Buildings and Hecke algebras

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Abstract

In this paper we establish a strong connection between buildings and Hecke algebras by studying two algebras of averaging operators on buildings. To each locally finite regular building we associate a natural algebra $\mathcal{B}$ of chamber set averaging operators, and when the building is affine we also define an algebra $\mathcal{A}$ of vertex set averaging operators. We show that for appropriately parametrised Hecke algebras $\mathcal{H}$ and $\tilde{\mathcal{H}}$, the algebra $\mathcal{B}$ is isomorphic to $\mathcal{H}$ and the algebra $\mathcal{A}$ is isomorphic to the centre of $\tilde{\mathcal{H}}$. On the one hand these results give a thorough understanding of the algebras $\mathcal{A}$ and $\mathcal{B}$. On the other hand they give a nice geometric and combinatorial understanding of Hecke algebras, and in particular of the Macdonald spherical functions and the centre of affine Hecke algebras. Our results also produce interesting examples of association schemes and polynomial hypergroups. In later work we use the results here to study random walks on affine buildings.
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Introduction

Let $G = PGL(n + 1, F)$ where F is a local field, and let $K = PGL(n + 1, \mathcal{O})$, where $\mathcal{O}$ is the valuation ring of F. The space of bi-K-invariant compactly supported functions on G forms a commutative convolution algebra (see [18, Corollary 3.3.7], for example). Associated to G there is a building $\mathcal{X}$ (of type $\tilde{A}_n$), and the above algebra is isomorphic to

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an algebra $\mathcal{A}$ of averaging operators defined on the space of all functions $G/K \to \mathbb{C}$. In [7] it was shown that these averaging operators may be defined in a natural way using only the geometric and combinatorial properties of $\mathcal{X}$, hence removing the group G entirely from the discussion. For example, in the case $n = 1$, $\mathcal{X}$ is a homogeneous tree and $\mathcal{A}$ is the algebra generated by the operator A_1, where for each vertex, $(A_1 f)(x)$ is given by the average value of f over the neighbours of x.

In [7], using this geometric approach, Cartwright showed that $\mathcal{A}$ is a commutative algebra, and that the algebra homomorphisms $h : \mathcal{A} \to \mathbb{C}$ can be expressed in terms of the classical Hall–Littlewood polynomials of [19, III, §2]. It was not assumed that $\mathcal{X}$ was constructed from a group G (although there always is such a group when $n \geqslant 3$). Although not entirely realised in [7], as a consequence of our work here we see that the commutativity of the algebra $\mathcal{A}$ and the description of the algebra homomorphisms $h : \mathcal{A} \to \mathbb{C}$ follow from the fact that $\mathcal{A}$ is isomorphic to the centre of an appropriately parametrised affine Hecke algebra.

One objective of this paper is to put the above observations into a more general setting. To do so we will demonstrate a close connection between buildings and Hecke algebras through the 'combinatorial' study of two algebras of averaging operators associated to buildings. Apart from establishing these important connections, our results also have applications to the theory of random walks on buildings, and provides interesting examples of association schemes and polynomial hypergroups. We will elaborate on the random walk applications in a later paper, where we generalise the results in [9]. Let us briefly describe the results we give here.

0.1. Regularity and parameter systems

To begin with we consider buildings as certain *chamber systems*. Thus a *building* $\mathcal{X}$ is a set $\mathcal{C}$ of *chambers* with an associated Coxeter system (W, S) and a *W-distance function* $\delta : \mathcal{C} \times \mathcal{C} \to W$. For each $c \in \mathcal{C}$ and $w \in W$, define $\mathcal{C}_w(c) = \{d \in \mathcal{C} \mid \delta(c, d) = w\}$. An important assumption we make throughout is that $\mathcal{X}$ is *regular*, by which we mean that for each $s \in S$, $|\mathcal{C}_s(c)| = |\mathcal{C}_s(d)|$ for all $c, d \in \mathcal{C}$. In a regular building we write $q_s = |\mathcal{C}_s(c)|$, and we call the set $\{q_s\}_{s \in S}$ the *parameter system* of the building. In Proposition 2.1 we show that regularity implies the stronger result that $|\mathcal{C}_w(c)| = |\mathcal{C}_w(d)|$ for all $c, d \in \mathcal{C}$ and $w \in W$, and as such we define $q_w = |\mathcal{C}_w(c)|$. In Theorem 2.4 we show that all *thick* buildings with no rank 2 residues of type $\tilde{A}_1$ are regular, generalising [30, Proposition 3.4.2]. This shows that regularity is a very weak hypothesis.

0.2. The algebra $\mathcal{B}$

Let $\mathcal{X}$ be any (locally finite) regular building. For each $w \in W$ we define an operator B_w, acting on the space of functions $f : \mathcal{C} \to \mathbb{C}$, by

$$(B_w f)(c) = \frac{1}{q_w} \sum_{d \in \mathcal{C}_w(c)} f(d) \quad \text{for all } c \in \mathcal{C}. \tag{0.1}$$

We call these operators *chamber set averaging operators*, and write $\mathcal{B}$ for the linear span of $\{B_w\}_{w \in W}$ over $\mathbb{C}$. Our main result here is Theorem 3.10, where we show that $\mathcal{B}$ is isomorphic to a suitably parametrised Hecke algebra (the parametrisation depending on the parameter system of the building). This result is a generalisation of results in [11, Chapter 6] where an analogous algebra is studied under the assumption that there is a group G (of label preserving simplicial complex automorphisms) acting *strongly transitively* on the building. We note that it is simple to see that all buildings admitting such a group are regular. However not all regular buildings admit such a group (see [28] for the $\tilde{A}_2$, $\tilde{C}_2$ and $\tilde{G}_2$ buildings). Since we only assume regularity, our results are more general. We require this additional generality to prove the more difficult results concerning the algebra $\mathcal{A}$ of *vertex set operators* in their full generality. We note that some of our results in Section 3 are proved in [34] using the quite different language of *association schemes*.

0.3. The algebra $\mathcal{A}$

The latter part of this paper is mainly devoted to the study of an algebra $\mathcal{A}$ of *vertex set averaging operators* associated to locally finite regular *affine* buildings, and the connections with *affine Hecke algebras*. We consider the study of $\mathcal{A}$ to be the main contribution of this paper. It is a considerably more complicated object than the algebra $\mathcal{B}$. Let us give a (simplified) description of this algebra.

We now consider a building $\mathcal{X}$ as a certain *simplicial complex* [5, Chapter IV], and we write V for the *vertex set* of $\mathcal{X}$. In Definition 4.17 we define a subset $V_P \subseteq V$ of *good* vertices, which, for the sake of this simplified description, can be thought of as the *special vertices* of $\mathcal{X}$.

To each (locally finite regular) affine building we associate a root system R. Let P be the *coweight* lattice of R and write P^+ for a set of dominant coweights. For each $x \in V_P$ and $\lambda \in P^+$ we define (Definition 5.5) sets $V_\lambda(x)$ in such a way that $\{V_\lambda(x)\}_{\lambda \in P^+}$ forms a partition of V_P. In Theorem 5.15 we show that regularity implies that the cardinalities $|V_\lambda(x)|$, $\lambda \in P^+$, are independent of the particular $x \in V_P$, and as such we write $N_\lambda = |V_\lambda(x)|$. For each $\lambda \in P^+$ we define an averaging operator A_λ, acting on the space of functions $f : V_P \to \mathbb{C}$, by

$$(A_\lambda f)(x) = \frac{1}{N_\lambda} \sum_{y \in V_\lambda(x)} f(y) \quad \text{for all } x \in V_P. \tag{0.2}$$

These operators specialise to the operators studied in [7] when $\mathcal{X}$ is an $\tilde{A}_n$ building.

We write $\mathcal{A}$ for the linear span of $\{A_\lambda\}_{\lambda \in P^+}$ over $\mathbb{C}$. Our first main result concerning $\mathcal{A}$ is Theorem 5.24, where we show that $\mathcal{A}$ is a commutative algebra. We stress that we only assume regularity, and do not require the existence of groups or BN-pairs associated with the building. This puts our results in a very general setting.

To get a feel for the above definitions in a special case, let $\mathcal{X}$ be a homogeneous tree with degree $q + 1$, which is a special case of an $\tilde{A}_1$ building. Let $R = \{\alpha, -\alpha\}$, where $\alpha = e_1 - e_2$, be the usual root system of type A_1 in the vector space $E = \{x \in \mathbb{R}^2 \mid \langle x, e_1 + e_2 \rangle = 0\}$. Taking $\{\alpha\}$ as a base of R we have $P^+ = \{\frac{k}{2}\alpha\}_{k \in \mathbb{N}}$ where $\mathbb{N} = \{0, 1, \ldots\}$.

Here $V_P = V$, the set of all vertices, and, writing $V_k(x)$ in place of $V_\lambda(x)$ when $\lambda = \frac{k}{2}\alpha$ with $k \geqslant 0$, we see that $V_k(x)$ is the set of vertices of distance k from x. Thus we compute $N_k = 1$ if $k = 0$ and $(q + 1)q^{k-1}$ if $k \geqslant 1$. The algebra $\mathcal{A}$ in this case is a well-known object (see [10], for example). It is generated by A_1, where $(A_1 f)(x) = \frac{1}{q+1} \sum_{y \sim x} f(y)$ and the sum is over the neighbours of x.

Our results on the algebra $\mathcal{A}$ give interesting examples of *association schemes* (see Remarks 4.19 and 5.25) which generalises the well-known construction of association schemes from *infinite distance regular graphs*.

Remark 0.1. To increase the readability of this paper we have restricted our attention to *irreducible* affine buildings. Everything we do here goes through perfectly well for reducible affine buildings too, and the details will be given elsewhere. Put briefly, when $\mathcal{X}$ is a reducible building, it has a natural description as a *polysimplicial complex*, and by associating a reducible root system to $\mathcal{X}$ we can define the algebra $\mathcal{A}$ as in the irreducible case. It turns out that $\mathcal{X}$ decomposes (essentially uniquely) into the Cartesian product of certain *irreducible components* $\{\mathcal{X}_j\}_{j=1}^k$, each of which is an irreducible building. The results of this paper can be used on each irreducible component $\mathcal{X}_j$, thus obtaining a family $\{\mathcal{A}_j\}_{j=1}^k$ of algebras. It turns out that $\mathcal{A} \cong \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$, where $\times$ is *direct product*.

0.4. Connections with affine Hecke algebras

The main result of this paper is Theorem 6.16, where we considerably strengthen the commutativity result of Theorem 5.24 by showing that $\mathcal{A}$ is isomorphic to the centre of an appropriately parametrised *affine Hecke algebra* (the parametrisation depending on the parameters of the building). Let us briefly describe this important isomorphism. Let $\tilde{\mathcal{H}}$ be an *affine Hecke algebra*, and write $Z(\tilde{\mathcal{H}})$ for the centre of $\tilde{\mathcal{H}}$. It is well known that $Z(\tilde{\mathcal{H}})$ equals $\mathbb{C}[P]^{W_0}$, the algebra of W_0-invariant elements of the group algebra of P (here P is considered as a multiplicative group in exponential notation $\lambda \leftrightarrow x^\lambda$). For $\lambda \in P^+$ let $P_\lambda(x)$ denote the *Macdonald spherical function*. This is a special element of $\mathbb{C}[P]^{W_0}$ which arises naturally in connection with the *Satake isomorphism*. The isomorphism in Theorem 6.16 is then $A_\lambda \mapsto P_\lambda$.

This isomorphism serves two purposes. Firstly it gives us an essentially complete understanding the algebra $\mathcal{A}$. For example, in Theorem 6.17 we use rather simple facts about the Macdonald spherical functions to show that $\mathcal{A}$ is generated by $\{A_{\lambda_i}\}_{i \in I_0}$ where $\{\lambda_i\}_{i \in I_0}$ is a set of fundamental coweights of R. On the other hand, since $\mathcal{A}$ is a purely combinatorial object, the above isomorphism gives a nice combinatorial description of $Z(\tilde{\mathcal{H}})$ when a suitable building exists. In particular the *structure constants* $c_{\lambda,\mu;\nu}$ that appear in

$$P_\lambda(x)P_\mu(x) = \sum_{\nu \in P^+} c_{\lambda,\mu;\nu} P_\nu(x) \quad \text{are} \quad c_{\lambda,\mu;\nu} = \frac{N_\nu}{N_\lambda N_\mu} \big| V_\lambda(x) \cap V_{\mu^*}(y) \big|,$$

for some $\mu^* \in P^+$ (depending only on μ in a simple way). This shows that (when a suitable building exists) $c_{\lambda,\mu;\nu} \geqslant 0$.

In Theorem 7.2 we extend this result by showing that the $c_{\lambda,\mu;\nu}$'s are (up to positive normalisation factors) polynomials in the variables $\{q_s - 1\}_{s \in S}$ with nonnegative integer

coefficients (even when no building exists). This generalises the main theorem in [24], where the corresponding result for the A_n case (where the $c_{\lambda,\mu;\nu}$'s are certain *Hall polynomials*) is proved. Thus we see how to construct a polynomial hypergroup from the structure constants $c_{\lambda,\mu;\nu}$ as in [3] (see also [17]).

Since the submission of this paper we have learnt that Theorem 7.2 has been proved independently by Schwer in [31], where a formula for $c_{\lambda,\mu;\nu}$ is given.

In later papers we will use our results here to give a description of the algebra homomorphisms $h: \mathcal{A} \to \mathbb{C}$ in terms of the Macdonald spherical functions of [18, Chapter 4]. We will also provide an integral formula for these algebra homomorphisms (over the *boundary of* $\mathcal{X}$), and use these results to study local limit theorems, central limit theorems and rate of escape theorems for *radial* random walks on affine buildings.

1. Coxeter groups, chamber systems and buildings

Let I be an index set, which we assume throughout is finite, and for $i, j \in I$ let $m_{i,j}$ be an integer or ∞ such that $m_{i,j} = m_{j,i} \geqslant 2$ for all $i \neq j$, and $m_{i,i} = 1$ for all $i \in I$. We call $M = (m_{i,j})_{i,j \in I}$ a *Coxeter matrix*. The *Coxeter group* of type M is the group

$$W = \big\langle \{s_i\}_{i \in I} \mid (s_i s_j)^{m_{i,j}} = 1 \text{ for all } i, j \in I \big\rangle, \tag{1.1}$$

where the relation $(s_i s_j)^{m_{i,j}} = 1$ is omitted if $m_{i,j} = \infty$. Let $S = \{s_i \mid i \in I\}$. For subsets $J \subset I$ we write W_J for the subgroup of W generated by $\{s_i\}_{i \in J}$. Given $w \in W$, we define the *length* $\ell(w)$ of w to be smallest $n \in \mathbb{N}$ such that $w = s_{i_1} \cdots s_{i_n}$, with $i_1, \ldots, i_n \in I$.

It will be useful on occasion to work with I^*, the free monoid on I. Thus elements of I^* are *words* $f = i_1 \cdots i_n$ where $i_1, \ldots, i_n \in I$, and we write $s_f = s_{i_1} \cdots s_{i_n} \in W$. Recall [29, Chapter 2, §1] that an *elementary homotopy* is an alteration from a word of the form $f_1 p(i, j) f_2$ to a word of the form $f_1 p(j, i) f_2$, where $p(i, j) = \cdots iji j$ ($m_{i,j}$ terms). We say that the words f and f' are *homotopic* if f can be transformed into f' by a sequence of elementary homotopies, in which case we write $f \sim f'$. A word f is said to be *reduced* if it is not homotopic to a word of the form $f_1 i i f_2$ for any $i \in I$. Thus $f = i_1 \cdots i_n \in I^*$ is reduced if and only if $s_f = s_{i_1} \cdots s_{i_n}$ is a reduced expression in W (that is, $\ell(s_f) = n$).

The *Coxeter graph* of W is the graph $D = D(W)$ with vertex set I, such that vertices $i, j \in I$ are joined by an edge if and only if $m_{i,j} \geqslant 3$. If $m_{i,j} \geqslant 4$ then the edge $\{i, j\}$ is labelled by $m_{i,j}$.

By an *automorphism of D* we mean a permutation of the vertex set of D that preserves adjacency and edge labels, that is, a permutation σ of I such that $m_{\sigma(i),\sigma(j)} = m_{i,j}$ for all $i, j \in I$. We write $\mathrm{Aut}(D)$ for the group of all automorphisms of D.

An automorphism σ of D induces a group automorphism of W, which we will also denote by σ, via the (well-defined) action

$$\sigma(w) = s_{\sigma(i_1)} \cdots s_{\sigma(i_n)} \tag{1.2}$$

whenever $s_{i_1} \cdots s_{i_n}$ is an expression for w. Note that $\ell(\sigma(w)) = \ell(w)$ for all $w \in W$.

Recall [29, p. 1] that a set $\mathcal{C}$ is a *chamber system over a set I* if each $i \in I$ determines a partition of $\mathcal{C}$, two elements in the same block of this partition being called *i-adjacent*. The elements of $\mathcal{C}$ are called *chambers*, and we write $c \sim_i d$ to mean that the chambers c and d are i-adjacent. By a *gallery* of type $i_1 \cdots i_n \in I^*$ in $\mathcal{C}$ we mean a finite sequence $c_0, \ldots, c_n$ of chambers such that $c_{k-1} \sim_{i_k} c_k$ and $c_{k-1} \neq c_k$ for $1 \leqslant k \leqslant n$. If $J \subseteq I$, we say that $d \in \mathcal{C}$ is *J-connected* to $c \in \mathcal{C}$ if d can be joined to c by a gallery $c = c_0, \ldots, c_n = d$ of type $j_1 \cdots j_n$ with each $j_k \in J$. We call such a gallery a *J-gallery*, and for $c \in \mathcal{C}$ we write $R_J(c)$ for the set of all chambers that can be joined to c by a J-gallery. We call $R_J(c)$ the *J-residue* of c. If $\mathcal{C}$ and $\mathcal{D}$ are chamber systems over a common index set I, we call a map $\psi : \mathcal{C} \to \mathcal{D}$ an *isomorphism of chamber systems* if ψ is a bijection such that $c \sim_i d$ if and only if $\psi(c) \sim_i \psi(d)$.

To a Coxeter group W over the index set I we associate a chamber system $\mathcal{C}(W)$, called the *Coxeter complex* of W, by taking the elements $w \in W$ as chambers, and for each $i \in I$ define i-adjacency by declaring $w \sim_i w$ and $w \sim_i ws_i$.

For the present purpose it is most convenient to consider *buildings* as certain chamber systems. Thus we give the definition of buildings from [29].

Definition 1.1. [29]. Let M be the Coxeter matrix of a Coxeter group W over I. Then $\mathcal{X}$ is a *building of type M* if

 (i) $\mathcal{X}$ is a chamber system over I such that for each $c \in \mathcal{X}$ and $i \in I$, there is a chamber $d \neq c$ in $\mathcal{X}$ such that $d \sim_i c$, and
 (ii) there exists a *W-distance function* $\delta : \mathcal{X} \times \mathcal{X} \to W$ such that if f is a reduced word then $\delta(c, d) = s_f$ if and only if c and d can be joined by a gallery of type f.

We will always use the symbol $\mathcal{X}$ to denote a building. It is convenient to write $\mathcal{C} = \mathcal{C}(\mathcal{X})$ for the chamber set of $\mathcal{X}$, even though according to the above definition $\mathcal{X}$ is itself a set of chambers. We sometimes say that $\mathcal{X}$ is a building of type W if W is the Coxeter group of type M. A building $\mathcal{X}$ is said to be *thick* if for each $c \in \mathcal{C}$ and $i \in I$ there exist at least two distinct chambers $d \neq c$ such that $d \sim_i c$. The *rank* of a building of type M is the cardinality of the index set I. We sometimes call a building *irreducible* if the associated Coxeter group is irreducible (that is, has connected Coxeter graph).

