long-lived steadily propagating wave packets generated by these initial conditions. Thus we find a physical system that may allow experimental observation of propagating parametric solitons, while in nonlinear optics they are observed only as spatial solitons.

1. Introduction

Envelope solitons supported by parametric interactions [1], i.e., quadratic nonlinear interactions coupling three underlying harmonic waves, or only two waves in the case of second-harmonic generation (SHG) in diffractive media, have recently attracted a lot of attention in nonlinear optics [2-4], after being introduced long ago in the pioneering paper by Karamzin and Sukhorukov [5]. These solitons were observed experimentally in the form of stationary self-supported localized beams in a three- or two-dimensional bulk (in the former case, the beam is cylindrical) [6]. However, they have never been observed in the temporal domain, i.e., as propagating solitons. A principal difficulty is that it is virtually impossible to fabricate a sufficiently long optical fibre with a quadratic nonlinearity, which would be a natural medium to support a temporal-domain soliton. In this work we aim to demonstrate that such parametric solitons in the time-domain can be easily launched in other physical systems, where they may be interesting objects in their own right.

The model which will be considered in this work is a system of two coupled Korteweg-de Vries (KdV) equations,

$$u_t + u_{xxx} - 6uu_x = v_x, \tag{1.1}$$

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$$v_t + \delta v_{xxx} + \eta v_x - 6\mu v v_x = \kappa u_x, \tag{1.2}$$

where η is the long wave phase speed mismatch between the two components, δ and μ are relative dispersion and nonlinear coefficients, respectively, and the coupling constant κ is normalized to unity in the sequel.

This system appears in a theory describing near-resonant interaction of internal waves in a stratified fluid [7], and also for the analogous near-resonant interaction of planetary waves [8]. The relative dispersion δ plays a crucial role. In principle, it may be either positive or negative. When $\delta < 0$, there is a gap in the spectrum of the phase speeds of the linearized system, in which envelope gap-solitons, very different from the classical KdV solitary waves, may stably exist [9]. It is also noteworthy that for the case of negative relative dispersion ($\delta < 0$) but with opposite sign of the linear coupling (i.e. $\kappa = -1$) intersection of the dispersion curves of the corresponding uncoupled KdV subsystems may give rise, instead of the above-mentioned gap in the phase-speed spectrum, to a gap in the wavenumber spectrum, which actually implies a novel fundamental instability in the coupled-wave system [10]. If, on the other hand, the relative dispersion is positive (which is the more typical case for applications to internal waves), the gap is not opened, and strictly speaking solitons do not exist at all in this model, since, whatever the velocity of a wave packet, one can always find a linear wave with the same phase velocity, which can be expected to give rise to the emission of these linear waves by the nonlinear wave packet. Nevertheless, we will show below that this model can support, through the SHG process, very long-lived approximate envelope solitons of a new type. They can exist as stationary solutions within the framework of an asymptotic expression for the envelope amplitudes, very slowly decaying into linear waves.

Before proceeding to analysis of the SHG resonance in (1.1) and (1.2), it is relevant to note that the same system can generate a sort of a parametric soliton in a simpler way, viz., through the interaction between a fundamental and zero (rather than second) harmonic. Indeed, let us take, for simplicity, a purely symmetric system with $\delta = \mu = 1$, $\eta = 0$ and we recall that $\kappa = 1$. In this case, the spectrum of the linearized system is very simple, $c = \pm 1 - k^2$, where c is the phase velocity, k is the wavenumber, and the alternate signs correspond to two different branches of the dispersion curve. The corresponding group velocity as a function of k is

$$v = \pm 1 - 3k^2. \tag{1.3}$$

If we now start the analysis with a harmonic with wavenumber k, its quadratic self-interaction generates the zero harmonic. For the latter one, the group velocity is ± 1 (see (1.3)). Picking up the lower sign, one notices that it coincides with the group velocity of the fundamental harmonic belonging to the other branch at $k^2 = \frac{2}{3}$. Thus, the fundamental harmonic at this wavenumber can be resonantly coupled to the zero harmonic ("mean field"). After a straightforward analysis, the system of asymptotic equations for the slowly varying complex envelope U_1 of the fundamental and the real-valued mean field U_0 take the form