2. Regularity and parameter systems

In this section we write $\mathcal{X}$ for a building of type M, with associated Coxeter group W over index set I. We will assume that $\mathcal{X}$ is *locally finite*, by which we mean $|I| < \infty$ and $|\{b \in \mathcal{C} \mid a \sim_i b\}| < \infty$ for all $i \in I$ and $a \in \mathcal{C}$.

For each $a \in \mathcal{C}$ and $w \in W$, let

$$\mathcal{C}_w(a) = \big\{ b \in \mathcal{C} \mid \delta(a, b) = w \big\}. \tag{2.1}$$

Observe that for each fixed $a \in \mathcal{C}$, the family $\{\mathcal{C}_w(a)\}_{w \in W}$ forms a partition of $\mathcal{C}$.

We say that $\mathcal{X}$ is *regular* if for each $s \in S$, $|\mathcal{C}_s(a)|$ is independent of $a \in \mathcal{C}$. If $\mathcal{X}$ is a regular building we define $q_s = |\mathcal{C}_s(a)|$ for each $s \in S$ (this is independent of $a \in \mathcal{C}$ by definition), and we call $\{q_s\}_{s \in S}$ the *parameter system of the building*. Local finiteness implies that $q_s < \infty$ for all $s \in S$. We often write q_i in place of q_{s_i} for $i \in I$.

The two main results of this section are Proposition 2.1(ii), where we give a method for finding relationships that must hold between the parameters of buildings, and Theorem 2.4, where we generalise [30, Proposition 3.4.2] and show that all thick buildings with no rank 2 residues of type $\tilde{A}_1$ are regular.

Proposition 2.1. *Let $\mathcal{X}$ be a locally finite regular building.*

(i) $|\mathcal{C}_w(a)| = q_{i_1} q_{i_2} \cdots q_{i_n}$ *whenever $w = s_{i_1} \cdots s_{i_n}$ is a reduced expression, and*
(ii) $q_i = q_j$ *whenever $m_{i,j} < \infty$ is odd.*

Proof. We first prove (i). The result is true when $\ell(w) = 1$ by regularity. We claim that whenever $s = s_i \in S$ and $\ell(ws) = \ell(w) + 1$,

$$\mathcal{C}_{ws}(a) = \bigcup_{b \in \mathcal{C}_w(a)} \mathcal{C}_s(b) \tag{2.2}$$

where the union is disjoint, from which the result follows by induction.

First suppose that $c \in \mathcal{C}_{ws}(a)$ where $\ell(ws) = \ell(w) + 1$. Then there exists a minimal gallery $a = c_0, \ldots, c_k = c$ of type fi (where $w = s_f$ with $f \in I^*$ reduced) from a to c, and in particular $c \in \mathcal{C}_s(c_{k-1})$ where $c_{k-1} \in \mathcal{C}_w(a)$. On the other hand, if $c \in \mathcal{C}_s(b)$ for some $b \in \mathcal{C}_w(a)$ then $c \in \mathcal{C}_{ws}(a)$ since $\ell(ws) = \ell(w) + 1$, and so we have equality in (2.2). To see that the union is disjoint, suppose that $b, b' \in \mathcal{C}_w(a)$ and that $\mathcal{C}_s(b) \cap \mathcal{C}_s(b') \neq \emptyset$. Then if $b' \neq b$ we have $b' \in \mathcal{C}_s(b)$, and thus $b' \in \mathcal{C}_{ws}(a)$, a contradiction.

To prove (ii), suppose $m_{i,j} < \infty$ is odd. Since $s_i s_j s_i \cdots s_i = s_j s_i s_j \cdots s_j$ ($m_{i,j}$ factors on each side), by (i) we have $q_i q_j q_i \cdots q_i = q_j q_i q_j \cdots q_j$ ($m_{i,j}$ factors on each side), and the result follows. $\square$

Corollary 2.2. *Let $\mathcal{X}$ be a locally finite regular building of type W. If $s_j = w s_i w^{-1}$ for some $w \in W$ then $q_i = q_j$.*

Proof. By [4, IV, §1, No. 3, Proposition 3], $s_j = w s_i w^{-1}$ for some $w \in W$ if and only if there exists a sequence $s_{i_1}, \ldots, s_{i_p}$ such that $i_1 = i$, $i_p = j$, and $m_{i_k, i_{k+1}}$ is finite and odd for each $1 \leqslant k < p$. The result now follows from Proposition 2.1(ii). $\square$

Proposition 2.1(i) justifies the notation $q_w = q_{i_1} \cdots q_{i_n}$ whenever $s_{i_1} \cdots s_{i_n}$ is a reduced expression for w; it is independent of the particular reduced expression chosen. Clearly we have $q_{w^{-1}} = q_w$ for all $w \in W$.

Example 2.3. Using Proposition 2.1(ii) it is now a simple exercise to describe the relations between the parameters of any given (locally finite) regular building. For example, in a building of type $\bullet \overset{4}{\rule{1.5cm}{0.4pt}} \bullet \rule{1.5cm}{0.4pt} \bullet$ (with the nodes labelled 0, 1 and 2 from left to right) we

must have $q_1 = q_2$ since $m_{1,2} = 3$ is odd. Note that we cannot relate q_0 to q_1 since $m_{0,1} = 4$ is even.

The following theorem seems to be well known (see [30, Proposition 3.4.2] for the case $|W| < \infty$), but we have been unable to find a direct proof in the literature. For the sake of completeness we will provide a proof here.

Theorem 2.4. *Let $\mathcal{X}$ be a thick building such that $m_{i,j} < \infty$ for each pair $i, j \in I$. Then $\mathcal{X}$ is regular.*

Before giving the proof of Theorem 2.4 we make some preliminary observations. First we note that the assumption that $m_{i,j} < \infty$ in Theorem 2.4 is essential, for $\tilde{A}_1$ buildings are not in general regular, as they are just trees with no end vertices. Secondly we note that Theorem 2.4 shows that most 'interesting' buildings are regular, for examining the Coxeter graphs of the affine Coxeter groups, for example, we see that $m_{i,j} = \infty$ only occurs in $\tilde{A}_1$ buildings. Thus regularity is not a very restrictive hypothesis.

Recall that for $m \geqslant 2$ or $m = \infty$ a *generalised m-gon* is a connected bipartite graph with diameter m and girth $2m$. By [29, Proposition 3.2], a building of type $\bullet \overset{m}{\rule{1cm}{0.4pt}} \bullet$ is a generalised m-gon, and vice versa (where the edge set of the m-gon is taken to be the chamber set of the building, and vice versa).

In a generalised m-gon we define the *valency* of a vertex v to be the number of edges that contain v, and we call the generalised m-gon *thick* if every vertex has valency at least 3. By [29, Proposition 3.3], in a thick generalised m-gon with $m < \infty$, vertices in the same partition have the same valency. In the statement of [29, Proposition 3.3], the assumption $m < \infty$ is inadvertently omitted. The result is in fact false if $m = \infty$, for a thick generalised ∞-gon is simply a tree in which each vertex has valency at least 3.

Proof of Theorem 2.4. For each $a \in \mathcal{C}$ and each $i \in I$, let $q_i(a) = |\mathcal{C}_{s_i}(a)|$. By thickness, we have $q_i(a) > 1$. We will show that $q_i(a) = q_i(b)$ for all $a, b \in \mathcal{C}$ and for all $i \in I$.

Fix $a \in \mathcal{C}$. By [29, Theorem 3.5] we know that for $i, j \in I$, the residue $R_{\{i,j\}}(a)$ is a thick building of type $M_{\{i,j\}}$ which is in turn a thick generalised $m_{i,j}$-gon by [29, Proposition 3.2]. Thus, since $m_{i,j} < \infty$ by assumption, [29, Proposition 3.3] implies that

$$q_i(b) = q_i(a) \quad \text{for all } b \in R_{\{i,j\}}(a). \tag{2.3}$$

Now, with a fixed as before, let $b \in \mathcal{C}$ be any other chamber. Suppose firstly that $b \sim_k a$ for some $k \in I$. If $k = i$, then $q_i(b) = q_i(a)$ since $\sim_i$ is an equivalence relation. So suppose that $k \neq i$. Then

$$
\begin{aligned}
q_i(b) + 1 &= \big|\{c \in \mathcal{C} \mid c \sim_i b\}\big| \\
&= \big|\{c \in R_{\{i,k\}}(b) \mid c \sim_i b\}\big| \\
&= \big|\{c \in R_{\{i,k\}}(b) \mid c \sim_i a\}\big| \quad \text{by (2.3)} \\
&= \big|\{c \in R_{\{i,k\}}(a) \mid c \sim_i a\}\big| \quad \text{since } R_{\{i,k\}}(b) = R_{\{i,k\}}(a) \\
&= \big|\{c \in \mathcal{C} \mid c \sim_i a\}\big| = q_i(a) + 1,
\end{aligned}
$$

and so $q_i(b) = q_i(a)$. Induction now shows that $q_i(a)$ is independent of the particular a, and so the building is regular. $\square$

Remark 2.5. The description of parameter systems given in this section by no means comes close to *classifying* the parameter systems of buildings. For example, it is an open question as to whether thick $\tilde{A}_2$ buildings exist with parameters that are not prime powers. By the free construction of certain buildings given in [28] this is equivalent to the corresponding question concerning the parameters of projective planes (generalised 3-gons). See [2, Section 6.2] for a discussion of the known parameter systems of generalised 4-gons.

We conclude this section by recording a definition of later reference.

Definition 2.6. Let $\{q_s\}_{s \in S}$ be a set of indeterminates such that $q_{s'} = q_s$ whenever $s' = wsw^{-1}$ for some $w \in W$. Then [4, IV, §1, No. 5, Proposition 5] implies that for $w \in W$, the monomial $q_w = q_{s_{i_1}} \cdots q_{s_{i_n}}$ is independent of the particular reduced decomposition $w = s_{i_1} \cdots s_{i_n}$ of w. If U is a finite subset of W, the *Poincaré polynomial $U(q)$ of U* is

$$U(q) = \sum_{w \in U} q_w.$$

Usually the set $\{q_s\}_{s \in S}$ will be the parameters of a building (see Corollary 2.2).

3. Chamber set operators and chamber regularity

The results of this section generalise the results in [11, Chapter 6], where it is assumed that there is a group G (of type preserving simplicial complex automorphisms) acting *strongly transitively* on $\mathcal{X}$ (see [11, §5.2]). As noted in the introduction, all buildings admitting such a group action are necessarily regular, whereas the converse is not true. Our proofs work for all locally finite regular buildings, which, by Theorem 2.4, includes all thick buildings with no rank 2 residues of type $\tilde{A}_1$. It should be noted that our results also apply to thin buildings (where $q_i = 1$ for all $i \in I$), as well as to regular buildings that are neither thick nor thin (that is, buildings that have $q_i = 1$ for some but not all $i \in I$). We note that some of the results of this section are proved in [34] in the context of *association schemes*.

Let $\mathcal{X}$ be a locally finite regular building. We say that $\mathcal{X}$ is *chamber regular* if for all w_1 and w_2 in W,

$$\left| \mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b) \right| = \left| \mathcal{C}_{w_1}(c) \cap \mathcal{C}_{w_2}(d) \right| \quad \text{whenever} \quad \delta(a, b) = \delta(c, d),$$

where the sets $\mathcal{C}_w(a)$ are as in (2.1). In this section we will prove that regularity implies chamber regularity (Proposition 3.9), and we introduce an algebra $\mathcal{B}$ of chamber set averaging operators (Definition 3.7) and show that this algebra is isomorphic to a suitably parametrised Hecke algebra (Theorem 3.10). Recall that for a regular building we define $q_s = |\mathcal{C}_s(a)|$, and we write $q_{s_i} = q_i$.

Definition 3.1. For each $w \in W$, define an operator B_w, acting on the space of all functions $f : \mathcal{C} \to \mathbb{C}$ as in (0.1).

Observe that $b \in \mathcal{C}_w(a)$ if and only if $a \in \mathcal{C}_{w^{-1}}(b)$. If $\mathcal{C}' \subseteq \mathcal{C}$, write $1_{\mathcal{C}'} : \mathcal{C} \to \{0, 1\}$ for the characteristic function on $\mathcal{C}'$. Thus for $w_1, w_2 \in W$ we have

$$
\begin{aligned}
(B_{w_1} B_{w_2} f)(a) &= \frac{1}{q_{w_1}} \sum_{b \in \mathcal{C}_{w_1}(a)} (B_{w_2} f)(b) \\
&= \frac{1}{q_{w_1} q_{w_2}} \sum_{b \in \mathcal{C}_{w_1}(a)} \sum_{c \in \mathcal{C}_{w_2}(b)} f(c) \\
&= \frac{1}{q_{w_1} q_{w_2}} \sum_{b \in \mathcal{C}} \sum_{c \in \mathcal{C}} 1_{\mathcal{C}_{w_1}(a)}(b) 1_{\mathcal{C}_{w_2}(b)}(c) f(c) \\
&= \frac{1}{q_{w_1} q_{w_2}} \sum_{c \in \mathcal{C}} \left(\sum_{b \in \mathcal{C}} 1_{\mathcal{C}_{w_1}(a)}(b) 1_{\mathcal{C}_{w_2^{-1}}(c)}(b) \right) f(c) \\
&= \frac{1}{q_{w_1} q_{w_2}} \sum_{c \in \mathcal{C}} \left| \mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(c) \right| f(c). \tag{3.1}
\end{aligned}
$$

We wish to explicitly compute the above when $w_2 = s \in S$ (and so $w_2^{-1} = w_2$). Thus we have the following lemmas.

Lemma 3.2. *Let $w \in W$ and $s \in S$, and fix $a \in \mathcal{C}$. Then*

$$
\mathcal{C}_w(a) \cap \mathcal{C}_s(b) \neq \emptyset \quad \Rightarrow \quad \begin{cases} b \in \mathcal{C}_{ws}(a) & \text{if } \ell(ws) = \ell(w) + 1, \\ b \in \mathcal{C}_w(a) \cup \mathcal{C}_{ws}(a) & \text{if } \ell(ws) = \ell(w) - 1. \end{cases}
$$

Proof. Let $s = s_i$ where $i \in I$. Suppose first that $\ell(ws) = \ell(w) + 1$ and that $c \in \mathcal{C}_w(a) \cap \mathcal{C}_s(b)$. Let f be a reduced word in I^* so that $s_f = w$, and so there exists a gallery from a to c of type f. Since $b \in \mathcal{C}_s(c)$, there is a gallery of type fi from a to b, which is a reduced word by hypothesis. It follows that $b \in \mathcal{C}_{ws}(a)$.

Suppose now that $\ell(ws) = \ell(w) - 1$, and that $c \in \mathcal{C}_w(a) \cap \mathcal{C}_s(b)$. Since ws is not reduced, there exists a reduced word f' such that $f'i$ is a reduced word for w. This shows that there exist a minimal gallery $a = a_0, \ldots, a_m = c$ such that $a_{m-1} \in \mathcal{C}_s(c)$. Since $b \in \mathcal{C}_s(c)$ too, it follows that either $b = a_{m-1}$ or $b \in \mathcal{C}_s(a_{m-1})$. In the former case we have $b \in \mathcal{C}_{ws}(a)$ and in the latter we have $b \in \mathcal{C}_w(a)$. $\quad \square$

We now perform counts that will be used to demonstrate chamber regularity.

Lemma 3.3. *Let $w \in W$ and $s \in S$. Fix $a, b \in \mathcal{C}$. Then*

$$
\left| \mathcal{C}_w(a) \cap \mathcal{C}_s(b) \right| = \begin{cases} 1 & \text{if } \ell(ws) = \ell(w) + 1 \text{ and } b \in \mathcal{C}_{ws}(a), \\ q_s & \text{if } \ell(ws) = \ell(w) - 1 \text{ and } b \in \mathcal{C}_{ws}(a), \\ q_s - 1 & \text{if } \ell(ws) = \ell(w) - 1 \text{ and } b \in \mathcal{C}_w(a). \end{cases}
$$

Proof. Suppose first that $\ell(ws) = \ell(w) + 1$ and that $b \in \mathcal{C}_{ws}(a)$. Thus there is a minimal gallery $a = a_0, \ldots, a_m = b$ such that $a_{m-1} \in \mathcal{C}_s(b)$. There are q_s chambers c in $\mathcal{C}_s(b)$. One of these chambers is a_{m-1}, which lies in $\mathcal{C}_w(a)$, and the remaining $q_s - 1$ lie in $\mathcal{C}_{ws}(a)$, so a_{m-1} is the only element of $\mathcal{C}_w(a) \cap \mathcal{C}_s(b)$. Thus $|\mathcal{C}_w(a) \cap \mathcal{C}_s(b)| = 1$ as claimed in this case.

Suppose now that $\ell(ws) = \ell(w) - 1$ and that $b \in \mathcal{C}_{ws}(a)$. Write $s = s_i$, and let $w = s_f$ where $f \in I^*$ is reduced. Since $\ell(ws) = \ell(w) - 1$, there exists a reduced word f' such that $f'i$ is a reduced word for w, and thus there exists a minimal gallery of type f' from a to b. Thus each $c \in \mathcal{C}_s(b)$ can be joined to a by a gallery of type $f'i \sim f$, and hence $c \in \mathcal{C}_w(a)$, verifying the count in this case.

Finally, suppose that $\ell(ws) = \ell(w) - 1$ and $b \in \mathcal{C}_w(a)$. Then, as in the proof of Lemma 3.2, there exists a minimal gallery $a = a_0, \ldots, a_m = b$ such that $b \in \mathcal{C}_s(a_{m-1})$. Exactly one of the q_s chambers $c \in \mathcal{C}_s(b)$ equals a_{m-1}, and thus lies in $\mathcal{C}_{ws}(a)$. For the remaining $q_s - 1$ chambers we have $c \in \mathcal{C}_s(a_{m-1})$, and thus $c \in \mathcal{C}_w(a)$, completing the proof. $\quad\square$

Theorem 3.4. *Let $w \in W$ and $s \in S$. Then*

$$B_w B_s = \begin{cases} B_{ws} & \text{when } \ell(ws) = \ell(w) + 1, \\ \frac{1}{q_s} B_{ws} + \left(1 - \frac{1}{q_s}\right) B_w & \text{when } \ell(ws) = \ell(w) - 1. \end{cases}$$

Proof. Let us look at the case $\ell(ws) = \ell(w) - 1$. The case $\ell(ws) = \ell(w) + 1$ is similar. By (3.1) and Lemma 3.3 we have

$$B_w B_s = \frac{q_{ws}}{q_w} B_{ws} + \left(1 - \frac{1}{q_s}\right) B_w.$$

All that remains is to show that $q_{ws}/q_w = 1/q_s$. If f is a reduced word with $s_f = w$ and $s = s_i$, the hypothesis that $\ell(ws) = \ell(w) - 1$ implies that there exists a reduced word f' such that $f'i$ is a reduced word for w. The result now follows. $\quad\square$

Corollary 3.5. $B_{w_1} B_{w_2} = B_{w_1 w_2}$ *whenever* $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

Corollary 3.6. *Let $w_1, w_2 \in W$. There exist numbers $b_{w_1, w_2; w_3} \in \mathbb{Q}^+$ such that*

$$B_{w_1} B_{w_2} = \sum_{w_3 \in W} b_{w_1, w_2; w_3} B_{w_3} \quad \text{and} \quad \sum_{w_3 \in W} b_{w_1, w_2; w_3} = 1.$$

Moreover, $|\{w_3 \in W \mid b_{w_1, w_2; w_3} \neq 0\}|$ is finite for all $w_1, w_2 \in W$.

Proof. An induction on $\ell(w_2)$ shows existence of the numbers $b_{w_1, w_2; w_3} \in \mathbb{Q}^+$ such that $B_{w_1} B_{w_2} = \sum_{w_3} b_{w_1, w_2; w_3} B_{w_3}$, and shows that only finitely many of the $b_{w_1, w_2; w_3}$'s are nonzero for fixed w_1 and w_2. Evaluating both sides at the constant function $1_{\mathcal{C}} : \mathcal{C} \to \{1\}$ shows that $\sum_{w_3} b_{w_1, w_2; w_3} = 1$. $\quad\square$

Definition 3.7. Let $\mathscr{B}$ be the linear span of the set $\{B_w \mid w \in W\}$ over $\mathbb{C}$. Corollary 3.6 shows that $\mathscr{B}$ is an associative algebra.

Proposition 3.8. $\{B_w \mid w \in W\}$ *is a vector space basis of* $\mathscr{B}$, *and* $\mathscr{B}$ *is generated by* $\{B_s \mid s \in S\}$.

Proof. Suppose we have a relation $\sum_{k=1}^{n} b_k B_{w_k} = 0$, and fix $a, b \in \mathcal{C}$ with $\delta(a, b) = w_j$ with $1 \leqslant j \leqslant n$. Then writing $\delta_b = 1_{\{b\}}$ we have

$$0 = \sum_{k=1}^{n} b_k (B_{w_k} \delta_b)(a) = \sum_{k=1}^{n} b_k q_{w_k}^{-1} \delta_{k,j} = b_j q_{w_j}^{-1},$$

and so $b_j = 0$. From Corollary 3.5 we see that $\{B_s \mid s \in S\}$ generates $\mathscr{B}$. $\square$

We refer to the numbers $b_{w_1, w_2; w_3}$ from Corollary 3.6 as the *structure constants* of the algebra $\mathscr{B}$ (with respect to the natural basis $\{B_w \mid w \in W\}$).

Proposition 3.9. *Let* $\mathscr{X}$ *be a regular building of type* W, *and let* $w_1, w_2, w_3 \in W$. *For any pair* $a, b \in \mathcal{C}$ *with* $b \in \mathcal{C}_{w_3}(a)$ *we have*

$$\left| \mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(b) \right| = \frac{q_{w_1} q_{w_2}}{q_{w_3}} b_{w_1, w_2; w_3},$$

and so $\mathscr{X}$ *is chamber regular.*

Proof. Using (3.1) we compute $(B_{w_1} B_{w_2} \delta_b)(a) = q_{w_1}^{-1} q_{w_2}^{-1} | \mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(b)|$, whereas by Corollary 3.6 we have $(B_{w_1} B_{w_2} \delta_b)(a) = q_{w_3}^{-1} b_{w_1, w_2; w_3}$. $\square$

Those readers familiar with Hecke algebras will notice immediately from Theorem 3.4 the connection between $\mathscr{B}$ and Hecke algebras. For our purposes we define *Hecke algebras* as follows (see [14, Chapter 7]). For each $s \in S$, let a_s and b_s be complex numbers such that $a_{s'} = a_s$ and $b_{s'} = b_s$ whenever $s' = wsw^{-1}$ for some $w \in W$. The (*generic*) *Hecke algebra* $\mathcal{H}(a_s, b_s)$ is the algebra over $\mathbb{C}$ with presentation given by basis elements T_w, $w \in W$, and relations

$$T_w T_s = \begin{cases} T_{ws} & \text{when } \ell(ws) = \ell(w) + 1, \\ a_s T_{ws} + b_s T_w & \text{when } \ell(ws) = \ell(w) - 1. \end{cases} \tag{3.2}$$

Theorem 3.10. *Suppose a building* $\mathscr{X}$ *of type* W *exists with parameters* $\{q_s\}_{s \in S}$. *Then* $\mathscr{B} \cong \mathcal{H}(q_s^{-1}, 1 - q_s^{-1})$.

Proof. We note first that by Corollary 2.2, the numbers $a_s = q_s^{-1}$ and $b_s = 1 - q_s^{-1}$ satisfy the condition $a_{s'} = a_s$ and $b_{s'} = b_s$ whenever $s' = wsw^{-1}$ for some $w \in W$.

Since $\{T_w \mid w \in W\}$ is a vector space basis of $\mathcal{H}(q_s^{-1}, 1 - q_s^{-1})$ and $\{B_w \mid w \in W\}$ is a vector space basis of $\mathcal{B}$ (see Proposition 3.8) there exists a unique vector space isomorphism $\Phi : \mathcal{H}(q_s^{-1}, 1 - q_s^{-1}) \to \mathcal{B}$ such that $\Phi(T_w) = B_w$ for all $w \in W$. By (3.2) and Theorem 3.4 we have $\Phi(T_w T_s) = \Phi(T_w)\Phi(T_s)$ for all $w \in W$ and $s \in S$, and so Φ is an algebra homomorphism. It follows that Φ is an algebra isomorphism. $\quad\square$

Recall that we write D for the Coxeter graph of W.

Definition 3.11. Let $\mathcal{X}$ be a locally finite regular building. Define

$$\mathrm{Aut}_q(D) = \left\{ \sigma \in \mathrm{Aut}(D) \mid q_{\sigma(i)} = q_i \text{ for all } i \in I \right\}.$$

Lemma 3.12. *For all $w_1, w_2 \in W$ and $\sigma \in \mathrm{Aut}_q(D)$ we have*

$$\left| \mathcal{C}_{\sigma(w_1)}(a') \cap \mathcal{C}_{\sigma(w_2)}(b') \right| = \left| \mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b) \right|,$$

whenever $a, b, a', b' \in \mathcal{C}$ are chambers with $\delta(a', b') = \sigma(\delta(a, b))$.

Proof. We first show that, in the notation of Corollary 3.6,

$$b_{w_1, w_2; w_3} = b_{\sigma(w_1), \sigma(w_2); \sigma(w_3)} \tag{3.3}$$

for all $w_1, w_2, w_3 \in W$.

Theorem 3.4, the definition of $\mathrm{Aut}_q(D)$ and the fact that $\ell(\sigma(w)) = \ell(w)$ for all $w \in W$ show that this is true when $\ell(w_2) = 1$, beginning an induction. Suppose (3.3) holds whenever $\ell(w_2) < n$, and suppose $w = s_{i_1} \cdots s_{i_{n-1}} s_{i_n}$ has length n. Write $w' = s_{i_1} \cdots s_{i_{n-1}}$ and $s = s_{i_n}$. Observe that $\sigma(w) = \sigma(w')\sigma(s)$ so that $B_{\sigma(w)} = B_{\sigma(w')} B_{\sigma(s)}$ by Theorem 3.4, and so

$$
\begin{aligned}
B_{\sigma(w_1)} B_{\sigma(w)} &= \left(B_{\sigma(w_1)} B_{\sigma(w')} \right) B_{\sigma(s)} \\
&= \sum_{w_3 \in W} b_{\sigma(w_1), \sigma(w'); \sigma(w_3)} B_{\sigma(w_3)} B_{\sigma(s)} \\
&= \sum_{w_3 \in W} \left(b_{w_1, w'; w_3} \sum_{w_4 \in W} b_{\sigma(w_3), \sigma(s); \sigma(w_4)} B_{\sigma(w_4)} \right) \\
&= \sum_{w_4 \in W} \left(\sum_{w_3 \in W} b_{w_1, w'; w_3} b_{w_3, s; w_4} \right) B_{\sigma(w_4)}.
\end{aligned}
$$

Thus

$$b_{\sigma(w_1), \sigma(w); \sigma(w_4)} = \sum_{w_3 \in W} b_{w_1, w'; w_3} b_{w_3, s; w_4} \quad \text{for all } w_4 \in W. \tag{3.4}$$

The same calculation without the σ's shows that this is also $b_{w_1, w; w_4}$. This completes the induction step, and so (3.3) holds for all w_1, w_2 and w_3 in W.

Thus for any chambers a, b, a', b' with $\delta(a, b) = w_3$, and $\delta(a', b') = \sigma(w_3)$ we have (using Proposition 3.9)

$$\left| \mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b) \right| = \frac{q_{w_1} q_{w_2^{-1}}}{q_{w_3}} b_{w_1, w_2^{-1}; w_3}$$

$$= \frac{q_{\sigma(w_1)} q_{\sigma(w_2^{-1})}}{q_{\sigma(w_3)}} b_{\sigma(w_1), \sigma(w_2^{-1}); \sigma(w_3)}$$

$$= \left| \mathcal{C}_{\sigma(w_1)}(a') \cap \mathcal{C}_{\sigma(w_2)}(b') \right|. \qquad \square$$

4. Preliminary material

This section is preparation for our study of the vertex set averaging operators associated to locally finite regular affine buildings.

4.1. Chamber systems and simplicial complexes

Recall that a *simplicial complex* with vertex set V is a collection X of finite subsets of V (called *simplices*) such that for every $v \in V$, the singleton $\{v\}$ is a simplex, and every subset of a simplex x is a simplex (called a *face of x*). If x is a simplex which is not a proper subset of any other simplex, then we call x a *maximal simplex*, or *chamber* of X.

A *labelled simplicial complex* with vertex set V is a simplicial complex equipped with a set I of *types*, and a *type map* $\tau : V \to I$ such that the restriction $\tau|_C : C \to I$ of τ to any chamber C is a bijection.

An *isomorphism* of simplicial complexes is a bijection of the vertex sets that maps simplices, and only simplices, to simplices. If both simplicial complexes are labelled by the same set, then an isomorphism which preserves types is said to be *type preserving*.

There is a well-known method of producing labelled simplicial complexes from chamber systems, and vice versa (see [6, §1.4] for details). This allows us to consider buildings and Coxeter complexes as certain labelled simplicial complexes (with canonical labellings). The following is an alternative (and of course equivalent) definition of buildings from a simplicial complex approach.

Definition 4.1. [5]. Let W be a Coxeter group of type M. A *building of type M* is a non-empty simplicial complex X which contains a family of subcomplexes called *apartments* such that

 (i) each apartment is isomorphic to the (simplicial) Coxeter complex of W,
 (ii) given any two chambers of X there is an apartment containing both, and
(iii) given any two apartments $\mathcal{A}$ and $\mathcal{A}'$ that contain a common chamber, there exists an isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ fixing $\mathcal{A} \cap \mathcal{A}'$ pointwise.

We remark that Definition 4.1(iii) can be replaced with the following [5, p. 76].

(iii)' If $\mathcal{A}$ and $\mathcal{A}'$ are apartments both containing simplices ρ and σ, then there is an isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ fixing ρ and σ pointwise.

It is easy to see that $\mathcal{X}$ is in fact a labellable simplicial complex, and all the isomorphisms in the above definition may be taken to be label preserving.

4.2. Root systems

For the purpose of fixing notation we will give a brief discussion of root systems. A thorough reference to this well-known material is [4]. Let E be an n-dimensional vector space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle$, and for $\alpha \in E \setminus \{0\}$ define $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$.

Let R be an *irreducible*, but not necessarily *reduced*, *root system* in E (see [4, VI, §1, Nos. 1, 2]).

The elements of R are called *roots*, and the *rank* of R is n, the dimension of E. A root system that is not reduced is said to be *nonreduced*. See [4, VI, §4, Nos. 5–14] for the classification of irreducible root systems.

Let $B = \{\alpha_i \mid i \in I_0\}$ be a *base* of R, where $I_0 = \{1, 2, \ldots, n\}$. Thus B is a subset of R such (i) a vector space basis of E, and (ii) each root in R can be written as a linear combination of elements of B with integer coefficients which are either all nonnegative or all nonpositive. We say that $\alpha \in R$ is *positive* (respectively *negative*) if the expression for α from (ii) has only nonnegative (respectively nonpositive) coefficients. Let R^+ (respectively R^-) be the set of all positive (respectively negative) roots. Thus $R^- = -R^+$ and $R = R^+ \cup R^-$, where the union is disjoint.

Define the *height* (*with respect to B*) of $\alpha = \sum_{i \in I_0} k_i \alpha_i \in R$ by $\mathrm{ht}(\alpha) = \sum_{i \in I_0} k_i$. By [4, VI, §1, No. 8, Proposition 25] there exists a unique root $\tilde{\alpha} \in R$ whose height is maximal, and defining numbers m_i by

$$\tilde{\alpha} = \sum_{i \in I_0} m_i \alpha_i \tag{4.1}$$

we have $m_i \geqslant 1$ for all $i \in I_0$. To complete the notation we define $m_0 = 1$.

The *dual* (or *inverse*) of R is $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$. By [4, VI, §1, No. 1, Proposition 2] $R^\vee$ is an irreducible root system which is reduced if and only if R is.

We define a dual basis $\{\lambda_i\}_{i \in I_0}$ of E by $\langle \lambda_i, \alpha_j \rangle = \delta_{i,j}$. Recall that the *coroot lattice* Q of R is the $\mathbb{Z}$-span of R, and the *coweight lattice* P of R is the $\mathbb{Z}$-span of $\{\lambda_i\}_{i \in I_0}$. Elements of P are called *coweights* (of R), and it is clear that $Q \subseteq P$. Note that in the literature Q and P are also called the *root* and *weight* lattices of $R^\vee$. We call a coweight $\lambda = \sum_{i \in I_0} a_i \lambda_i$ *dominant* if $a_i \geqslant 0$ for all $i \in I_0$, and we write P^+ for the set of all dominant coweights.

For each $n \geqslant 1$ there is exactly one irreducible nonreduced root system (up to isomorphism) of rank n, denoted by BC_n [4, VI, §4, No. 14]. We may take $E = \mathbb{R}^n$ with the usual inner product, and let $\alpha_j = e_j - e_{j+1}$ for $1 \leqslant j < n$ and $\alpha_n = e_n$. Then $B = \{\alpha_j\}_{j=1}^n$, and

$$R^+ = \{e_k, 2e_k, e_i + e_j, e_i - e_j \mid 1 \leqslant k \leqslant n,\ 1 \leqslant i < j \leqslant n\}.$$

Notice that $R^\vee = R$, and one easily sees that $Q = P$.

4.3. Hyperplane arrangements and reflection groups

Let R be an irreducible root system, and for each $\alpha \in R$ and $k \in \mathbb{Z}$ let $H_{\alpha;k} = \{x \in E \mid \langle x, \alpha \rangle = k\}$. Let $\mathcal{H}$ denote the family of these (affine) *hyperplanes* $H_{\alpha;k}$, $\alpha \in R$, $k \in \mathbb{Z}$. We write H_α in place of $H_{\alpha;0}$, and denote by $\mathcal{H}_0$ the family of these hyperplanes H_α, $\alpha \in R$.

Given $H_{\alpha;k} \in \mathcal{H}$, the associated *orthogonal reflection* is the map $s_{\alpha;k} : E \to E$ given by $s_{\alpha;k}(x) = x - (\langle x, \alpha \rangle - k)\alpha^\vee$ for all $x \in E$. We write s_α in place of $s_{\alpha;0}$, and s_i in place of s_{α_i}. The *Weyl group of R*, denoted $W_0(R)$, or simply W_0, is the subgroup of $\mathrm{GL}(E)$ generated by the reflections s_α, $\alpha \in R$, and the *affine Weyl group of R*, denoted $W(R)$, or simply W, is the subgroup of $\mathrm{Aff}(E)$ generated by the reflections $s_{\alpha;k}$, $\alpha \in R$, $k \in \mathbb{Z}$. Here $\mathrm{Aff}(E)$ is the set of maps $x \mapsto Tx + v$, $T \in \mathrm{GL}(E)$, $v \in E$. Writing t_v for the translation $x \mapsto x + v$, we consider E as a subgroup of $\mathrm{Aff}(E)$ by identifying v and t_v. We have $\mathrm{Aff}(E) = \mathrm{GL}(E) \ltimes E$, and $W \cong W_0 \ltimes Q$. Note that $W_0(R^\vee) = W_0(R)$ [4, VI, §1, No. 1].

Let $s_0 = s_{\tilde{\alpha};1}$, define $I = I_0 \cup \{0\}$, and let $S_0 = \{s_i \mid i \in I_0\}$ and $S = \{s_i \mid i \in I\}$. The group W_0 (respectively W) is a Coxeter group over I_0 (respectively I) generated by S_0 (respectively S).

We write $\Sigma = \Sigma(R)$ for the vector space E equipped with the sectors, chambers and vertices as defined below. The open connected components of $E \setminus \bigcup_{H \in \mathcal{H}} H$ are called the *chambers* of Σ (this terminology is motivated by building theory, and differs from that used in [4] where there are *chambers* and *alcoves*), and we write $\mathcal{C}(\Sigma)$ for the set of chambers of Σ. Since R is irreducible, each $C \in \mathcal{C}(\Sigma)$ is an open (geometric) simplex [4, V, §3, No. 9, Proposition 8]. Call the extreme points of the sets $\bar{C}$, $C \in \mathcal{C}(\Sigma)$, *vertices* of Σ, and write $V(\Sigma)$ for the set of all vertices of Σ.

In choice of B gives a natural *fundamental chamber*

$$C_0 = \{x \in E \mid \langle x, \alpha_i \rangle > 0 \text{ for all } i \in I_0 \text{ and } \langle x, \tilde{\alpha} \rangle < 1\}, \qquad (4.2)$$

where we use the notation of (4.1).

The *fundamental sector* of Σ is

$$\mathcal{S}_0 = \{x \in E \mid \langle x, \alpha_i \rangle > 0 \text{ for all } i \in I_0\}, \qquad (4.3)$$

and the *sectors* of Σ are the sets $\lambda + w\mathcal{S}_0$, where $\lambda \in P$ and $w \in W_0$. The sector $\mathcal{S} = \lambda + w\mathcal{S}_0$ is said to have *base vertex* λ (we will see in Section 4.5 that λ is indeed a vertex of Σ).

The group W_0 acts simply transitively on the set of sectors based at 0, and $\bar{\mathcal{S}}_0$ is a fundamental domain for the action of W_0 on E. Similarly, W acts simply transitively on $\mathcal{C}(\Sigma)$, and $\bar{C}_0$ if a fundamental domain for the action of W on E [4, VI, §1–3].

The following fact follows easily from [4, VI, §2, No. 2, Proposition 4(ii)].

Lemma 4.2. W_0 *acts simply transitively on the set of $C \in \mathcal{C}(\Sigma)$ with $0 \in \bar{C}$.*

4.4. A geometric realisation of the Coxeter complex

The set $\mathcal{C}(\Sigma)$ from Section 4.3 forms a chamber system over I if we declare $wC_0 \sim_i wC_0$ and $wC_0 \sim_i ws_iC_0$ for each $w \in W$ and each $i \in I$. The map $w \mapsto wC_0$ is an iso-

morphism of the Coxeter complex $\mathcal{C}(W)$ of Section 1 onto this chamber system, and so Σ may be regarded as a *geometric realisation* of $\mathcal{C}(W)$.

The vertices of $\bar{C}_0$ are $\{0\} \cup \{\lambda_i/m_i \mid i \in I_0\}$ (see [4, VI, §2, No. 2]), and we declare $\tau(0) = 0$ and $\tau(\lambda_i/m_i) = i$ for $i \in I_0$. This extends to a unique labelling $\tau : V(\Sigma) \to I$ (see [6, Lemma 1.5]), and the action of W on Σ is type preserving.