$$\mathrm{i}(U_1)_t + \tfrac{1}{2}(U_1)_{xx} + U_0 U_1 = 0, \tag{1.4}$$

$$(U_0)_t + \sigma(|U_1|^2)_x = 0, \tag{1.5}$$

where the variables x and t are scaled versions of the original variables in a frame moving with the common group velocity, and σ is a constant. These equations are exactly integrable and support an obvious soliton solution similar to the soliton of the nonlinear Schrödinger equation [11]. In this work we will consider within the framework of the general system (1.1) and (1.2), another way to support a parametric soliton, which is based on the resonance between the fundamental and second harmonics (SH). In the lowest-order approximation, this resonance demands that the phase and group velocities of the fundamental and second harmonics coincide for certain wavenumbers, which proves to be possible in general, but incidentally is not possible in the above-mentioned symmetric system.

The rest of the paper is organized as follows. In Section 2, we analyze the dispersion curves for the general system and find the wavenumbers where this double resonance can occur. In Section 3, which is the technical core of this work, we derive by means of a standard multiscale technique, an asymptotic system for the fundamental and secondary envelopes (similar to (1.4) and (1.5) for the resonance with the zero harmonic). In their eventual form, these equations coincide with "canonical" equations that give rise to parametric solitons in nonlinear optics [2,3]. Therefore, we can expect existence of the known one-parametric family of stationary soliton solutions [3,4]. However, as we mentioned above, the coupled KdV equations with dispersions of the same sign (i.e. $\delta > 0$), which is the case under consideration, cannot strictly speaking have exact soliton solutions because there is no gap in the spectrum of the linear waves. Usually, the solitons obtained as exact solutions to the asymptotic amplitude equations exist for very long times as almost exact solitons to the underlying system (here the coupled KdV equations (1.1) and (1.2)), suffering from an extremely slow radiative decay (see, e.g., Ref. [9]). In order to understand the meaning of these solutions, in Section 4 we report results of direct numerical simulations of these coupled KdV equations, (1.1) and (1.2). As an initial state, we take wave forms suggested by the soliton solutions of the asymptotic equations. The result is that, over fairly long times, these initial wave forms undergo practically no evolution except for translation at a constant velocity (an arbitrary localized wave packet completely decays on the same time scale).

An important practical problem in doing these simulations is representing the soliton solutions to the asymptotic amplitude equations, because they are known in an exact analytical form for only a single parameter value [5], while at all other parameter values they were found numerically. However, in Ref. [4] it was demonstrated that one can approximate the general soliton solutions to a very reasonable accuracy by analytical expressions based on a variational approximation. This is exactly the form in which we take solutions to the asymptotic equations in order to generate the initial states for our direct simulations. The final result is that we indeed obtain "almost genuine" envelope solitons governed by the linearly coupled KdV equations (1.1) and (1.2). Concluding remarks are collected in Section 5.

2. Dispersion curves and the double resonance

The condition for second-harmonic resonance (i.e. equality of the phase-velocities of the fundamental and second harmonics) requires that

$$2\omega(k_r) = \omega(2k_r), \tag{2.1}$$

where k_r is the resonant wavenumber. Here $\omega(k)$ is the dispersion relation, generally multi-valued with two or more branches, which may depend on several parameters. In order to guarantee a sufficiently strong interaction the group velocities must also coincide so that

$$\frac{\partial\omega}{\partial k}(k_r) = \frac{\partial\omega}{\partial k}(2k_r). \tag{2.2}$$