4.5. Special and good vertices of Σ

Following [4, V, §3, No. 10], a point $v \in E$ is said to be *special* if for every $H \in \mathcal{H}$ there exists a hyperplane $H' \in \mathcal{H}$ parallel to H such that $v \in H'$. Note that in our set-up $0 \in E$ is special. Each special point is a vertex of Σ [4, V, §3, No. 10], and thus we will call the special points *special vertices*. Note that in general not all vertices are special (for example, in the $\tilde{C}_2$ and $\tilde{G}_2$ complexes). When R is reduced P is the set of special vertices of Σ [4, VI, §2, No. 2, Proposition 3]. When R is nonreduced then P is a proper subset of the special vertices of Σ (see Example 4.5).

To deal with the reduced and nonreduced cases simultaneously, we define the *good* vertices of Σ to be the elements of P. On the first reading the reader is encouraged to think of P as the set of all special vertices, for this is true unless R is of type BC_n. Note that, according to our definitions, every sector of Σ is based at a good vertex of Σ.

We write I_P for the set of *good types*. That is, $I_P = \{\tau(\lambda) \mid \lambda \in P\} \subseteq I$.

Lemma 4.3. *Let the numbers m_i be as in* (4.1). *Then $I_P = \{i \in I \mid m_i = 1\}$.*

Proof. The vertices of C_0 are $\{0\} \cup \{\lambda_i/m_i \mid i \in I_0\}$. The good vertices of C_0 are those in P, and thus have type 0 or i for some i with $m_i = 1$. $\quad\square$

4.6. Examples

Example 4.4. ($R = C_2$). Take $E = \mathbb{R}^2$, $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2$. Then $B = \{\alpha_1, \alpha_2\}$ and $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$.

The dotted lines in Fig. 1 are the hyperplanes $\{H_{w\alpha_1;k} \mid w \in W_0, k \in \mathbb{Z}\}$, and the dashed lines are the hyperplanes $\{H_{w\alpha_2;k} \mid w \in W_0, k \in \mathbb{Z}\}$. In this example $\lambda_1 = e_1$ and $\lambda_2 = \frac{1}{2}(e_1 + e_2)$, and $\tau(0) = 0$, $\tau(\frac{1}{2}e_1) = 1$ and $\tau(\frac{1}{2}(e_1 + e_2)) = 2$. We have $P = \{(x, y) \in (\frac{1}{2}\mathbb{Z})^2 \mid x + y \in \mathbb{Z}\}$, which coincides with the set of all special vertices (as expected, since R is reduced here). Thus $I_P = \{0, 2\}$.

Example 4.5. ($R = BC_2$). Take $E = \mathbb{R}^2$, $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2$. Then $B = \{\alpha_1, \alpha_2\}$ and $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_2, 2\alpha_1 + 2\alpha_2\}$.

The dotted and solid lines in Fig. 2 represent the hyperplanes in the sets $\{H_{w\alpha_1;k} \mid w \in W_0, k \in \mathbb{Z}\}$ and $\{H_{w\alpha_2;k} \mid w \in W_0, k \in \mathbb{Z}\}$ respectively. The union of the dashed and solid lines represent the hyperplanes in $\{H_{w(2\alpha_2);k} \mid w \in W_0, k \in \mathbb{Z}\}$.

In contrast to the previous example, here we have $\lambda_1 = e_1$ and $\lambda_2 = e_1 + e_2$. The set of special vertices and the vertex types are as in Example 4.4, but here $P = \mathbb{Z}^2$ (and so $I_P = \{0\}$).
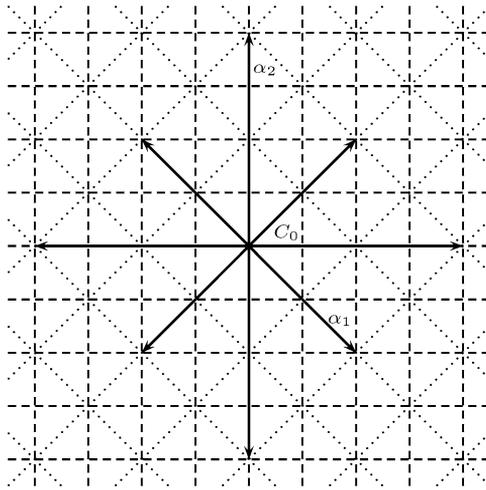
Fig. 1.
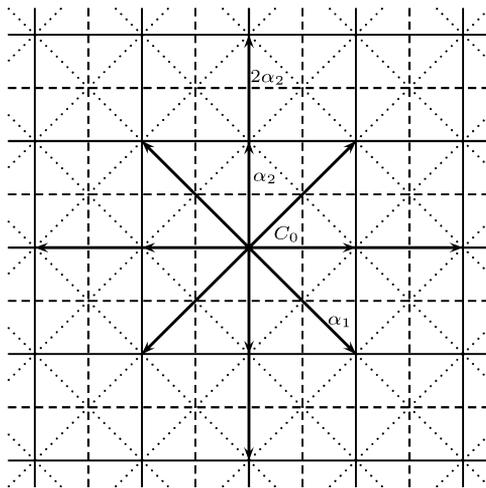
Fig. 2.

4.7. The extended affine Weyl group

The *extended affine Weyl group of R*, denoted $\tilde{W}(R)$ or simply $\tilde{W}$, is $\tilde{W} = W_0 \ltimes P$. In general $\tilde{W}$ is larger than W. In fact, $\tilde{W}/W \cong P/Q$ [4, VI, §2, No. 3]. We note that while $W(C_n) = W(BC_n)$, $\tilde{W}(C_n)$ is not isomorphic to $\tilde{W}(BC_n)$.

In particular, notice that for each $\lambda \in P$, the translation $t_\lambda : E \to E$, $t_\lambda(x) = x + \lambda$, is in $\tilde{W}$.

The group $\tilde{W}$ permutes the chambers of Σ, but in general does not act simply transitively. Recall [21, §2.2] that for $w \in \tilde{W}$, the *length* of w is defined by

$$\ell(w) = \left| \left\{ H \in \mathcal{H} \mid H \text{ separates } C_0 \text{ and } w^{-1}C_0 \right\} \right|.$$

When $w \in W$, this definition agrees with the definition of $\ell(w)$ given previously for Coxeter groups.

The subgroup $G = \{g \in \tilde{W} \mid \ell(g) = 0\}$ will play an important role; it is the stabiliser of C_0 in $\tilde{W}$. We have $\tilde{W} \cong W \ltimes G$ [4, VI, §2, No. 3], and furthermore, $G \cong P/Q$, and so G is a finite abelian group. Let w_0 and $w_{0\lambda}$ denote the longest elements of W_0 and $W_{0\lambda}$ respectively, where for $\lambda \in P$,

$$W_{0\lambda} = \{w \in W_0 \mid w\lambda = \lambda\}. \tag{4.4}$$

Recall the definition of the numbers m_i (with $m_0 = 1$) from (4.1). Then

$$G = \{g_i \mid m_i = 1\} \tag{4.5}$$

where $g_0 = 1$ and $g_i = t_{\lambda_i} w_{0\lambda_i} w_0$ for $i \in I_P \setminus \{0\}$ (see [4, VI, §2, No. 3] in the reduced case and note that $G = \{1\}$ in the nonreduced case since $G \cong P/Q$).

4.8. Automorphisms of Σ and D

An *automorphism* of Σ is a bijection ψ of E that maps chambers, and only chambers, to chambers with the property that $C \sim_i D$ if and only if $\psi(C) \sim_{i'} \psi(D)$ for some $i' \in I$ (depending on C, D and i). Let $\mathrm{Aut}(\Sigma)$ denote the automorphism group of Σ. Clearly W_0, W and $\tilde{W}$ can be considered as subgroups of $\mathrm{Aut}(\Sigma)$, and we have $W_0 \leqslant W \leqslant \tilde{W} \leqslant \mathrm{Aut}(\Sigma)$. Note that in some cases $\tilde{W}$ is a proper subgroup of $\mathrm{Aut}(\Sigma)$. For example, if R is of type A_2, then the map $a_1\lambda_1 + a_2\lambda_2 \mapsto a_1\lambda_2 + a_2\lambda_1$ is in $\mathrm{Aut}(\Sigma)$ but is not in $\tilde{W}$.

Write D for the Coxeter graph of W (see Section 1). Recall the definition of the type map $\tau : V(\Sigma) \to I$ from Section 4.4.

Proposition 4.6. *Let $\psi \in \mathrm{Aut}(\Sigma)$. Then there exists $\sigma \in \mathrm{Aut}(D)$ such that $(\tau \circ \psi)(v) = (\sigma \circ \tau)(v)$ for all $v \in V(\Sigma)$. If $C \sim_i D$, then $\psi(C) \sim_{\sigma(i)} \psi(D)$.*

Proof. The result follows from [5, pp. 64–65]. $\quad\square$

For each $g_i \in G$ (see (4.5)), let $\sigma_i \in \mathrm{Aut}(D)$ be the automorphism induced as in Proposition 4.6. We call the automorphisms $\sigma_i \in \mathrm{Aut}(D)$ *type rotating* (for in the $\tilde{A}_n$ case they are the permutations $k \mapsto k + i \bmod n + 1$), and we write $\mathrm{Aut}_{\mathrm{tr}}(D)$ for the group of all type rotating automorphisms of D. Thus

$$\mathrm{Aut}_{\mathrm{tr}}(D) = \{\sigma_i \mid i \in I_P\}. \tag{4.6}$$

Note that since $g_0 = 1$, $\sigma_0 = \mathrm{id}$.

Let D_0 be the Coxeter graph of W_0. We have [4, VI, §4, No. 3]

$$\text{Aut}(D) = \text{Aut}(D_0) \ltimes \text{Aut}_{\text{tr}}(D). \tag{4.7}$$

The group $\tilde{W}$ has a presentation with generators s_i, $i \in I$, and g_j, $j \in I_P$, and relations (see [25, (1.20)])

$$(s_i s_j)^{m_{i,j}} = 1 \quad \text{for all } i, j \in I, \quad \text{and}$$

$$g_j s_i g_j^{-1} = s_{\sigma_j(i)} \quad \text{for all } i \in I \text{ and } j \in I_P. \tag{4.8}$$

Proposition 4.7. *Let $i \in I_P$ and $\sigma \in \text{Aut}_{\text{tr}}(D)$.*

(i) $\sigma_i(0) = i$.
(ii) *If $\sigma(i) = i$, then $\sigma = \sigma_0 = \text{id}$.*
(iii) $\text{Aut}_{\text{tr}}(D)$ *acts simply transitively on the good types of D.*

Proof. (i) follows from the formula $g_i = t_{\lambda_i} w_{0\lambda_i} w_0$ ($i \in I_0$) given in Section 4.7. By (i) we have $(\sigma_i^{-1} \circ \sigma \circ \sigma_i)(0) = 0$, and so $\sigma_i^{-1} \circ \sigma \circ \sigma_i = \sigma_0 = \text{id}$. Thus (ii) holds, and (iii) is now clear. □

Proposition 4.8. *Let $\psi \in \text{Aut}(\Sigma)$.*

(i) *The image under ψ of a gallery in Σ is again a gallery in Σ.*
(ii) *A gallery in Σ is minimal if and only if its image under ψ is minimal.*
(iii) *There exists a unique $\sigma \in \text{Aut}(D)$ so that ψ maps galleries of type f to galleries of type $\sigma(f)$. If $\psi = w \in \tilde{W}$ then $\sigma \in \text{Aut}_{\text{tr}}(D)$. If $w = w' g_i$, where $w' \in W$, then $\sigma = \sigma_i$.*
(iv) *If $\psi \in \tilde{W}$ maps $\lambda \in P$ to $\mu \in P$, then the induced automorphism from (iii) is $\sigma = \sigma_m \circ \sigma_l^{-1}$, where $l = \tau(\lambda)$ and $m = \tau(\mu)$.*

Proof. (i) and (ii) are obvious.

(iii) The first statement follows easily from Proposition 4.6, and the remaining statements follow from the definition of $\text{Aut}_{\text{tr}}(D)$.

(iv) Since $\sigma(l) = m$, we have $(\sigma \circ \sigma_l)(0) = \sigma_m(0)$, and so $\sigma = \sigma_m \circ \sigma_l^{-1}$ by Proposition 4.7. □

Proposition 4.9. $x \mapsto -x$ *is an automorphism of Σ.*

Proof. The map $x \mapsto -x$ maps $\mathcal{H}$ to itself and is continuous, and so maps chambers to chambers. If $C \sim_i D$ and $C \neq D$ then there is only one $H \in \mathcal{H}$ separating C and D, and then $-H$ is the only hyperplane in $\mathcal{H}$ separating $-C$ and $-D$, and so $-C \sim_{i'} -D$ for some $i' \in I$. □

Definition 4.10. Let $\sigma_* \in \text{Aut}(D)$ be the automorphism of D induced by the automorphism $x \mapsto -x$ of Σ (see Proposition 4.9). Furthermore, for $\lambda \in P$ let $\lambda^* = w_0(-\lambda)$, where w_0

is the longest element of W_0. Finally, for $l \in I_P$ let $l^* = \tau(\lambda^*)$, where $\lambda \in P$ is any vertex with $\tau(\lambda) = l$.

We need to check that the definition of l^* is unambiguous. If $\tau(\lambda) = \tau(\mu)$, then $\lambda = w\mu$ for some $w \in W$. Since $W = W_0 \ltimes Q$ we have $w = w't_\gamma$ for some $w' \in W_0$ and $\gamma \in Q$, and so $-\lambda = -w'(\gamma + \mu) = w't_{-\gamma}(-\mu) = w''(-\mu)$ for some $w'' \in W$. Thus $\tau(-\lambda) = \tau(-\mu)$, and so $\tau(\lambda^*) = \tau(\mu^*)$.

Note that in general σ_* is not an element of $\mathrm{Aut}_{\mathrm{tr}}(D)$. In the BC_n case, σ_* is the identity, for the map $x \mapsto -x$ fixes the good type 0, implying that $\sigma_* = \mathrm{id}$ by direct consideration of the Coxeter graph.

Proposition 4.11. *If $\lambda \in P^+$, then $\lambda^* \in P^+$.*

Proof. Observe that $w_0(-\mathcal{S}_0) = \mathcal{S}_0$ since $-\mathcal{S}_0$ is a sector that lies on the opposite side of every wall to $\mathcal{S}_0$. Thus $w_0(-\lambda) \in P^+$. $\quad\square$

4.9. Special group elements and technical results

For $i \in I$, let $W_i = W_{I \setminus \{i\}}$ (this extends our notation for W_0). Given $\lambda \in P^+$, define t'_λ to be the unique element of W such that $t_\lambda = t'_\lambda g$ for some $g \in G$, and, using [4, VI, §1, Exercise 3], let w_λ be the unique minimum length representative of the double coset $W_0 t'_\lambda W_l$, where $l = \tau(\lambda)$. Fix a reduced word $f_\lambda \in I^*$ such that $s_{f_\lambda} = w_\lambda$.

Proposition 4.12. *Let $\lambda \in P^+$ and $i \in I_P$. Suppose that $\tau(\lambda) = l$, and write $j = \sigma_i(l)$. Then $g_j = g_i g_l$ and $t_\lambda = t'_\lambda g_l$.*

Proof. We see that $g_j = g_i g_l$ since the image of 0 under both functions is the same. Temporarily write $t_\lambda = t'_\lambda g_\lambda$, and so $g_\lambda = t'^{-1}_\lambda t_\lambda$. Observe that $g_\lambda(0) = v_k$ for some $k \in I_P$ (here v_k is the type k vertex of C_0). But $(t'^{-1}_\lambda t_\lambda)(0) = t'^{-1}_\lambda(\lambda) = v_l$, since t'_λ is type preserving. Thus $v_k = v_l$, so $k = l$, and so $g_\lambda = g_l$. $\quad\square$

Recall that $\sigma \in \mathrm{Aut}(D)$ induces an automorphism (which we also denote by σ) of W as in (1.2). From (4.8) we have the following.

Lemma 4.13. *Let $\lambda \in P$ and $l = \tau(\lambda)$. Then $g_l W_0 g_l^{-1} = W_l = \sigma_l(W_0)$, and so W_l is the stabiliser of the type l vertex v_l of C_0.*

Proposition 4.14. *Let $\lambda \in P^+$. Then:*

(i) $w_\lambda = t_\lambda w_{0\lambda} w_0 g_l^{-1} = t'_\lambda \sigma_l(w_{0\lambda} w_0)$, *where $l = \tau(\lambda)$, and $w_{0\lambda}$ and w_0 are the longest elements of $W_{0\lambda}$ and W_0 respectively.*
(ii) $\lambda \in w_\lambda \bar{C}_0$, *and $w_\lambda C_0$ is the unique chamber nearest C_0 with this property.*
(iii) $w_\lambda C_0 \subseteq \mathcal{S}_0$.

Proof. (i) By Proposition 4.12 and Lemma 4.13 we have $W_0 t_\lambda W_0 = W_0 t'_\lambda g_l W_0 = W_0 t'_\lambda W_l g_l$, and so the double coset $W_0 t_\lambda W_0$ has unique minimal length representative $m_\lambda = w_\lambda g_l$. By [21, (2.4.5)] (see also [25, (2.16)]) we have $m_\lambda = t_\lambda w_{0\lambda} w_0$, proving the first equality in (i). Then

$$w_\lambda = m_\lambda g_l^{-1} = t_\lambda w_{0\lambda} w_0 g_l^{-1} = t'_\lambda g_l w_{0\lambda} w_0 g_l^{-1} = t'_\lambda \sigma_l(w_{0\lambda} w_0).$$

(ii) With m_λ as above we have $m_\lambda(0) = (t_\lambda w_{0\lambda} w_0)(0) = \lambda$, so $\lambda \in m_\lambda \bar{C}_0$. Now $w_\lambda = m_\lambda g_l^{-1}$, and since $g_l^{-1} \in G$ fixes C_0 we have $\lambda \in w_\lambda \bar{C}_0$. To see that w_λ is the unique chamber nearest C_0 that contains λ in its closure, notice that by Lemma 4.13 the stabiliser of λ in W is $t'_\lambda W_l t'^{-1}_\lambda$, which acts simply transitively on the set of chambers containing λ in their closure. So if wC_0 is a chamber containing λ in its closure, then $wC_0 = (t'_\lambda w_l t'^{-1}_\lambda) t'_\lambda(C_0) = t'_\lambda w_l C_0$ for some $w_l \in W_l$. Thus $w = t'_\lambda w_l \in t'_\lambda W_l \subset W_0 t'_\lambda W_l$, and so $\ell(w_\lambda) \leqslant \ell(w)$. The uniqueness follows from [29, Theorem 2.9].

We now prove (iii). The result is clear if $\lambda = 0$, so let $\lambda \in P^+ \setminus \{0\}$. If $\lambda \in \mathcal{S}_0$ then $\mathcal{S}_0 \cap w_\lambda C_0 \neq \emptyset$, and so $w_\lambda C_0 \subseteq \mathcal{S}_0$ since $w_\lambda C_0$ is connected and contained in $E \setminus \bigcup_{H \in \mathcal{H}_0} H$.