In general, the two resonance conditions impose a certain relation between the parameters. For the coupled Korteweg–de Vries equations (1.1) and (1.2) the linear dispersion relation is

$$(\omega + k^3)(\omega + \delta k^3 - \eta k) - k^2 = 0. \tag{2.3}$$

The two solution branches are given by

$$2\omega = -(1+\delta)k^3 + \eta k \pm k\sqrt{[(1-\delta)k^2 + \eta]^2 + 4}, \tag{2.4}$$

where we recall that here $\kappa = 1$ in the system (1.1) and (1.2). We shall assume that $\delta > 0$, so that each KdV component in the coupled system (1.1) and (1.2) has the same-signed dispersion. The resonance conditions (2.1) and (2.2) can be met if the first and second harmonics belong to the lower and upper branches respectively. We find that the resonant wavenumber k_r is given by

$$k_r = \sqrt{\frac{5}{8}\frac{\eta^2+4}{\eta(\delta-1)}}, \tag{2.5}$$

provided that the parameters satisfy the condition

$$\frac{3}{5}\frac{\delta+1}{|\delta-1|} = \sqrt{1 - \frac{16}{25}\frac{\eta^2}{\eta^2+4}}, \tag{2.6}$$

and we require that $\eta(\delta - 1) > 0$, which is possible only for a nonzero value of η, and for $\delta \neq 1$. A typical example of the dispersion curves is shown in Fig. 1 at the same parameter values which we will use in Section 4 for the numerical simulations.

3. Derivation of the amplitude equations

To investigate the dynamics of the interaction of a fundamental harmonic with the wavenumber k_r with

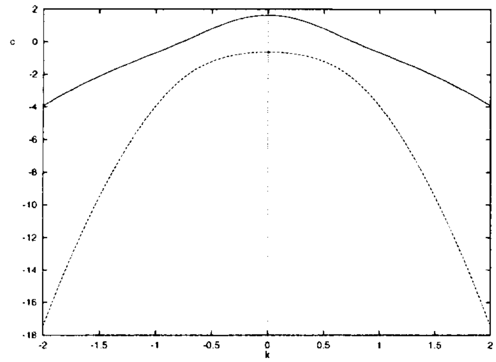

Fig. 1. Dispersion curves, c as a function of k, where $c = \omega/k$, for the parameter values $\mu = \eta = 1$ and the corresponding resonant value of the relative dispersion $\delta = 4.6$. The resonant wavenumber is $k_r = 0.9$.

its second harmonic with wavenumber $2k_r$ in the system (1.1) and (1.2), we will perform a multiscale perturbation analysis for the resonant waves. Thus, introducing a small parameter ϵ and setting $X = \epsilon x, T_1 = \epsilon t, T_2 = \epsilon^2 t$, we write

$$u = \epsilon^2 A(X,T)e^{i\theta_1} + \epsilon^2 B(X,T)e^{i\theta_2} + \epsilon^3 u_3 + \epsilon^4 u_4 + \text{c.c.},$$
$$v = \epsilon^2 \xi_1 A(X,T)e^{i\theta_1} + \epsilon^2 \xi_2 B(X,T)e^{i\theta_2} + \epsilon^3 v_3 + \epsilon^4 v_4 + \text{c.c.}, \qquad (3.1)$$

where

$$\theta_1 = k_1 x - \omega_1 t, \quad \theta_2 = k_2 x - \omega_2 t,$$

with the carrier-wavenumbers k_1, k_2 and $\omega_i = \omega(k_i)$, $i = 1,2$ and $T = (T_1, T_2, \ldots)$. Recall that we choose the lower (upper) branch, respectively, for $i = 1$ and $i = 2$. Here $k_1 \equiv k_r + \epsilon\Delta k$ and $k_2 \equiv 2(k_r + \epsilon\Delta k) = 2k_1$, where Δk allows a small deviation from the exact resonance-condition (2.2). The system (1.1) and (1.2) can be written as

$$\hat{\mathcal{L}}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{N}(u,v), \qquad (3.2)$$

where the linear operator matrix is

$$\hat{\mathcal{L}}_{uu} = \partial_t + \partial_{xxx}, \quad \hat{\mathcal{L}}_{uv} = -\partial_x,$$
$$\hat{\mathcal{L}}_{vu} = -\partial_x, \quad \hat{\mathcal{L}}_{vv} = \partial_t + \eta\partial_x + \delta\partial_{xxx}$$

and the nonlinear right-hand-side is

$$\mathcal{N}_1(u,v) = 6uu_x, \quad \mathcal{N}_2(u,v) = 6\mu v v_x.$$

Thus, at the lowest order, $O(\epsilon^2)$, (3.2) is trivially satisfied by the linear dispersion relation for each mode, yielding

$$\xi_1(k_1) = -\frac{\omega_1 + k_1^3}{k_1}, \quad \xi_2(k_2) = -\frac{\omega_2 + k_2^3}{k_2}.$$

At the next order, $O(\epsilon^3)$, linearity again allows us to separate the two modes, and we obtain for the first harmonic component

$$\hat{\mathcal{L}}_{uu}u_3 + \hat{\mathcal{L}}_{uv}v_3$$
$$= \left(-A_{T_1} + 3k_1^2 A_X + \left(\xi_1 A_X + k_1 \frac{d\xi}{dk} A_X\right)\right)e^{i\theta_1}$$
$$= \left(-A_{T_1} - \frac{\partial\omega_1}{\partial k}(k_1)A_X\right)e^{i\theta_1}, \qquad (3.3)$$

and

$$\hat{\mathcal{L}}_{vu}u_3 + \hat{\mathcal{L}}_{vv}v_3 = \left(-\xi_1 A_{T_1} - \xi_1\frac{\partial\omega_1}{\partial k}(k_1)A_X\right)e^{i\theta_1}. \qquad (3.4)$$

Note that here we are regarding $\xi_i(k_i)$ as an operator, with $k_i \to k_i - i\epsilon\partial_X$, for reasons which we shall discuss below. There are similar expressions for the second harmonic on the right-hand side. Since the eigenfunctions of the homogeneous adjoint operator are proportional to $e^{i\theta_1}$ and $e^{i\theta_2}$, we can set $u_3 = v_3 = 0$, to obtain

$$A_{T_1} + \frac{\partial\omega_1}{\partial k}(k_r)A_X = -\epsilon\frac{\partial^2\omega_1}{\partial k^2}(k_r)\Delta k A_X + O(\epsilon^2), \qquad (3.5)$$

$$B_{T_1} + \frac{\partial\omega_2}{\partial k}(2k_r)B_X = -\epsilon\frac{\partial^2\omega_2}{\partial k^2}(2k_r)2\Delta k B_X + O(\epsilon^2). \qquad (3.6)$$

The corrections on the right-hand side will be absorbed into higher-order terms of the expansion. We note that the ansatz (3.1), with $\xi_i(k_i)$ interpreted as an operator as described above ensures vanishing of the right-hand sides of (3.3) and (3.4) at the leading order, and so $u_3 = v_3 = 0$. Here, of course, the form (3.1) implicitly includes an $O(\epsilon^3)$ correction due to $\xi_i(k_i)$. Next, the form of (3.5) and (3.6) implies that we make the transformation

$$X' = X - v_G T_1,$$

where

$$v_G \equiv \frac{\partial\omega_1}{\partial k}(k_r) = \frac{\partial\omega_2}{\partial k}(2k_r),$$

because at the leading order, A and B are functions of X' and T_2. Henceforth we omit the primes on X'.

At the next order, $O(\epsilon^4)$, separation of the two waves is no longer possible because of the nonlinear terms. Thus, we have to take into account the nonlinearity-induced phase-mismatch

$$\begin{aligned}\chi &\equiv \theta_2 - 2\theta_1 \\ &\approx -(\Delta k)^2\left(2\frac{\partial^2\omega_2}{\partial k^2}(2k_r) - \frac{\partial^2\omega_1}{\partial k^2}(k_r)\right)T_2.\end{aligned} \tag{3.7}$$

Using the explicit form of ξ as described above, one obtains for the resonant components,

$$\begin{aligned}&\hat{\mathcal{L}}_{uu}u_4 + \hat{\mathcal{L}}_{uv}v_4 \\ &= \left(-A_{T_2} + \frac{\mathrm{i}}{2}\frac{\partial^2\omega_1}{\partial k^2}(k_r)A_{XX}\right. \\ &\quad \left. - \frac{\partial^2\omega_1}{\partial k^2}(k_r)\Delta k A_X\right)\mathrm{e}^{\mathrm{i}\theta_1} \\ &\quad + \left(-B_{T_2} + \frac{\mathrm{i}}{2}\frac{\partial^2\omega_2}{\partial k^2}(2k_r)B_{XX}\right. \\ &\quad \left. - \frac{\partial^2\omega_2}{\partial k^2}(2k_r)2\Delta k B_X\right)\mathrm{e}^{\mathrm{i}\theta_2} \\ &\quad + 6\mathrm{i}k_r A^2\mathrm{e}^{\mathrm{i}\theta_2 - \mathrm{i}\chi} + 6\mathrm{i}k_r A^* B\mathrm{e}^{\mathrm{i}\theta_1 + \mathrm{i}\chi},\end{aligned} \tag{3.8}$$