Now suppose that $\lambda \in \bar{\mathcal{S}}_0 \setminus \mathcal{S}_0$, so $\lambda \in H_\alpha$ for some $\alpha \in B$. Let $C_0, C_1, \ldots, C_m = w_\lambda C_0$ be the gallery of type f_λ from C_0 to $w_\lambda C_0$. If $w_\lambda C_0 \not\subseteq \mathcal{S}_0$ then this gallery crosses the wall H_α, so let C_k be the first chamber on the opposite side of H_α to C_0. The sequence $C_0, \ldots, C_{k-1}, s_\alpha(C_k), \ldots, s_\alpha(w_\lambda C_0)$ joins 0 to λ as $s_\alpha(\lambda) = \lambda$. Since $C_{k-1} = s_\alpha(C_k)$, we can construct a gallery joining 0 to λ of length strictly less than m, a contradiction. $\square$

Each coset $wW_{0\lambda}$, $w \in W_0$, has a unique minimal length representative. To see this, notice that by Lemma 5.4, $W_{0\lambda}$ is the subgroup of W_0 generated by $S_{0\lambda} = \{s \in S_0 \mid s\lambda = \lambda\}$, and apply [4, IV, §1, Exercise 3]. We write W_0^λ for the set of minimal length representatives of elements of $W_0 / W_{0\lambda}$. The following proposition records some simple facts.

Proposition 4.15. *Let $\lambda \in P^+$ and write $l = \tau(\lambda)$. Then:*

(i) $t'_\lambda = w_\lambda w_l$ *for some $w_l \in W_l$, and $\ell(t'_\lambda) = \ell(w_\lambda) + \ell(w_l)$.*

(ii) *Each $w \in W_0$ can be written uniquely as $w = uv$ with $u \in W_0^\lambda$ and $v \in W_{0\lambda}$, and moreover $\ell(w) = \ell(u) + \ell(v)$.*

(iii) *For $v \in W_{0\lambda}$, $vw_\lambda = w_\lambda w_l \sigma_l(v) w_l^{-1}$ where $w_l \in W_l$ is as in (i). Moreover,*

$$\ell(vw_\lambda) = \ell(v) + \ell(w_\lambda) = \ell(w_\lambda) + \ell\big(w_l \sigma_l(v) w_l^{-1}\big).$$

(iv) *Each $w \in W_0 w_\lambda W_l$ can be written uniquely as $w = uw_\lambda w'$ for some $u \in W_0^\lambda$ and $w' \in W_l$, and moreover $\ell(w) = \ell(u) + \ell(w_\lambda) + \ell(w')$.*

Proof. (i) follows from the proof of Proposition 4.14 and [4, VI, §1, Exercise 3].

(ii) is immediate from the definition of W_0^λ, and [4, VI, §1, Exercise 3].

(iii) Observe first that $vt_\lambda = t_\lambda v$ in the extended affine Weyl group, for $vt_\lambda v^{-1} = t_{v\lambda}$ for all $v \in W_0$, and $t_{v\lambda} = t_\lambda$ if $v \in W_{0\lambda}$. Since $t_\lambda = t'_\lambda g_l$ (see Proposition 4.12) we have

$$vt'_\lambda = vt_\lambda g_l^{-1} = t_\lambda v g_l^{-1} = t'_\lambda \big(g_l v g_l^{-1}\big) = t'_\lambda \sigma_l(v),$$

and so from (i), $vw_\lambda = w_\lambda w_l \sigma_l(v) w_l^{-1}$. By [4, IV, §1, Exercise 3] we have $\ell(vw_\lambda) = \ell(v) + \ell(w_\lambda)$; in fact, $\ell(ww_\lambda) = \ell(w) + \ell(w_\lambda)$ for all $w \in W_0$. Observe now that $w s_\alpha w^{-1} = s_{w\alpha}$ for $w \in W_0$, and it follows that $\ell(w_l \sigma_l(v) w_l^{-1}) = \ell(v)$.

(iv) By [4, IV, §1, Exercise 3] each $w \in W_0 w_\lambda W_l$ can be written as $w = w_1 w_\lambda w_2$ for some $w_1 \in W_0$ and $w_2 \in W_l$ with $\ell(w) = \ell(w_1) + \ell(w_\lambda) + \ell(w_2)$. Write $w_1 = uv$ where $u \in W_0^\lambda$ and $v \in W_{0\lambda}$ as in (ii). Then by (iii)

$$w_1 w_\lambda w_2 = uv w_\lambda w_2 = u w_\lambda \big(w_l \sigma_l(v) w_l^{-1} w_2\big),$$

and so each $w \in W_0 w_\lambda W_l$ can be written as $w = u w_\lambda w'$ for some $u \in W_0^\lambda$ and $w' \in W_l$ with $\ell(w) = \ell(u) + \ell(w_\lambda) + \ell(w')$. Suppose that we have two such expressions $w = u_1 w_\lambda w_1' = u_2 w_\lambda w_2'$ where $u_1, u_2 \in W_0^\lambda$ and $w_1', w_2' \in W_l$. Write v_l for the type l vertex of C_0. Then $(u_1 w_\lambda w_l')(v_l) = (u_1 w_\lambda)(v_l) = u_1 \lambda$, and similarly $(u_2 w_\lambda w_2')(v_l) = u_2 \lambda$. Thus $u_1^{-1} u_2 \in W_{0\lambda}$, and so $u_1 W_{0\lambda} = u_2 W_{0\lambda}$, forcing $u_1 = u_2$. This clearly implies that $w_1' = w_2'$ too, completing the proof. $\quad\square$

Recall the definitions of σ_*, λ^* and l^* from Definition 4.10.

Proposition 4.16. *Let $\lambda \in P^+$ (so $\lambda^* \in P^+$ too), and write $\tau(\lambda) = l$.*

(i) $\sigma_*^2 = \text{id}$ *and* $\sigma_*(0) = 0$.
(ii) $\sigma_*(w_\lambda) = w_{\lambda^*}$ *and* $\sigma_*(l) = l^*$.
(iii) $\sigma_* \circ \sigma_i \circ \sigma_*^{-1} = \sigma_{i^*}$ *for all* $i \in I_P$.
(iv) $w_{\lambda^*} = \sigma_l^{-1}(w_\lambda^{-1})$.

Proof. (i) is clear, since $-(-x) = x$ for all $x \in E$.

(ii) Let ψ be the automorphism of Σ given by $\psi(x) = w_0(-x)$ for all $x \in E$. Then the automorphism of D induced by ψ is σ_* (see Proposition 4.6). Let $C_0, \ldots, C_m = w_\lambda C_0$ be the gallery of type f_λ in Σ starting at C_0, and so $\psi(C_0), \ldots, \psi(C_m)$ is a minimal gallery of type $\sigma_*(f_\lambda)$ (see Proposition 4.8). Observe that $\psi(C_0) = C_0$ and $\lambda^* \in \psi(\bar{C}_m)$. The gallery $\psi(C_0), \ldots, \psi(C_m)$ from C_0 to λ^* cannot be replaced by any shorter gallery joining C_0 and λ^*, for if so, by applying ψ^{-1} we could obtain a gallery from C_0 to λ of length $< \ell(w_\lambda)$. Thus $\psi(C_m) = C_{\lambda^*}$ by Proposition 4.14, and so $\sigma_*(f_\lambda) \sim f_{\lambda^*}$. Therefore $\sigma_*(w_\lambda) = w_{\lambda^*}$, and so $\sigma_*(l) = l^*$.

(iii) Since $\text{Aut}_{\text{tr}}(D)$ is normal in $\text{Aut}(D)$ (see (4.7)) we know that $\sigma_* \circ \sigma_i \circ \sigma_*^{-1} = \sigma_k$ for some $k \in I_P$. By (i) and (ii) we have $(\sigma_* \circ \sigma_i \circ \sigma_*^{-1})(0) = i^*$ and the result follows.

(iv) Let $C_0, \ldots, C_m$ be the gallery from (ii) and write $f_\lambda = i_1 \cdots i_m$. Then $C_m, \ldots, C_0$ is a gallery of type $\text{rev}(f_\lambda) = i_m \cdots i_1$ joining λ to 0. Let $\psi = w_0 \circ w_{0\lambda}^{-1} \circ t_{-\lambda} : \Sigma \to \Sigma$ where $w_{0\lambda}$ is the longest element of $W_{0\lambda}$. By Proposition 4.14(i) we have

$$\psi(C_m) = \big(w_0 \circ w_{0\lambda}^{-1} \circ t_{-\lambda} \circ w_\lambda\big)(C_0) = C_0.$$

Thus by Proposition 4.8 $C_0 = \psi(C_m), \ldots, \psi(C_0)$ is a gallery of type $\sigma_l^{-1}(\mathrm{rev}(f_\lambda))$ joining 0 to λ^* (since $\lambda^* \in \psi(\bar{C}_0)$). Since no shorter gallery joining 0 to λ^* exists (for if so apply ψ^{-1} to obtain a contradiction) it follows that

$$w_{\lambda^*} = \sigma_l^{-1}(s_{\mathrm{rev}(f_\lambda)}) = \sigma_l^{-1}\big(s_{f_\lambda}^{-1}\big) = \sigma_l^{-1}\big(w_\lambda^{-1}\big). \qquad \square$$

4.10. Affine buildings

A building $\mathcal{X}$ is called *affine* if the associated Coxeter group W is an affine Weyl group. To study the algebra $\mathcal{A}$ of the next section, it is convenient to associate a root system R to each irreducible locally finite regular affine building. If $\mathcal{X}$ is of type W, we wish to choose R so that (among other things) (i) the affine Weyl group of R is isomorphic to W, and (ii) $q_{\sigma(i)} = q_i$ for all $i \in I$ and $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$ (note that $\mathrm{Aut}_{\mathrm{tr}}(D)$ depends on the choice of R, see (4.6)).

It turns out (as should be expected) that the choice of R is in most cases straight forward; for example, if $\mathcal{X}$ is of type $\tilde{F}_4$ then choose R to be a root system of type F_4 (and call $\mathcal{X}$ an *affine building of type F_4*). The regular buildings of types $\tilde{A}_1$ and $\tilde{C}_n$ ($n \geqslant 2$) are the only exceptions to this rule, and in these cases the nonreduced root systems BC_n ($n \geqslant 1$) play an important role. Let us briefly describe why.

Using Proposition 2.1(ii) we see that the parameters of a regular $\tilde{C}_n$ ($n \geqslant 2$) building must be as follows:

$$\underset{\bullet}{q_0} \ 4 \ \underset{\bullet}{q_1} \quad \underset{\bullet}{q_1} \quad \bullet \ \bullet \ \bullet \quad \underset{\bullet}{q_1} \quad \underset{\bullet}{q_1} \ 4 \ \underset{\bullet}{q_n}$$

If we choose R to be a C_n root system then the automorphism $\sigma_n \in \mathrm{Aut}_{\mathrm{tr}}(D)$ interchanges the left most and right most nodes, and so condition (ii) is not satisfied (unless $q_0 = q_n$). If, however, we take R to be a BC_n root system, then $\mathrm{Aut}_{\mathrm{tr}}(D) = \{\mathrm{id}\}$, and so both conditions (i) and (ii) are satisfied.

Thus, in order to facilitate the statements of later results, we rename regular $\tilde{C}_n$ ($n \geqslant 2$) buildings, and call them *affine buildings of type BC_n* (or $\widetilde{BC}_n$ ($n \geqslant 2$) buildings). We reserve the name '$\tilde{C}_n$ building' for the special case when $q_0 = q_n$ in the above parameter system. For a similar reason we rename regular $\tilde{A}_1$ buildings (which are *semi-homogeneous trees*) and call them *affine buildings of type BC_1* (or $\widetilde{BC}_1$ buildings), and reserve the name '$\tilde{A}_1$ building' for homogeneous trees. With these conventions we make the following definitions.

Definition 4.17. Let $\mathcal{X}$ be an affine building of type R with vertex set V, and let $\Sigma = \Sigma(R)$. Let $V_{\mathrm{sp}}(\Sigma)$ denote the set of all special vertices of Σ (see Section 4.5), and let $I_{\mathrm{sp}} = \{\tau(\lambda) \mid \lambda \in V_{\mathrm{sp}}(\Sigma)\}$.

(i) A vertex $x \in V$ is said to be *special* if $\tau(x) \in I_{\mathrm{sp}}$. We write V_{sp} for the set of all special vertices of $\mathcal{X}$.

(ii) A vertex $x \in V$ is said to be *good* if $\tau(x) \in I_P$, where I_P is as in Section 4.5. We write V_P for the set of all good vertices of $\mathcal{X}$.

Clearly $V_P \subset V_{\mathrm{sp}}$. In fact if R is reduced, then by the comments made in Section 4.5, $V_P = V_{\mathrm{sp}}$. If R is nonreduced (so R is of type BC_n for some $n \geqslant 1$), then V_P is the set of all type 0 vertices of $\mathcal{X}$, whereas V_{sp} is the set of all type 0 and type n vertices of $\mathcal{X}$.

Proposition 4.18. *A vertex $x \in V$ is good if and only if there exists an apartment $\mathcal{A}$ containing x and a type preserving isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) \in P$.*

Proof. Let $x \in V_P$, and choose any apartment $\mathcal{A}$ containing x. Let $\psi : \mathcal{A} \to \Sigma$ be a type preserving isomorphism (from the building axioms). Then $\psi(x)$ is a vertex in Σ with type $\tau(x) \in I_P$, and so $\psi(x) \in P$. The converse is obvious. $\quad\square$

Remark 4.19. We note that *infinite distance regular graphs* are just $\widetilde{BC}_1$ buildings in very thin disguise. To see the connection, given any $p, q \geqslant 1$, construct a $\widetilde{BC}_1$ building (that is, a semi-homogeneous tree) with parameters $q_0 = p$ and $q_1 = q$. Construct a new graph $\Gamma_{p,q}$ with vertex set V_P and vertices $x, y \in V_P$ connected by an edge if and only if $d(x, y) = 2$. It is simple to see that $\Gamma_{p,q}$ is the (graph) free product $\mathbb{K}_q * \cdots * \mathbb{K}_q$ (p copies) where $\mathbb{K}_q$ is the complete graph on q letters. By the classification [15,22] $\Gamma_{p,q}$ is infinite distance regular, and all infinite distance regular graphs occur in this way.

Recall the definition of $\mathrm{Aut}_q(D)$ from Definition 3.11.

Theorem 4.20. *The diagrams in Appendix* A *characterise the parameter systems of the locally finite regular affine buildings. In each case* $\mathrm{Aut}_{\mathrm{tr}}(D) \cup \{\sigma_*\} \subseteq \mathrm{Aut}_q(D)$.

Proof. These parameter systems are found case by case using Proposition 2.1(ii) and the classification of the irreducible affine Coxeter graphs. Note that $\mathrm{Aut}_{\mathrm{tr}}(D) \cup \{\sigma_*\} = \{\mathrm{id}\}$ if $\mathcal{X}$ is a $\widetilde{BC}_n$ building. Thus the final result follows by considering each Coxeter graph. $\quad\square$

5. Vertex set operators and vertex regularity

Let $\mathcal{X}$ be a locally finite regular affine building of type R (see Section 4.10). Recall (Definition 4.17) that we write V_P for the set of all good vertices of $\mathcal{X}$.

For each $\lambda \in P^+$ we will define an averaging operator A_λ acting on the space of all functions $f : V_P \to \mathbb{C}$, and we will introduce an algebra $\mathcal{A}$ of these operators. The operators A_λ were defined in [32, II, §1.1.2, Exercise 3] for homogeneous trees, [8,23] for $\tilde{A}_2$ buildings, and [7] for $\tilde{A}_n$ buildings. Our definition gives the generalisation of the operators A_λ and the algebra $\mathcal{A}$ to all (irreducible) root systems.

5.1. Initial observations

Recall the definition of type preserving isomorphisms of simplicial complexes.

Definition 5.1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be apartments of $\mathcal{X}$.

(i) An isomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ is called *type-rotating* if it is of the form $\psi = \psi_2^{-1} \circ w \circ \psi_1$ where $\psi_1 : \mathcal{A}_1 \to \Sigma$ and $\psi_2 : \mathcal{A}_2 \to \Sigma$ are type preserving isomorphisms, and $w \in \tilde{W}$.

(ii) We have an analogous definition to (i) for isomorphisms $\psi : \mathcal{A}_1 \to \Sigma$ by omitting ψ_2.

Proposition 5.2. *Let $\mathcal{A}$, $\mathcal{A}'$ be any apartments and suppose that $\psi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism. Then*:

(i) *The image under ψ of a gallery in $\mathcal{A}$ is a gallery in $\mathcal{A}'$.*

(ii) *A gallery in $\mathcal{A}$ is minimal if and only if its image under ψ is minimal in $\mathcal{A}'$.*

(iii) *There exists a unique $\sigma \in \mathrm{Aut}(D)$ so that ψ maps galleries of type f in $\mathcal{A}$ to galleries of type $\sigma(f)$ in $\mathcal{A}'$. If ψ is type rotating, then $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$, and $(\tau \circ \psi)(x) = (\sigma \circ \tau)(x)$ for all vertices x of $\mathcal{A}$.*

(iv) *If ψ is type rotating and maps a type $i \in I_P$ vertex in $\mathcal{A}$ to a type $j \in I_P$ vertex in $\mathcal{A}'$, then the induced automorphism from* (iii) *is $\sigma = \sigma_j \circ \sigma_i^{-1}$.*

Proof. This follows from Proposition 4.8 and the definition of type rotating isomorphisms. $\square$

Lemma 5.3. *Suppose $x \in V_P$ is contained in the apartments $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{X}$, and suppose that $\psi : \mathcal{A} \to \Sigma$ and $\psi' : \mathcal{A}' \to \Sigma$ are type rotating isomorphisms such that $\psi(x) = 0 = \psi'(x)$. Let $\psi'' : \mathcal{A} \to \mathcal{A}'$ be a type preserving isomorphism mapping x to x (the existence of which is guaranteed by Definition 4.1). Then $\phi = \psi' \circ \psi'' \circ \psi^{-1}$ is in W_0.*

Proof. Observe that $\phi : \Sigma \to \Sigma$ has $\phi(0) = 0$. Since ψ and ψ' are type rotating isomorphisms we have $\psi = w \circ \psi_1$ and $\psi' = w' \circ \psi_1'$ for some $w, w' \in \tilde{W}$ and $\psi_1 : \mathcal{A} \to \Sigma$, $\psi_1' : \mathcal{A}' \to \Sigma$ type preserving isomorphisms. Therefore,

$$\phi = w' \circ \psi_1' \circ \psi'' \circ \psi_1^{-1} \circ w^{-1} = w' \circ \phi' \circ w^{-1}, \quad \text{say.}$$

Now $\phi' = \psi_1' \circ \psi'' \circ \psi_1^{-1} : \Sigma \to \Sigma$ is a type preserving automorphism, as it is a composition of type preserving isomorphisms. By [29, Lemma 2.2] we have $\phi' = v$ for some $v \in W$, and hence $\phi = w' \circ v \circ w^{-1} \in \tilde{W}$. Since $\phi(0) = 0$ and $\tilde{W} = W_0 \ltimes P$ we in fact have $\phi \in W_0$, completing the proof. $\square$

5.2. The sets $V_\lambda(x)$

The following definition gives the analogue of the partition $\{\mathcal{C}_w(a)\}_{w \in W}$ used for the chamber set of $\mathcal{X}$. Let us first record the following lemma from [5, p. 24] (or [13, §10.3, Lemma B]). Recall the definition of the fundamental sector $\mathcal{S}_0$ from (4.3).

Lemma 5.4. *Let $w \in W_0$ and $\lambda \in E$. If $\lambda' = w\lambda \in \bar{\mathcal{S}}_0 \cap w\bar{\mathcal{S}}_0$ then $\lambda' = \lambda$, and $w \in \langle \{s_i \mid s_i \lambda = \lambda\} \rangle$.*

Definition 5.5. Given $x \in V_P$ and $\lambda \in P^+$, we define $V_\lambda(x)$ to be the set of all $y \in V_P$ such that there exists an apartment $\mathcal{A}$ containing x and y and a type rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$.