where the asterisk stands for the complex conjugate. Analogously, one obtains the second equation

$$\begin{aligned}&\hat{\mathcal{L}}_{vu}u_4 + \hat{\mathcal{L}}_{vv}v_4 \\ &= \left(-\xi_1(k_r)A_{T_2} + \xi_1(k_r)\frac{\mathrm{i}}{2}\frac{\partial^2\omega_1}{\partial k^2}(k_r)A_{XX}\right. \\ &\quad \left. - \xi_1\frac{\partial^2\omega_1}{\partial k^2}(k_r)\Delta k A_X\right)\mathrm{e}^{\mathrm{i}\theta_1} \\ &\quad + \left(-\xi_2(2k_r)B_{T_2} + \xi_2(2k_r)\frac{\mathrm{i}}{2}\frac{\partial^2\omega_2}{\partial k^2}(2k_r)B_{XX}\right. \\ &\quad \left. - \xi_2\frac{\partial^2\omega_2}{\partial k^2}(2k_r)2\Delta k B_X\right)\mathrm{e}^{\mathrm{i}\theta_2} \\ &\quad + 6\mu\mathrm{i}k_r\xi_1^2(k_r)A^2\mathrm{e}^{\mathrm{i}\theta_2 - \mathrm{i}\chi} \\ &\quad + 6\mu\mathrm{i}k_r\xi_1(k_r)\xi_2(2k_r)A^* B\mathrm{e}^{\mathrm{i}\theta_1 + \mathrm{i}\chi}.\end{aligned} \tag{3.9}$$

The compatibility conditions for solution of the coupled system of Eqs. (3.8) and (3.9) yields a pair of evolution equations for the amplitudes A and B,

$$\begin{aligned}&\mathrm{i}A_{T_2} + \Phi_1 A_{XX} + 2\mathrm{i}\Phi_1\Delta k A_X + N_1 A^* B\mathrm{e}^{\mathrm{i}\chi} = 0, \\ &\mathrm{i}B_{T_2} + \Phi_2 B_{XX} + 4\mathrm{i}\Phi_2\Delta k B_X + N_2 A^2\mathrm{e}^{-\mathrm{i}\chi} = 0,\end{aligned} \tag{3.10}$$

where the coefficients are defined by

$$\Phi_1 = \frac{1}{2}\frac{\partial^2\omega_1}{\partial k^2}(k_r), \tag{3.11}$$

$$\Phi_2 = \frac{1}{2}\frac{\partial^2\omega_2}{\partial k^2}(2k_r), \tag{3.12}$$

$$N_1 = 6k_r\frac{1 + \mu\xi_1^2\xi_2}{1 + \xi_1^2}, \tag{3.13}$$

$$N_2 = 6k_r\frac{1 + \mu\xi_1^2\xi_2}{1 + \xi_2^2}. \tag{3.14}$$

The phase-mismatch of the nonlinear terms can be eliminated by the transformation

$$A = A'\exp(-\mathrm{i}\Delta k X - \mathrm{i}\sigma_1 T_2), \tag{3.15}$$

$$B = B'\exp(-2\mathrm{i}\Delta k X - \mathrm{i}\sigma_2 T_2). \tag{3.16}$$

Substituting this transformation into the system (3.10), the phase-mismatch can be removed by choosing

$$2\sigma_1 - \sigma_2 = (4\Phi_2 - 2\Phi_1)(\Delta k)^2. \tag{3.17}$$

Finally, we obtain, omitting the primes,

$$\begin{aligned}&\mathrm{i}A_{T_2} + \Phi_1 A_{XX} + S_1 A + N_1 A^* B = 0, \\ &\mathrm{i}B_{T_2} + \Phi_2 B_{XX} + S_2 B + N_2 A^2 = 0,\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}&S_1 \equiv \sigma_1 + \Phi_1(\Delta k)^2, \\ &S_2 \equiv \sigma_2 + 4\Phi_2(\Delta k)^2 = 2S_1.\end{aligned} \tag{3.19}$$

We now see that we can set $S_1 = S_2 = 0$ by choosing $\sigma_1 = -\Phi_1(\Delta k)^2, \sigma_2 = -4\Phi_2(\Delta k)^2$, which satisfies the constraint (3.17). It is readily seen that in terms of the new variables, the effective phases are $\theta_1 = \frac{1}{2}\theta_2 = k_r - \omega(k_r)t$ and are evaluated at precisely the resonant wavenumber, satisfying both resonance conditions (2.1) and (2.2). However, it is useful to retain S_1 and S_2 in (3.18), which is equivalent to adding the terms $-S_1T_2$ and $-S_2T_2$ to the phases θ_1 and θ_2, respectively. Such terms in effect replace the resonant frequencies $\omega_i(k_r)$ with $\omega_i(k_r)+\epsilon^2 S_i$, $i=1,2$. Thus, each S_i can be regarded as an $O(\epsilon^2)$ frequency detuning term, for which the parameters in the coupled KdV equations (1.1) and (1.1) fail to satisfy (2.6) by terms of $O(\epsilon^2)$. Note that (3.19) implies that we can either satisfy the resonance condition (2.1) exactly, i.e. we can choose $\Delta k=0$ while introducing a frequency detuning via the remaining free parameter σ_1, or we can satisfy (2.2) exactly introducing the detuning via the wavenumber mismatch Δk. We will choose the frequency detuning and set $\Delta k=0$. It is important to mention that stationary-wave solutions of (3.18) only exist in the detuned case $S_1\neq 0$. In the exactly resonant case we can obtain oscillatory solitary wave solutions, for which the phase of the oscillations introduces a necessary term balancing the dispersion in the asymptotic limit.

To obtain stationary solitary wave solutions of (3.18), we must assume that $\Phi_1 S_1<0$ and $\Phi_2 S_2<0$. Then, using the transformations

$$A=\sqrt{\frac{S_1^2\Phi_2}{2N_1N_2\Phi_1}}A',\quad B=\tau\frac{|S_1|}{N_1}B',$$

$$X=\sqrt{\left|\frac{\Phi_1}{S_1}\right|}X',\quad T_2=\frac{1}{S_1}T',$$

where $\tau=\pm 1$ for $N_1N_2\Phi_1\Phi_2>0$ or <0 respectively, and omitting the primes, the system (3.18) is cast into the final form,

$$\begin{aligned}&\mathrm{i}A_T+A_{XX}-A+A^*B=0,\\ &\tfrac{1}{2}\mathrm{i}\zeta B_T+B_{XX}-\zeta B+\tfrac{1}{2}A^2=0,\end{aligned}\qquad(3.20)$$

where

$$\zeta=2\frac{\Phi_1}{\Phi_2}.$$

For the system (1.1) and (1.2) we have $\Phi_1<0$ and $\Phi_2<0$ and so $\zeta>0$.

In the stationary case a particular solitary wave solution of (3.20) for $\zeta=1$ was found in Ref. [5] to be

$$A_s(X)=\pm\frac{3\sqrt{2}}{2\cosh^2(x/2)},\quad B_s(X)=\frac{3}{2\cosh^2(x/2)}.$$

For $\zeta\gg 1$ localized solutions of the system (3.20) can be obtained using an asymptotic expansion around $1/\zeta$ (see, e.g., Ref. [2]). For arbitrary $\zeta\neq 1$, Steblina et al. [4] have found good agreement of the numerically obtained solution with a Gaussian ansatz, whose parameters were determined by means of a variational approximation. We will use their result for the ansatz

$$A=a\mathrm{e}^{-\rho X^2},\quad B=b\mathrm{e}^{-\gamma X^2},\qquad(3.21)$$

where the parameters were found to be

$$a^2=\frac{(\rho+1)(2\rho+\gamma)(\gamma+\zeta)}{\sqrt{\rho\gamma}},$$

$$b=\frac{(\rho+1)\sqrt{2\rho+\gamma}}{\sqrt{2\rho}},\quad \gamma=\frac{4\rho^2}{1-\rho}.$$