Proposition 5.6. *Let $V_\lambda(x)$ be as in Definition 5.5.*

  (i) *Given $x, y \in V_P$, there exists some $\lambda \in P^+$ such that $y \in V_\lambda(x)$.*
 (ii) *If $y \in V_\lambda(x) \cap V_{\lambda'}(x)$ then $\lambda = \lambda'$.*
(iii) *Let $y \in V_\lambda(x)$. If $\mathcal{A}$ is any apartment containing x and y, then there exists a type-rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$.*

Proof. First we prove (i). By Definition 4.1 there exists an apartment $\mathcal{A}$ containing x and y and a type preserving isomorphism $\psi_1 : \mathcal{A} \to \Sigma$. Let $\mu = \psi_1(x)$ and $\nu = \psi_1(y)$, so $\mu, \nu \in P$. There exists a $w \in W_0$ such that $w(\nu - \mu) \in \bar{\mathcal{S}}_0 \cap P$ [13, p. 55, Exercise 14], and so the isomorphism $\psi = w \circ t_{-\mu} \circ \psi_1$ satisfies $\psi(x) = 0$ and $\psi(y) = w(\nu - \mu) \in P^+$, proving (i).

We now prove (ii). Suppose that there are apartments $\mathcal{A}$ and $\mathcal{A}'$ containing x and y, and type-rotating isomorphisms $\psi : \mathcal{A} \to \Sigma$ and $\psi' : \mathcal{A}' \to \Sigma$ such that $\psi(x) = \psi'(x) = 0$ and $\psi(y) = \lambda \in P^+$ and $\psi'(y) = \lambda' \in P^+$. We claim that $\lambda = \lambda'$.

By Definition 4.1(iii)$'$ there exists a type preserving isomorphism $\psi'' : \mathcal{A} \to \mathcal{A}'$ which fixes x and y. Then $\phi = \psi' \circ \psi'' \circ \psi^{-1} : \Sigma \to \Sigma$ is a type-rotating automorphism of Σ that fixes 0 and maps λ to λ'. By Lemma 5.3 we have $\phi = w$ for some $w \in W_0$, and so we have $\lambda' = w\lambda \in \bar{\mathcal{S}}_0 \cap w\bar{\mathcal{S}}_0$. Thus by Lemma 5.4 we have $\lambda' = \lambda$.

Note first that (iii) is not immediate from the definition of $V_\lambda(x)$. To prove (iii), by the definition of $V_\lambda(x)$ there exists an apartment $\mathcal{A}'$ containing x and y, and a type-rotating isomorphism $\psi' : \mathcal{A}' \to \Sigma$ such that $\psi'(x) = 0$ and $\psi'(y) = \lambda$. Then by Definition 4.1(iii)$'$ there is a type preserving isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ fixing x and y. Then $\psi = \psi' \circ \phi : \mathcal{A} \to \Sigma$ has the required properties. $\square$

Remark 5.7. Note that the assumption that ψ is type-rotating in Definition 5.5 is essential for Proposition 5.6(ii) to hold. To see this we only need to look at an apartment of an $\tilde{A}_2$ building. The map $a_1\lambda_1 + a_2\lambda_2 \mapsto a_1\lambda_2 + a_2\lambda_1$ is an automorphism which maps λ_1 to λ_2. Thus if we omitted the hypothesis that ψ is type-rotating in Definition 5.5, part (ii) of Proposition 5.6 would be false.

Proposition 5.8. *If $y \in V_\lambda(x)$, then $x \in V_{\lambda^*}(y)$ where λ^* is as in Definition 4.10.*

Proof. If $\psi : \mathcal{A} \to \Sigma$ is a type rotating isomorphism mapping x to 0 and y to λ, then $w_0 \circ t_{-\lambda} \circ \psi : \mathcal{A} \to \Sigma$ is a type rotating isomorphism mapping y to 0 and x to $\lambda^* = w_0(-\lambda) \in P^+$ (see Proposition 4.11). $\square$

Lemma 5.9. *Let $x \in V_P$ and $\lambda \in P^+$. If $y, y' \in V_\lambda(x)$ then $\tau(y) = \tau(y')$.*

Proof. Let $\mathcal{A}$ be an apartment containing x and y, and $\mathcal{A}'$ be an apartment containing x and y'. Let $\psi : \mathcal{A} \to \Sigma$ and $\psi' : \mathcal{A}' \to \Sigma$ be type rotating isomorphisms with $\psi(x) = \psi'(x) = 0$ and $\psi(y) = \psi'(y') = \lambda$. Thus $\chi = \psi'^{-1} \circ \psi : \mathcal{A} \to \mathcal{A}'$ is a type preserving

automorphism since $\chi(x) = x$ (see Proposition 4.7). Since $\chi(y) = y'$ we have $\tau(y) = \tau(y')$.   □

In light of the above lemma we define $\tau(V_\lambda(x)) = \tau(y)$ for any $y \in V_\lambda(x)$.

Clearly the sets $V_\lambda(x)$ are considerably more complicated objects than the sets $\mathcal{C}_w(a)$. The following theorem provides an important connection between the sets $V_\lambda(x)$ and $\mathcal{C}_w(a)$ that will be relied on heavily in subsequent work. Given a chamber $c \in \mathcal{C}$ and an index $i \in I$, we define $\pi_i(c)$ to be the type i vertex of c. For the following theorem the reader is reminded of the definition of $w_\lambda \in W$ and $f_\lambda \in I^*$ from Section 4.9.

Theorem 5.10. *Let $x \in V_P$ and $\lambda \in P^+$. Suppose $\tau(x) = i$ and $\tau(V_\lambda(x)) = j$, and let $a \in \mathcal{C}$ be any chamber with $\pi_i(a) = x$. Then*

$$\{b \in \mathcal{C} \mid \pi_j(b) \in V_\lambda(x)\} = \bigcup_{w \in W_i \sigma_i(w_\lambda) W_j} \mathcal{C}_w(a),$$

where the union is disjoint.

Proof. Suppose first that $y = \pi_j(b) \in V_\lambda(x)$. Let $a = c_0, c_1, \ldots, c_n = b$ be a minimal gallery from a to b of type f, say. By [29, Theorem 3.8], all the c_k lie in some apartment, $\mathcal{A}$, say. Let $\psi : \mathcal{A} \to \Sigma$ be a type rotating isomorphism such that $\psi(x) = 0$ and $\psi(y) = \lambda$. Then $\psi(c_0), \psi(c_1), \ldots, \psi(c_n)$ is a minimal gallery of type $\sigma_i^{-1}(f)$ by Proposition 5.2.

Recall the definition of the fundamental chamber C_0 from (4.2). Since 0 is a vertex of $\psi(c_0)$, we can construct a gallery from $\psi(c_0)$ to C_0 of type e_1, say, where $s_{e_1} \in W_0$. Similarly there is a gallery from $w_\lambda C_0$ to $\psi(c_n)$ of type e_2, where $s_{e_2} \in W_{\sigma_i^{-1}(j)}$. Thus we have a gallery

$$\psi(c_0) \xrightarrow{e_1} C_0 \xrightarrow{f_\lambda} w_\lambda C_0 \xrightarrow{e_2} \psi(c_n)$$

of type $e_1 f_\lambda e_2$. Since Σ is a Coxeter complex, galleries (reduced or not) from one chamber to another of types f_1 and f_2, say, satisfy $s_{f_1} = s_{f_2}$ [29, p. 12], so $s_{\sigma_i^{-1}(f)} = s_{e_1 f_\lambda e_2}$. Thus

$$\delta(a, b) = s_f = \sigma_i(s_{\sigma_i^{-1}(f)}) = \sigma_i(s_{e_1 f_\lambda e_2}) = s_{e_1'} s_{\sigma_i(f_\lambda)} s_{e_2'},$$

where $e_1' \in W_i$ and $e_2' \in W_j$. Thus $b \in \mathcal{C}_w(a)$ for some $w \in W_i \sigma_i(w_\lambda) W_j$.

Now suppose that $b \in \mathcal{C}_w(a)$ for some $w \in W_i \sigma_i(w_\lambda) W_j$. Let $y = \pi_j(b)$. By [29, p. 35, Exercise 1], there exists a gallery of type $e_1' \sigma_i(f_\lambda) e_2'$ from a to b where $e_1' \in W_i$ and $e_2' \in W_j$. Let $c_k, c_{k+1}, \ldots, c_l$ be the subgallery of type $\sigma_i(f_\lambda)$. Note that $\pi_i(c_k) = x$ and $\pi_j(c_l) = y$. Observe that $\sigma_i(f_\lambda)$ is reduced since $\sigma_i \in \mathrm{Aut}(D)$, and so all of the chambers c_m, $k \leqslant m \leqslant l$, lie in an apartment $\mathcal{A}$, say. Let $\psi : \mathcal{A} \to \Sigma$ be a type rotating isomorphism such that $\psi(x) = 0$. Thus $\psi(c_k), \ldots, \psi(c_l)$ is a gallery of type f_λ in Σ (Proposition 5.2). Since W_0 acts transitively on the chambers $C \in \mathcal{C}(\Sigma)$ with $0 \in \bar{C}$ (Lemma 4.2) there exists $w \in W_0$ such that $w(\psi(c_k)) = C_0$. Then $\psi' = w \circ \psi : \mathcal{A} \to \Sigma$ is a type rotating isomorphism that takes the gallery $c_k, \ldots, c_l$ in $\mathcal{A}$ of type $\sigma_i(f_\lambda)$ to a gallery $C_0, \ldots, \psi'(c_l)$ of type f_λ. But in a Coxeter complex there is only one gallery of each type. So $\psi'(c_l)$ must be $w_\lambda(C_0)$, and by considering types $\psi'(y) = \lambda$, and so $y \in V_\lambda(x)$.   □

For $x \in V$ we write $\mathrm{st}(x)$ for the set of all chambers that have x as a vertex. Recall the definition of Poincaré polynomials from Definition 2.6.

Lemma 5.11. *Let $x \in V_P$. Then $|\mathrm{st}(x)| = W_0(q)$. In particular, this value is independent of the particular $x \in V_P$.*

Proof. Suppose $\tau(x) = i \in I_P$ and let c_0 be any chamber that has x as a vertex. Then

$$\mathrm{st}(x) = \left\{ c \in \mathcal{C} \mid \delta(c_0, c) \in W_i \right\} = \bigcup_{w \in W_i} \mathcal{C}_w(c_0)$$

where the union is disjoint, and so $|\mathrm{st}(x)| = \sum_{w \in W_i} q_w$. Theorem 4.20 now shows that

$$\left| \mathrm{st}(x) \right| = \sum_{w \in W_0} q_{\sigma_i(w)} = \sum_{w \in W_0} q_w = W_0(q). \qquad \square$$

Note that if the hypothesis 'let $x \in V_P$' in Lemma 5.11 is replaced by the hypothesis 'let x be a special vertex', then in the nonreduced case it is no longer true in general that $|\mathrm{st}(x)| = W_0(q)$.

5.3. The cardinalities $|V_\lambda(x)|$

In this subsection we will find a closed form for $|V_\lambda(x)|$. We need to return to the operators B_w introduced in Section 3.

For each $i \in I$ define an element $\mathbb{1}_i \in \mathcal{B}$ by

$$\mathbb{1}_i = \frac{1}{W_i(q)} \sum_{w \in W_i} q_w B_w. \tag{5.1}$$

Lemma 5.12. *Let $i \in I$. Then $\mathbb{1}_i B_w = B_w \mathbb{1}_i = \mathbb{1}_i$ for all $w \in W_i$, and $\mathbb{1}_i^2 = \mathbb{1}_i$.*

Proof. Suppose s is a generator of W_i and set $W_i^{\pm} = \{w \in W_i \mid \ell(ws) = \ell(w) \pm 1\}$. Then

$$
\begin{aligned}
W_i(q)\mathbb{1}_i B_s &= \sum_{w \in W_i^+} q_w B_{ws} + \sum_{w' \in W_i^-} q_w \left(\frac{1}{q_s} B_{ws} + \left(1 - \frac{1}{q_s}\right) B_w \right) \\
&= \sum_{w \in W_i^-} \frac{q_w}{q_s} B_w + \sum_{w' \in W_i^-} q_w \left(\frac{1}{q_s} B_{ws} + \left(1 - \frac{1}{q_s}\right) B_w \right) \\
&= \sum_{w \in W_i^-} \left(\frac{q_w}{q_s} B_{ws} + q_w B_w \right) \\
&= \sum_{w \in W_i^+} q_w B_w + \sum_{w \in W_i^-} q_w B_w = W_i(q)\mathbb{1}_i.
\end{aligned}
$$

A similar calculation works for $B_s \mathbb{1}_i$ too. It follows that $\mathbb{1}_i B_w = B_w \mathbb{1}_i = \mathbb{1}_i$ for all $w \in W_i$ and so $\mathbb{1}_i^2 = \mathbb{1}_i$. $\square$

Recall the definition of $W_{0\lambda}$ from (4.4).

Theorem 5.13. *Let $\lambda \in P^+$ and write $l = \tau(\lambda)$. Then*

$$\sum_{w \in W_0 w_\lambda W_l} q_w B_w = \frac{W_0^2(q)}{W_{0\lambda}(q)} q_{w_\lambda} \mathbb{1}_0 B_{w_\lambda} \mathbb{1}_l.$$

Proof. Recall from Corollary 3.5 that $B_{w_1} B_{w_2} = B_{w_1 w_2}$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Then by Proposition 4.15(ii), Proposition 4.15(iii), Lemma 5.12 and Proposition 4.15(iv) (in that order)

$$\mathbb{1}_0 B_{w_\lambda} \mathbb{1}_l = \frac{1}{W_0(q)} \sum_{u \in W_0^\lambda} \sum_{v \in W_{0\lambda}} q_u q_v B_u B_v B_{w_\lambda} \mathbb{1}_l$$

$$= \frac{1}{W_0(q)} \sum_{u \in W_0^\lambda} \sum_{v \in W_{0\lambda}} q_u q_v B_u B_{w_\lambda} B_{w_l \sigma_l(v) w_l^{-1}} \mathbb{1}_l$$

$$= \frac{1}{W_0(q)} \sum_{u \in W_0^\lambda} \sum_{v \in W_{0\lambda}} q_u q_v B_u B_{w_\lambda} \mathbb{1}_l$$

$$= \frac{W_{0\lambda}(q)}{W_0(q) W_l(q)} q_{w_\lambda}^{-1} \sum_{w \in W_0 w_\lambda W_l} q_w B_w,$$

and the result follows, since

$$W_l(q) = \sum_{w \in W_l} q_w = \sum_{w \in W_0} q_{\sigma_l(w)} = W_0(q)$$

by Proposition 4.20. $\square$

Lemma 5.14. *Let $\lambda \in P^+$, $x \in V_P$, and $y \in V_\lambda(x)$. Write $\tau(x) = i$, $\tau(y) = j$ and $\tau(\lambda) = l$. Then $\sigma_i^{-1}(j) = l$, and so $\sigma_j = \sigma_i \circ \sigma_l$.*

Proof. Since $y \in V_\lambda(x)$, there exists an apartment $\mathcal{A}$ containing x and y and a type rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$. Since $\psi(x) = 0$, the σ from Proposition 5.2(iii) maps i to 0 and so is σ_i^{-1}. Thus $\lambda = \psi(y)$ has type $\sigma(j) = \sigma_i^{-1}(j)$ and so $l = \sigma_i^{-1}(j)$. Thus $\sigma_j(0) = (\sigma_i \circ \sigma_l)(0)$, and so $\sigma_j = \sigma_i \circ \sigma_l$. $\square$

Theorem 5.15. *Let $x \in V_P$ and $\lambda \in P^+$ with $\tau(\lambda) = l \in I_P$. Then*

$$\big|V_\lambda(x)\big| = \frac{1}{W_0(q)} \sum_{w \in W_0 w_\lambda W_l} q_w = \frac{W_0(q)}{W_{0\lambda}(q)} q_{w_\lambda} = \big|V_{\lambda^*}(x)\big|.$$

Proof. Suppose $\tau(x) = i \in I_P$ and $\tau(y) = j \in I_P$ for all $y \in V_\lambda(x)$. Let $\mathcal{C}_\lambda(x) = \{c \in \mathcal{C} \mid \pi_j(c) \in V_\lambda(x)\}$ and construct a map $\psi : \mathcal{C}_\lambda(x) \to V_\lambda(x)$ by $c \mapsto \pi_j(c)$ for all $c \in \mathcal{C}_\lambda(x)$. Clearly ψ is surjective.

Observe that for each $y \in V_\lambda(x)$ the set $\{c \in \mathcal{C}_\lambda(x) \mid \psi(c) = y\}$ has $|\mathrm{st}(y)|$ distinct elements, and so by Lemma 5.11 we see that $\psi : \mathcal{C}_\lambda(x) \to V_\lambda(x)$ is a $W_0(q)$-to-one surjection. Let $c_0 \in \mathcal{C}$ be any chamber that has x as a vertex. Then by the above and Theorem 5.10 we have

$$\bigl|V_\lambda(x)\bigr| = \frac{|\mathcal{C}_\lambda(x)|}{W_0(q)} = \frac{1}{W_0(q)} \sum_{w \in W_i \sigma_i(w_\lambda) W_j} \bigl|\mathcal{C}_w(c_0)\bigr| = \frac{1}{W_0(q)} \sum_{w \in W_i \sigma_i(w_\lambda) W_j} q_w.$$

Since $\sigma_i^{-1}(j) = l$ (Lemma 5.14) we have $W_i \sigma_i(w_\lambda) W_j = \sigma_i(W_0 w_\lambda W_l)$, and so by Theorem 4.20

$$\bigl|V_\lambda(x)\bigr| = \frac{1}{W_0(q)} \sum_{w \in W_0 w_\lambda W_l} q_{\sigma_i(w)} = \frac{1}{W_0(q)} \sum_{w \in W_0 w_\lambda W_l} q_w.$$

Let $1_\mathcal{C} : \mathcal{C} \to \{1\}$ be the constant function. Then $(B_w 1_\mathcal{C})(c) = 1$ for all $c \in \mathcal{C}$, and so we compute $(\mathbb{1}_l 1_\mathcal{C})(c) = 1$ for all $c \in \mathcal{C}$. Thus by Theorem 5.13

$$\sum_{w \in W_0 w_\lambda W_l} q_w = \frac{W_0^2(q)}{W_{0\lambda}(q)} q_{w_\lambda}.$$

Now, by Proposition 4.16 and Theorem 4.20 we have

$$\bigl|V_{\lambda^*}(x)\bigr| = \frac{1}{W_0(q)} \sum_{w \in \sigma_*(W_0 w_\lambda W_l)} q_w = \frac{1}{W_0(q)} \sum_{w \in W_0 w_\lambda W_l} q_w = \bigl|V_\lambda(x)\bigr|. \qquad \square$$

Definition 5.16. For $\lambda \in P^+$ we define $N_\lambda = |V_\lambda(x)|$, which is independent of $x \in V_P$ by Theorem 5.15.

By the above we have $N_\lambda = N_{\lambda^*}$.

5.4. The operators A_λ and the algebra $\mathscr{A}$

We now define the *vertex set averaging operators* on $\mathcal{X}$.

Definition 5.17. For each $\lambda \in P^+$, define an operator A_λ, acting on the space of all functions $f : V_P \to \mathbb{C}$ as in (0.2).