The parameter ρ is determined as a real positive root of the cubic equation

$$20\rho^3+(4-3\zeta)\rho^2+4\zeta\rho=\zeta.\qquad(3.22)$$

In the parameter range of interest, there is only one real root, which is positive, i.e. ρ is uniquely determined.

4. Numerical simulations

In this section we address the question of the validity of our theoretical results; to be specific we investigate whether the coupled system of equations (3.20) and their approximate solutions (3.21) is an appropriate representation for an SHG resonance in the full coupled KdV equations (1.1) and (1.1). To show this, we integrate the coupled KdV equations numerically using the approximate solution (3.21) as an initial condition, suitably transformed back into the original variables of the KdV system. We use a pseudo-spectral code, where the linear terms are treated with a semi-implicit Crank–Nicholson scheme, and the nonlinear terms with an explicit leapfrog. There are only two free parameters, μ and η, since δ is given through (2.6) as

$$\delta=-\frac{3+\sqrt{(9\eta^2+100)/(\eta^2+4)}}{3-\sqrt{(9\eta^2+100)/(\eta^2+4)}},\qquad(4.1)$$

which implies $\delta>4$. Note that $\mu=0$ is also a possible case, i.e. one equation of the system (1.1) and (1.1) can be linear. It is pertinent to mention that when the parameters take $O(1)$-values the extension of the

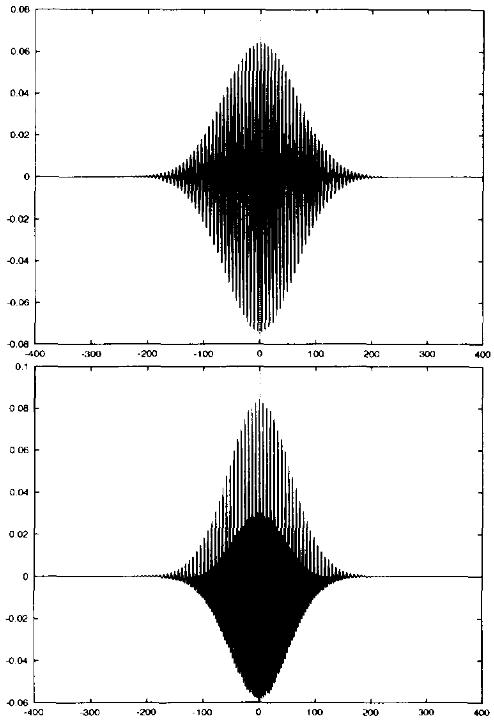

Fig. 2. Left panel: initial condition, right panel: evolved wavepacket at $t = 20$. The upper pictures refer to u, the lower to v. The parameters are chosen as $\mu = \eta = 1$, and the corresponding $\delta = 4.6$. The resonant wavenumber is $k_r = 0.9$.

wavepacket is very large, whereas the amplitude is very small to satisfy $\epsilon\Delta k \ll k_r$. Fig. 2 shows the initial condition determined as described above, and its evolution. During the evolution, the wavepacket readjusts slightly to get closer to an exact solution. It has been checked that over this time an arbitrary initial wave packet not satisfying Eqs. (3.20) or the system (1.1) and (1.1) strongly disperses (see Fig. 3). Similar numerical experiments have been performed at other parameter values, all revealing that the soliton solution to (3.20) approximates very closely a solution of the original coupled KdV system.

5. Conclusion

In this work we have analyzed a resonance between the fundamental and second harmonics in the linearly coupled KdV system (1.1) and (1.2), which is a general model of resonant interaction of internal waves in stratified fluids. We have shown that the resonance may indeed take place. By means of a multiscale expansion, we have derived asymptotic envelope amplitude equations which coincide with the standard second-harmonic generation equations in nonlinear optics. We have then performed direct simulations of the coupled KdV equation, with initial states corresponding to the parametric solitons of the asymptotic equations. The latter were taken in the form of a Gaussian ansatz provided by the variational approximation. The result is that these initial states give rise to almost exact envelope solitons of the coupled KdV equation. This provides the possibility to directly produce parametric solitons in the temporal domain, which is not possible in available nonlinear optical media.

Finally, we note that in realising the physical applicability of the theoretical results obtained here, we should recall that the coupled KdV equations (1.1) and (1.2) have been derived under a long-wave hypothesis. Eqs. (1.1) and (1.2) are in nondimensional form, but, for example their derivation in the internal wave context requires that $k_d h \ll 1$ where k_d is the dimensional wavenumber, and h is a length scale for the vertical stratification (e.g. the total fluid depth).

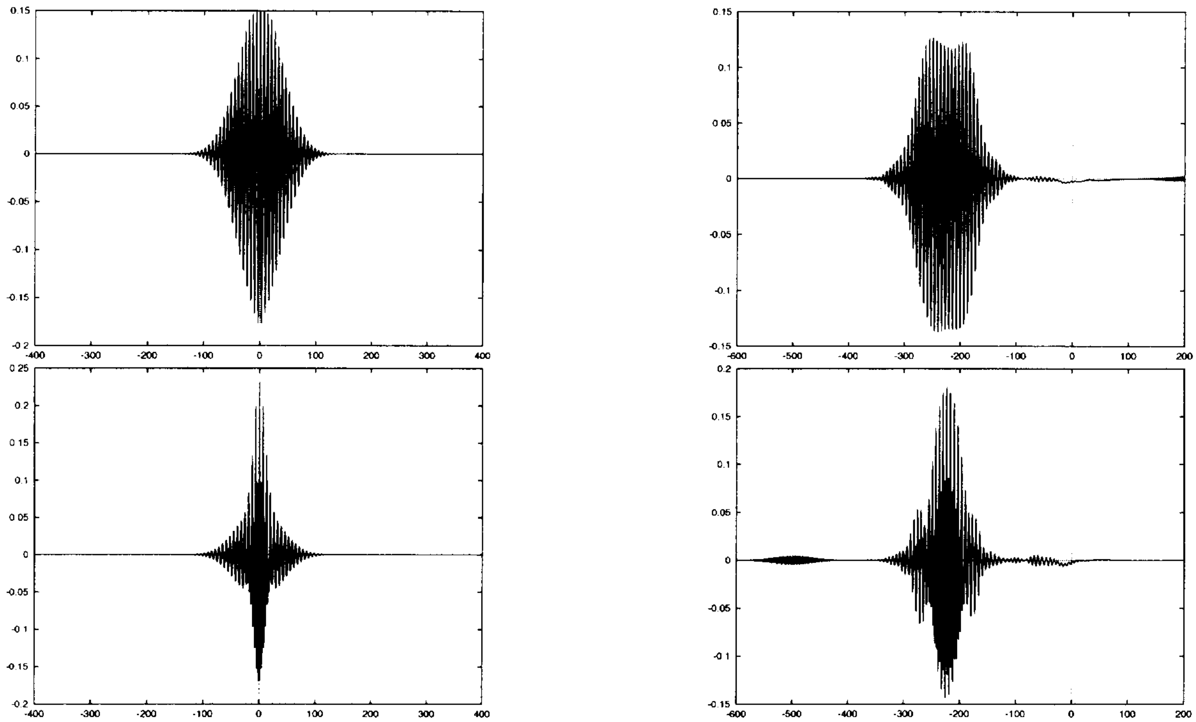

Fig. 3. Left panel: initial condition, right panel: evolved wavepacket at $t = 20$. The upper pictures refer to u, the lower to v. Same parameters as in Fig. 2, but with a disturbed ρ.

The condition for second-harmonic resonance (2.5) requires that the nondimensional wavenumber k_r be of order unity, which implies that the dimensional resonant wavenumber k_{rd} should be finite, but $k_{rd} \ll 1$. This condition can readily be achieved in oceanic, atmospheric or laboratory situations.

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