Lemma 5.18. *The operators A_λ are linearly independent.*

Proof. Suppose we have a relation $\sum_{\lambda \in P^+} a_\lambda A_\lambda = 0$, and fix $x, y \in V_P$ with $y \in V_\mu(x)$. Then writing δ_y for the function taking the value 1 at y and 0 elsewhere,

$$0 = \sum_{\lambda \in P^+} a_\lambda (A_\lambda \delta_y)(x) = \sum_{\lambda \in P^+} a_\lambda N_\lambda^{-1} \delta_{\lambda,\mu} = a_\mu N_\mu^{-1},$$

and so $a_\mu = 0$. □

Following the same technique used in (3.1) for the chamber set averaging operators, we have

$$(A_\lambda A_\mu f)(x) = \frac{1}{N_\lambda N_\mu} \sum_{y \in V_P} \big| V_\lambda(x) \cap V_{\mu^*}(y) \big| f(y) \quad \text{for all } x \in V_P. \tag{5.2}$$

Our immediate goal now is to understand the cardinalities $|V_\lambda(x) \cap V_{\mu^*}(y)|$.

Definition 5.19. We say that $\mathcal{X}$ is *vertex regular* if, for all $\lambda, \mu, \nu \in P^+$,

$$\big| V_\lambda(x) \cap V_{\mu^*}(y) \big| = \big| V_\lambda(x') \cap V_{\mu^*}(y') \big| \quad \text{whenever } y \in V_\nu(x) \text{ and } y' \in V_\nu(x'),$$

and *strongly vertex regular* if for all $\lambda, \mu, \nu \in P^+$

$$\big| V_\lambda(x) \cap V_{\mu^*}(y) \big| = \big| V_{\lambda^*}(x') \cap V_\mu(y') \big| \quad \text{whenever } y \in V_\nu(x) \text{ and } y' \in V_{\nu^*}(x').$$

Strong vertex regularity implies vertex regularity. To see this, suppose we are given $x, y, x', y' \in V_P$ with $y \in V_\nu(x)$ and $y' \in V_\nu(x')$, and choose any pair $x'', y'' \in V_P$ with $y'' \in V_{\nu^*}(x'')$. Then if strong vertex regularity holds, we have

$$\big| V_\lambda(x) \cap V_{\mu^*}(y) \big| = \big| V_{\lambda^*}(x'') \cap V_\mu(y'') \big| = \big| V_\lambda(x') \cap V_{\mu^*}(y') \big|.$$

Lemma 5.20. *Let $y \in V_\nu(x)$ and suppose that $z \in V_\lambda(x) \cap V_{\mu^*}(y)$. Write $\tau(x) = i$, $\tau(y) = j$, $\tau(z) = k$, $\tau(\lambda) = l$, $\tau(\mu) = m$, and $\tau(\nu) = n$.*

(i) *$\sigma_i^{-1}(k) = l$, $\sigma_k^{-1}(j) = m$ and $\sigma_i^{-1}(j) = n$. Thus $\sigma_i^{-1} \circ \sigma_k = \sigma_l$, $\sigma_k^{-1} \circ \sigma_j = \sigma_m$ and $\sigma_i^{-1} \circ \sigma_j = \sigma_n$.*
(ii) *$\sigma_n = \sigma_l \circ \sigma_m$.*

Proof. (i) follows immediately from Lemma 5.14. To prove (ii), we have

$$\sigma_l \circ \sigma_m = \sigma_i^{-1} \circ \sigma_k \circ \sigma_k^{-1} \circ \sigma_j = \sigma_i^{-1} \circ \sigma_j = \sigma_n. \qquad □$$

Recall the definition of the automorphism $\sigma_* \in \mathrm{Aut}(D)$ from Section 4.8.

Theorem 5.21. *$\mathcal{X}$ is strongly vertex regular.*

Proof. Let $x, y \in V_P$ with $y \in V_\nu(x)$ and suppose that $z \in V_\lambda(x) \cap V_{\mu^*}(y)$. Let $\tau(x) = i$, $\tau(y) = j$ and $\tau(z) = k$. With the notation used in the proof of Theorem 5.15, define a map $\psi : \mathcal{C}_\lambda(x) \cap \mathcal{C}_{\mu^*}(y) \to V_\lambda(x) \cap V_{\mu^*}(y)$ by the rule $\psi(c) = \pi_k(c)$. As in the proof of Theorem 5.15 we see that this is a $W_0(q)$-to-one surjection, and thus by Theorem 5.10

$$\big| V_\lambda(x) \cap V_{\mu^*}(y) \big| = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_i \sigma_i(w_\lambda) W_k \\ w_2 \in W_j \sigma_j(w_{\mu^*}) W_k}} \big| \mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b) \big|,$$

where a and b are any chambers with $\pi_i(a) = x$ and $\pi_j(b) = y$. Notice that this implies that $\delta(a, b) \in W_i \sigma_i(w_\nu) W_j$, by Theorem 5.10.

Writing $\tau(\lambda) = l$ and $\tau(\nu) = n$, Lemma 5.20(i) implies that

$$W_i \sigma_i(w_\lambda) W_k = \sigma_i \big(W_0 w_\lambda \sigma_i^{-1}(W_k) \big) = \sigma_i(W_0 w_\lambda W_{\sigma_i^{-1}(k)}) = \sigma_i(W_0 w_\lambda W_l),$$

$$W_j \sigma_j(w_{\mu^*}) W_k = \sigma_i \big(W_{\sigma_i^{-1}(j)} \big(\sigma_i^{-1} \circ \sigma_j \big)(w_{\mu^*}) W_{\sigma_i^{-1}(k)} \big) = \sigma_i \big(W_n \sigma_n(w_{\mu^*}) W_l \big)$$

and similarly $W_i \sigma_i(w_\nu) W_j = \sigma_i(W_0 w_\nu W_n)$. Applying Lemma 3.12 (with $\sigma = \sigma_i$) we therefore have

$$\big| V_\lambda(x) \cap V_{\mu^*}(y) \big| = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_n \sigma_n(w_{\mu^*}) W_l}} \big| \mathcal{C}_{w_1}(a') \cap \mathcal{C}_{w_2}(b') \big|, \tag{5.3}$$

where a', b' are any chambers with $\delta(a', b') \in W_0 w_\nu W_n$.

Vertex regularity follows from (5.3), for the value of $|V_\lambda(x) \cap V_{\mu^*}(y)|$ is seen to only depend on λ, μ and ν. To see that strong vertex regularity holds, we use Proposition 4.16 to see that

$$W_0 w_\lambda W_l = \sigma_* \big(W_{\sigma_*^{-1}(0)} \sigma_*^{-1}(w_\lambda) W_{\sigma_*^{-1}(l)} \big) = \sigma_*(W_0 w_{\lambda^*} W_{l^*}),$$

$$W_n \sigma_n(w_{\mu^*}) W_l = \sigma_* \big(W_{n^*} \big(\sigma_*^{-1} \circ \sigma_n \circ \sigma_* \big)(w_\mu) W_{l^*} \big) = \sigma_* \big(W_{n^*} \sigma_{n^*}(w_\mu) W_{l^*} \big),$$

and similarly $W_0 w_\nu W_n = \sigma_*(W_0 w_{\nu^*} W_{n^*})$. A further application of Lemma 3.12 (with $\sigma = \sigma_*$) implies that

$$\big| V_\lambda(x) \cap V_{\mu^*}(y) \big| = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_0 w_{\lambda^*} W_{l^*} \\ w_2 \in W_{n^*} \sigma_{n^*}(w_\mu) W_{l^*}}} \big| \mathcal{C}_{w_1}(a'') \cap \mathcal{C}_{w_2}(b'') \big|,$$

where a'', b'' are any chambers with $\delta(a'', b'') \in W_0 w_{\nu^*} W_{n^*}$. Thus by comparison with (5.3) we have

$$\big| V_\lambda(x) \cap V_{\mu^*}(y) \big| = \big| V_{\lambda^*}(x') \cap V_\mu(y') \big|,$$

where $x', y' \in V_P$ are any vertices with $y' \in V_{\nu^*}(x')$; that is, strong vertex regularity holds. □

Corollary 5.22. *There exist numbers* $a_{\lambda,\mu;\nu} \in \mathbb{Q}^+$ *such that for* $\lambda, \mu \in P^+$,

$$A_\lambda A_\mu = \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} A_\nu \quad and \quad \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} = 1.$$

Moreover, $|\{\nu \in P^+ \mid a_{\lambda,\mu;\nu} \neq 0\}|$ *is finite for all* $\lambda, \mu \in P^+$.

Proof. Let $v \in V_\nu(u)$ and set

$$a_{\lambda,\mu;\nu} = \frac{N_\nu}{N_\lambda N_\mu} |V_\lambda(u) \cap V_{\mu^*}(v)|, \tag{5.4}$$

which is independent of the particular pair u, v by vertex regularity. The numbers $a_{\lambda,\mu;\nu}$ are clearly nonnegative and rational, and from (5.2) we have

$$(A_\lambda A_\mu f)(x) = \sum_{\nu \in P^+} \left(\sum_{y \in V_\nu(x)} \frac{|V_\lambda(x) \cap V_{\mu^*}(y)|}{N_\lambda N_\mu} f(y) \right)$$

$$= \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} \left(\frac{1}{N_\nu} \sum_{y \in V_\nu(x)} f(y) \right)$$

$$= \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} (A_\nu f)(x).$$

When $f = 1_{V_P} : V_P \to \{1\}$ we see that $\sum a_{\lambda,\mu;\nu} = 1$.

We now show that only finitely many of the $a_{\lambda,\mu;\nu}$'s are nonzero for each fixed pair $\lambda, \mu \in P^+$. Fix $x \in V_P$ and observe that $a_{\lambda,\mu;\nu} \neq 0$ if and only if $V_\lambda(x) \cap V_{\mu^*}(y) \neq \emptyset$ for each $y \in V_\nu(x)$. Applying $(N_\lambda A_\lambda)(N_\mu A_\mu)$ to the constant function $1_{V_P} : V_P \to \{1\}$, we obtain

$$\sum_{y \in V_P} |V_\lambda(x) \cap V_{\mu^*}(y)| = N_\lambda N_\mu,$$

and hence $V_\lambda(x) \cap V_{\mu^*}(y) \neq \emptyset$ for only finitely many $y \in V_P$. □

Definition 5.23. Let $\mathcal{A}$ be the linear span of $\{A_\lambda \mid \lambda \in P^+\}$ over $\mathbb{C}$. The previous corollary shows that $\mathcal{A}$ is an associative algebra.

We refer to the numbers $a_{\lambda,\mu;\nu}$ in Corollary 5.22 as the *structure constants* of the algebra $\mathcal{A}$.

Theorem 5.24. *The algebra* $\mathcal{A}$ *is commutative.*

Proof. We need to show that $a_{\lambda,\mu;\nu} = a_{\mu,\lambda;\nu}$ for all $\lambda, \mu, \nu \in P^+$. Fixing any pair u, v in V_P with $v \in V_\nu(u)$, strong vertex regularity implies that

$$a_{\lambda,\mu;\nu} = \frac{N_\nu}{N_\lambda N_\mu} |V_\lambda(u) \cap V_{\mu^*}(v)| = \frac{N_\nu}{N_\lambda N_\mu} |V_{\lambda^*}(v) \cap V_\mu(u)| = a_{\mu,\lambda;\nu}$$

completing the proof. $\square$

We note that a similar calculation using Theorem 5.15 (specifically the fact that $N_\lambda = N_{\lambda^*}$) shows that $a_{\lambda,\mu;\nu} = a_{\lambda^*,\mu^*;\nu^*}$ for all $\lambda, \mu, \nu \in P^+$.

Remark 5.25. Let X be a set and let K be a partition of $X \times X$ such that $\emptyset \notin K$ and $\{(x, x) \mid x \in X\} \in K$. For $k \in K$, define $k^* = \{(y, x) \mid (x, y) \in k\}$, and for each $x \in X$ and $k \in K$ define $k(x) = \{y \in X \mid (x, y) \in k\}$. Recall [34] that an *association scheme* is a pair (X, K) as above such that (i) $k \in K$ implies that $k^* \in K$, and (ii) for each $k, l, m \in K$ there exists a cardinal number $e_{k,l;m}$ such that

$$(x, y) \in m \quad \text{implies that} \quad |k(x) \cap l^*(y)| = e_{k,l;m}.$$

Let $X = V_P$, and for each $\lambda \in P^+$ let $\lambda' = \{(x, y) \mid y \in V_\lambda(x)\}$. The set $L = \{\lambda' \mid \lambda \in P^+\}$ forms a partition of $V_P \times V_P$, and $\lambda'(x) = V_\lambda(x)$ for $x \in V_P$.

By vertex regularity it follows that the pair (V_P, L) forms an association scheme, and the cardinal numbers $e_{\lambda',\mu';\nu'}$ are simply $N_\lambda N_\mu N_\nu^{-1} a_{\lambda,\mu;\nu}$. By strong vertex regularity this association scheme also satisfies the condition $e_{\lambda',\mu';\nu'} = e_{\mu',\lambda';\nu'}$ for all $\lambda, \mu, \nu \in P^+$ (see [34, p. 1, footnote]).

Note that the algebra $\mathcal{A}$ is essentially the *Bose–Mesner algebra* of the association scheme (V_P, L) (see [1, Chapter 2]). With reference to Remark 4.19, the above construction generalises the familiar construction of association schemes from infinite distance regular graphs (see [1, §1.4.4] for the case of *finite* distance regular graphs).

Recall the definition of the numbers $b_{w_1,w_2;w_3}$ given in Corollary 3.6.

Proposition 5.26. *Let* $\tau(\lambda) = l$ *and* $\tau(\nu) = n$. *Suppose that* $y \in V_\nu(x)$ *and* $V_\lambda(x) \cap V_{\mu^*}(y) \neq \emptyset$. *Then*

$$a_{\lambda,\mu;\nu} = \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_{0\nu}(q) W_0^2(q) q_{w_\lambda} q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} b_{w_1,w_2;w_\nu}.$$

Proof. By Lemma 5.20(ii) we have $\sigma_n = \sigma_l \circ \sigma_m$. Thus by Proposition 4.16(iv) we have $W_n \sigma_n(w_{\mu^*}) W_l = (W_l \sigma_l(w_\mu) W_n)^{-1}$, and so by (5.3) we see that

$$|V_\lambda(x) \cap V_{\mu^*}(y)| = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} |\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(b)| \tag{5.5}$$

whenever $\delta(a, b) \in W_0 w_\nu W_n$.

By Proposition 3.9 (and the proof thereof) we have

$$\left| \mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2^{-1}}(b) \right| = q_{w_1} q_{w_2} (B_{w_1} B_{w_2} \delta_b)(a),$$

and the result now follows from (5.5) by using Theorem 5.15 and the definitions of $a_{\lambda,\mu;\nu}$ and $b_{w_1,w_2;w_3}$, by choosing $b \in \mathcal{C}_{w_\nu}(a)$.   □

6. Affine Hecke algebras and Macdonald spherical functions

In Section 6.3 we make an important connection between the algebra $\mathcal{A}$ and affine Hecke algebras. In particular, in Theorem 6.16 we show that $\mathcal{A}$ is isomorphic to $Z(\tilde{\mathcal{H}})$, the centre of an appropriately parametrised affine Hecke algebra $\tilde{\mathcal{H}}$.

In Sections 6.1 and 6.2 we give an outline of some known results regarding affine Hecke algebras. The main references for this material are [21,25]. Note that in [25] there is only one parameter q, although the results there go through without any serious difficulty in the more general case of multiple parameters $\{q_s\}_{s \in S}$. Note also that in [25] $Q = Q(R)$ and $P = P(R)$, whereas for us $Q = Q(R^\vee)$ and $P = P(R^\vee)$.

6.1. Affine Hecke algebras

Let $\{q_s\}_{s \in S}$ be a set of positive real numbers with $q_{s_i} = q_{s_j}$ whenever s_i and s_j are conjugate in $\tilde{W}$. The *affine Hecke algebra* $\tilde{\mathcal{H}}$ with parameters $\{q_s\}_{s \in S}$ is the algebra over $\mathbb{C}$ with presentation given by the generators T_w, $w \in \tilde{W}$, and relations

$$T_{w_1} T_{w_2} = T_{w_1 w_2} \quad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2), \tag{6.1}$$

$$T_w T_s = \frac{1}{q_s} T_{ws} + \left(1 - \frac{1}{q_s} \right) T_w \quad \text{if } \ell(ws) < \ell(w) \text{ and } s \in S. \tag{6.2}$$

By (6.1), $T_1 T_w = T_w T_1 = T_w$ for all $w \in \tilde{W}$, and hence $T_1 = I$ since $\{T_w\}_{w \in \tilde{W}}$ generates $\tilde{\mathcal{H}}$. Then (6.2) implies that each T_s, $s \in S$, is invertible, and from (6.1) we see that each T_g, $g \in G$, is invertible, with inverse $T_{g^{-1}}$ (recall the definition of G from Section 4.7). Since each $w \in \tilde{W}$ can be written as $w = w'g$ for $w' \in W$ and $g \in G$ it follows that each T_w, $w \in \tilde{W}$, is invertible.

Remark 6.1. (i) In [21] the numbers $\{q_s\}_{s \in S}$ are taken as positive real variables. Our choice to fix the numbers $\{q_s\}_{s \in S}$ does not change the algebraic structure of $\tilde{\mathcal{H}}$ in any serious way (for our purposes, at least).

(ii) The condition that $q_{s_i} = q_{s_j}$ whenever $s_i = w s_j w^{-1}$ for some $w \in \tilde{W}$ is equivalent to the condition that $q_{s_i} = q_{s_j}$ whenever $s_i = u s_{\sigma(j)} u^{-1}$ for some $\sigma \in \mathrm{Aut}_{\mathrm{tr}}(D)$ and $u \in W$. This condition is quite restrictive, and it is easy to see that we obtain the parameter systems given in Appendix A. Thus connections with our earlier results on the algebra $\mathcal{A}$ will become apparent when we take the numbers $\{q_s\}_{s \in S}$ to be the parameters of a locally finite regular affine building.

Definition 6.2. (i) We write $q_w = q_{s_{i_1}} \cdots q_{s_{i_m}}$ if $s_{i_1} \cdots s_{i_m}$ is a reduced expression for $w \in W$. This is easily seen to be independent of the particular reduced expression (see [4, IV, §1, No. 5, Proposition 5]). Each $\tilde{w} \in \tilde{W}$ can be written uniquely as $\tilde{w} = wg$ for $w \in W$ and $g \in G$, and we define $q_{\tilde{w}} = q_w$. In particular $q_g = 1$ for all $g \in G$. Furthermore, if $s = s_i$ we write $q_s = q_i$.

(ii) To conveniently state later results we make the following definitions. Let $R_1 = \{\alpha \in R \mid 2\alpha \notin R\}$, $R_2 = \{\alpha \in R \mid \frac{1}{2}\alpha \notin R\}$ and $R_3 = R_1 \cap R_2$ (so $R_1 = R_2 = R_3 = R$ if R is reduced). For $\alpha \in R_2$, write $q_\alpha = q_i$ if $\alpha \in W_0\alpha_i$ (note that if $\alpha \in W_0\alpha_i$ then necessarily $\alpha \in R_2$). It follows easily from Corollary 2.2 that this definition is unambiguous.

Note that R is the disjoint union of R_3, $R_1 \setminus R_3$ and $R_2 \setminus R_3$, and define set of numbers $\{\tau_\alpha\}_{\alpha \in R}$ by

$$\tau_\alpha = \begin{cases} q_\alpha & \text{if } \alpha \in R_3, \\ q_0 & \text{if } \alpha \in R_1 \setminus R_3, \\ q_\alpha q_0^{-1} & \text{if } \alpha \in R_2 \setminus R_3, \end{cases}$$

where $q_0 = q_{s_0}$ (with $s_0 = s_{\tilde{\alpha};1}$ and $\tilde{\alpha}$ is as in (4.1)). It is convenient to also define $\tau_\alpha = 1$ if $\alpha \notin R$. The reader only interested in the reduced case can simply read τ_α as q_α. Note that $\tau_{w\alpha} = \tau_\alpha$ for all $\alpha \in R$ and $w \in W_0$.

Remark 6.3. We have chosen a slight distortion of the usual definition of the algebra $\tilde{\mathcal{H}}$. This choice has been made so as to make the connection between the algebras $\mathcal{A}$ and $\tilde{\mathcal{H}}$ more transparent, as the reader will shortly see. To allow the reader to convert between our notation and that in [21], we provide the following instructions. With reference to our presentation for $\tilde{\mathcal{H}}$ given above, let $\tau_i = \sqrt{q_i}$ and $T'_w = \sqrt{q_w} T_w$ (these τ's are unrelated to those in Definition 6.2(ii)). Our presentation then transforms into that given in [21, 4.1.2] (with the T's there replaced by T''s). This transformation also makes it clear why the $\sqrt{q_w}$'s appear in the following discussion.

If $\lambda \in P^+$ let $x^\lambda = \sqrt{q_{t_\lambda}} T_{t_\lambda}$, and if $\lambda = \mu - \nu$ with $\mu, \nu \in P^+$ let $x^\lambda = x^\mu (x^\nu)^{-1}$. This is well defined by [21, p. 40], and for all $\lambda, \mu \in P$ we have $x^\lambda x^\mu = x^{\lambda+\mu} = x^\mu x^\lambda$.

We write $\mathbb{C}[P]$ for the $\mathbb{C}$-span of $\{x^\lambda \mid \lambda \in P\}$. The group W_0 acts on $\mathbb{C}[P]$ by linearly extending the action $wx^\lambda = x^{w\lambda}$. We write $\mathbb{C}[P]^{W_0}$ for the set of elements of $\mathbb{C}[P]$ that are invariant under the action of W_0. By Corollary 6.7, the centre $Z(\tilde{\mathcal{H}})$ of $\tilde{\mathcal{H}}$ is $\mathbb{C}[P]^{W_0}$.

Let $\mathcal{H}$ be the subalgebra of $\tilde{\mathcal{H}}$ generated by $\{T_s \mid s \in S\}$. The following relates the algebra $\mathcal{H}$ to the algebra $\mathcal{B}$ of chamber set averaging operators on an irreducible affine building.

Proposition 6.4. *Suppose a building $\mathcal{X}$ of type R exists with parameters $\{q_s\}_{s\in S}$. Then $\mathcal{H} \cong \mathcal{B}$.*

Proof. This follows in the same way as Theorem 3.10. $\square$

We make the following parallel definition to (5.1). Recall the definition of Poincaré polynomials from Definition 2.6. For each $i \in I$, let

$$\mathbb{1}_i = \frac{1}{W_i(q)} \sum_{w \in W_i} q_w T_w, \tag{6.3}$$

where $W_i = W_{I \setminus \{i\}}$ (as before). Thus $\mathbb{1}_i$ is an element of $\mathcal{H}$. As a word of warning, we have used the same notation as in (5.1) where we defined the analogous element in $\mathcal{B}$. There should be no confusion caused by this decision.

The following lemma follows in exactly the same way as Lemma 5.12.

Lemma 6.5. $\mathbb{1}_i T_w = T_w \mathbb{1}_i = \mathbb{1}_i$ *for all* $w \in W_i$ *and* $i \in I$. *Furthermore* $\mathbb{1}_i^2 = \mathbb{1}_i$.

6.2. The Macdonald spherical functions

The following relations are of fundamental significance.

Theorem 6.6. *Let* $\lambda \in P$ *and* $i \in I_0$.

(i) *If* $(R, i) \neq (BC_n, n)$ *for any* $n \geqslant 1$, *then*

$$x^\lambda T_{s_i} - T_{s_i} x^{s_i \lambda} = \left(1 - q_i^{-1}\right) \frac{x^\lambda - x^{s_i \lambda}}{1 - x^{-\alpha_i^\vee}}.$$

(ii) *If* $R = BC_n$ *for some* $n \geqslant 1$ *and* $i = n$, *then*

$$x^\lambda T_{s_n} - T_{s_n} x^{s_n \lambda} = \left[1 - q_n^{-1} + q_n^{-1/2}\left(q_0^{1/2} - q_0^{-1/2}\right)x^{-(2\alpha_n)^\vee}\right] \frac{x^\lambda - x^{s_n \lambda}}{1 - x^{-2(2\alpha_n)^\vee}}.$$

Proof. This follows from [21, (4.2.4)] (see Remark 6.3), taking into account [21, (1.4.3) and (2.1.6)] in case (ii). □

We note that the fractions appearing in Theorem 6.6 are in fact finite linear combinations of the x^μ's [21, (4.2.5)]. We refer to the relations in Theorem 6.6 as the *Bernstein relations*, for they are a crucial ingredient in the so-called *Bernstein presentation* of the Hecke algebra.

Corollary 6.7. *The centre* $Z(\tilde{\mathcal{H}})$ *of* $\tilde{\mathcal{H}}$ *is* $\mathbb{C}[P]^{W_0}$.

Proof. This well-known fact can be proved using the Bernstein relations, exactly as in [21, (4.2.10)]. □

For each $\lambda \in P^+$, define an element $P_\lambda(x) \in \mathbb{C}[P]^{W_0}$ by

$$P_\lambda(x) = \frac{q_{t_\lambda}^{-1/2}}{W_0(q)} \sum_{w \in W_0} w \left(x^\lambda \prod_{\alpha \in R^+} \frac{\tau_\alpha \tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1}{\tau_{\alpha/2}^{1/2} x^{\alpha^\vee} - 1} \right). \tag{6.4}$$

We call the elements $P_\lambda(x)$ the *Macdonald spherical functions* of $\tilde{\mathcal{H}}$.

Remark 6.8. (i) We have chosen a slightly different normalisation of the Macdonald spherical function from that in [21]. Our formula uses the normalisation of [18, Theorem 4.1.2].

(ii) Notice that the formula simplifies in the reduced case (namely, $\tau_{\alpha/2} = 1$). (iii) It is not immediately clear that $P_\lambda(x)$ as defined in (6.4) is in $\mathbb{C}[P]^{W_0}$, although this is a consequence of [4, VI, §3, No. 3, Proposition 2].

The proof of Theorem 6.9 below follows [25, Theorem 2.9] very closely.

Theorem 6.9. [25, Theorem 2.9]. *For $\lambda \in P^+$ we have $q_{t_\lambda}^{1/2} P_\lambda(x) \mathbb{1}_0 = \mathbb{1}_0 x^\lambda \mathbb{1}_0$.*

Proof. By the Satake isomorphism (see [25, Theorem 2.4] and [16, 5.2], for example) there exists some $P'_\lambda(x) \in \mathbb{C}[P]^{W_0}$ such that $P'_\lambda(x) \mathbb{1}_0 = \mathbb{1}_0 x^\lambda \mathbb{1}_0$. If $i \in I_0$ and $(R, i) \neq (BC_n, n)$, then by Theorem 6.6(i) (and using Lemma 6.5) we have

$$(1 + q_i T_{s_i}) x^\lambda \mathbb{1}_0 = x^\lambda \mathbb{1}_0 + q_i x^{s_i \lambda} T_{s_i} \mathbb{1}_0 + (q_i - 1) \frac{x^\lambda - x^{s_i \lambda}}{1 - x^{-\alpha_i}} \mathbb{1}_0$$

$$= \frac{q_i x^\lambda - x^{\lambda - \alpha_i^\vee} - q_i x^{s_i \lambda - \alpha_i^\vee} + x^{s_i \lambda}}{1 - x^{-\alpha_i^\vee}} \mathbb{1}_0$$

$$= \left(\frac{q_i x^{\alpha_i^\vee} - 1}{x^{\alpha_i^\vee} - 1} x^\lambda + \frac{q_i x^{-\alpha_i^\vee} - 1}{x^{-\alpha_i^\vee} - 1} x^{s_i \lambda} \right) \mathbb{1}_0$$

$$= (1 + s_i) \frac{q_i x^{\alpha_i^\vee} - 1}{x^{\alpha_i^\vee} - 1} x^\lambda \mathbb{1}_0. \tag{6.5}$$

A similar calculation, using Theorem 6.6(ii), shows that if $i \in I_0$ and $(R, i) = (BC_n, n)$, then

$$(1 + q_n T_{s_n}) x^\lambda \mathbb{1}_0 = (1 + s_n) \frac{\left(\sqrt{q_0 q_n}\, x^{(2\alpha_n)^\vee} - 1 \right) \left(\sqrt{q_n/q_0}\, x^{(2\alpha_n)^\vee} + 1 \right)}{x^{2(2\alpha_n)^\vee} - 1} x^\lambda \mathbb{1}_0. \tag{6.6}$$

It will be convenient to write (6.5) and (6.6) as one equation, as follows. In the reduced case, let $\beta_i = \alpha_i$ for all $i \in I_0$, and in the nonreduced case (so $R = BC_n$ for some $n \geq 1$) let $\beta_i = \alpha_i$ for $1 \leq i \leq n - 1$ and let $\beta_n = 2\alpha_n$. For $\alpha \in R$ and $i \in I_0$, write

$$a_i(x^{\alpha^\vee}) = \frac{\left(\tau_{\beta_i} \tau_{\beta_i/2}^{1/2} x^{\alpha^\vee} - 1 \right) \left(\tau_{\beta_i/2}^{1/2} x^{\alpha^\vee} + 1 \right)}{x^{2\alpha^\vee} - 1},$$

and so in all cases

$$(1 + q_i T_{s_i}) x^\lambda \mathbb{1}_0 = (1 + s_i) a_i \left(x^{\beta_i^\vee}\right) x^\lambda \mathbb{1}_0. \tag{6.7}$$

By induction we see that (writing T_i for T_{s_i})

$$\left[\prod_{k=1}^m (1 + q_{i_k} T_{i_k})\right] x^\lambda \mathbb{1}_0 = \left[\prod_{k=1}^m (1 + s_{i_k}) a_{i_k} \left(x^{\beta_{i_k}^\vee}\right)\right] x^\lambda \mathbb{1}_0, \tag{6.8}$$

where we write $\prod_{k=1}^m x_k$ for the ordered product $x_1 \cdots x_m$. Therefore $\mathbb{1}_0 x^\lambda \mathbb{1}_0$ can be written as $f x^\lambda \mathbb{1}_0$, where f is independent of λ and is a finite linear combination of terms of the form

$$(1 + s_{i_1}) a_{i_1} \left(x^{\beta_{i_1}^\vee}\right) \cdots (1 + s_{i_m}) a_{i_m} \left(x^{\beta_{i_m}^\vee}\right),$$

where $i_1, \ldots, i_m \in I_0$.

Thus we have

$$P_\lambda'(x) = \sum_{w \in W_0} w\left(b_w(x) x^\lambda\right),$$

where each $b_w(x)$ is a linear combination of products of terms $a_i(x^{\beta_i^\vee})$ and is independent of $\lambda \in P^+$. It is easily seen that this expression is unique, and since $P_\lambda'(x) \in \mathbb{C}[P]^{W_0}$ it follows that $b_w(x) = b_{w'}(x)$ for all $w, w' \in W_0$, and we write $b(x)$ for this common value. Thus

$$P_\lambda'(x) = \sum_{w \in W_0} w\left(b(x) x^\lambda\right) = \sum_{w \in W_0} w\left(x^{w_0 \lambda} w_0 b(x)\right),$$

where w_0 is the longest element of W_0.

We now compute the coefficient of $x^{w_0 \lambda}$ in the above expression. Since this coefficient is independent of $\lambda \in P^+$ we may assume that $\langle \lambda, \alpha_i \rangle > 0$ for all $i \in I_0$ and so $w\lambda \neq w_0 \lambda$ for all $w \in W_0 \setminus \{w_0\}$.

If $w_0 = s_{i_1} \cdots s_{i_m}$ is a reduced expression, then

$$\mathbb{1}_0 = \frac{1}{W_0(q)} \big((1 + q_{i_1} T_{i_1}) \cdots (1 + q_{i_m} T_{i_m})$$
$$+ \text{terms } (1 + q_{j_1} T_{j_1}) \cdots (1 + q_{j_l} T_{j_l}) \text{ with } j_k \in I_0 \text{ and } l < m\big).$$

Thus, by (6.8)

$$\mathbb{1}_0 x^\lambda \mathbb{1}_0 = \frac{1}{W_0(q)} \left[\left(\prod_{k=1}^m s_{i_k} a_{i_k} \left(x^{\beta_{i_k}^\vee}\right)\right) x^\lambda \mathbb{1}_0\right.$$
$$\left. + \text{terms } \left(\prod_{k=1}^l s_{j_k} a_{j_k} \left(x^{\beta_{j_k}^\vee}\right)\right) x^\lambda \mathbb{1}_0 \text{ with } j_k \in I_0 \text{ and } l < m\right].$$

Thus the coefficient of $x^{w_0\lambda}$ is

$$
w_0 b(x) = \frac{1}{W_0(q)} s_{i_1} a_{i_1} \left(x^{\beta_{i_1}^\vee} \right) \cdots s_{i_m} a_{i_m} \left(x^{\beta_{i_m}^\vee} \right)
$$

$$
= \frac{1}{W_0(q)} w_0 \prod_{\beta \in R_1^+} \frac{\left(\tau_\beta \tau_{\beta/2}^{1/2} x^{\beta^\vee} - 1 \right) \left(\tau_{\beta/2}^{1/2} x^{\beta^\vee} + 1 \right)}{x^{2\beta^\vee} - 1},
$$

where we have used the fact that

$$
\left\{ \beta_{i_m}^\vee, s_{i_m} \beta_{i_{m-1}}^\vee, \dots, s_{i_m} s_{i_{m-1}} \cdots s_{i_2} \beta_{i_1}^\vee \right\} = \left(R_1^+ \right)^\vee
$$

(see [21, (2.2.9)]) and the fact that $\tau_{w\alpha} = \tau_\alpha$ for all $w \in W_0$ and $\alpha \in R$. The result now follows after an elementary manipulation. $\quad\square$

Since $x^\lambda = q_{t_\lambda}^{1/2} T_{t_\lambda}$ by definition, we have the following.

Corollary 6.10. *For $\lambda \in P^+$ we have*

$$
\mathbb{1}_0 T_{t_\lambda} \mathbb{1}_0 = P_\lambda(x) \mathbb{1}_0.
$$

We write Q^+ for the $\mathbb{N}$-span of $\{\alpha^\vee \mid \alpha \in R^+\}$. Define a partial order $\preccurlyeq$ on P by $\mu \preccurlyeq \lambda$ if and only if $\lambda - \mu \in Q^+$.

Theorem 6.11. *$\{P_\lambda(x) \mid \lambda \in P^+\}$ is a basis of $\mathbb{C}[P]^{W_0}$. Furthermore, the Macdonald spherical functions satisfy*

$$
P_\lambda(x) P_\mu(x) = \sum_{\nu \preccurlyeq \lambda + \mu} c_{\lambda,\mu;\nu} P_\nu(x)
$$

for some numbers $c_{\lambda,\mu;\nu}$, with $c_{\lambda,\mu;\lambda+\mu} > 0$.

Proof. This is a simple application of the triangularity condition of the Macdonald spherical functions, see [20, §10]. $\quad\square$

6.3. Connecting $\mathcal{A}$ and $Z(\tilde{\mathcal{H}})$

We can now see how to relate the vertex set averaging operators A_λ from Section 5 to the algebra elements $P_\lambda(x)$. Let us recall (and make) some definitions. For $\lambda, \mu, \nu \in P^+$ and $w_1, w_2, w_3 \in W$, define numbers $a_{\lambda,\mu;\nu}$, $b_{w_1,w_2;w_3}$, $c_{\lambda,\mu;\nu}$ and $d_{w_1,w_2;w_3}$ by

$$
A_\lambda A_\mu = \sum_{\nu \in P^+} a_{\lambda,\mu;\nu} A_\nu, \qquad B_{w_1} B_{w_2} = \sum_{w_3 \in W} b_{w_1,w_2;w_3} B_{w_3},
$$

$$
P_\lambda(x) P_\mu(x) = \sum_{\nu \in P^+} c_{\lambda,\mu;\nu} P_\nu(x), \qquad T_{w_1} T_{w_2} = \sum_{w_3 \in W} d_{w_1,w_2;w_3} T_{w_3}.
$$

Thus the numbers are the structure constants of the algebras $\mathcal{A}$, $\mathcal{B}$, $\mathbb{C}[P]^{W_0}$ and $\mathcal{H}$ with respect to the bases $\{A_\lambda \mid \lambda \in P^+\}$, $\{B_w \mid w \in W\}$, $\{P_\lambda(x) \mid \lambda \in P^+\}$ and $\{T_w \mid w \in W\}$ respectively.

Note that by Proposition 6.4 we have $b_{w_1,w_2;w_3} = d_{w_1,w_2;w_3}$ whenever a building with parameter system $\{q_s\}_{s \in S}$ exists. We stress that $d_{w_1,w_2;w_3}$ is a more general object, for it makes sense for a much more general set of q_s's.

Recall the definition of w_λ from Section 4.9, and recall the definition of $W_{0\lambda}$ from (4.4). We give the following lemma linking double cosets in W with double cosets in $\tilde{W}$.

Lemma 6.12. *Let* $\lambda \in P^+$ *and* $i \in I_P$. *Suppose that* $\tau(\lambda) = l$, *and write* $j = \sigma_i(l)$ (*so* $\sigma_j = \sigma_i \circ \sigma_l$). *Then*

$$W_i \sigma_i(t'_\lambda) W_j = g_i W_0 t_\lambda W_0 g_j^{-1},$$

where the elements g_i *are defined in* (4.5).

Proof. By Proposition 4.12, $g_j = g_i g_l$ and $t_\lambda = t'_\lambda g_l$, and by (4.8), $\sigma_k(w) = g_k w g_k^{-1}$ for all $w \in W$ and $k \in I_P$. Thus

$$W_i \sigma_i(t'_\lambda) W_j = \left(g_i W_0 g_i^{-1}\right)\left(g_i t_\lambda g_l^{-1} g_i^{-1}\right)\left(g_j W_0 g_j^{-1}\right) = g_i W_0 t_\lambda W_0 g_j^{-1}. \qquad \square$$

Lemma 6.13. [25, Lemma 2.7]. *Let* $\lambda \in P^+$. *Then*

$$\sum_{w \in W_0 t_\lambda W_0} q_w T_w = \frac{W_0^2(q)}{W_{0\lambda}(q)} q_{w_\lambda} \mathbb{1}_0 T_{t_\lambda} \mathbb{1}_0.$$

Proof. This can be deduced from Theorem 5.13, or see the proof in [25]. $\square$

The following important theorem will be used (along with Proposition 5.26) to prove that $\mathcal{A} \cong Z(\tilde{\mathcal{H}})$.

Theorem 6.14. *Let* $\lambda, \mu, \nu \in P^+$ *and write* $\tau(\lambda) = l$, $\tau(\mu) = m$ *and* $\tau(\nu) = n$. *Then if* $c_{\lambda,\mu;\nu} \neq 0$ *we have*

$$c_{\lambda,\mu;\nu} = \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_{0\nu}(q) W_0^2(q) q_{w_\lambda} q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} d_{w_1,w_2;w_\nu}.$$

Proof. To abbreviate notation we write $P_\lambda = P_\lambda(x)$. First observe that by Theorem 6.11 we have $c_{\lambda,\mu;\nu} = 0$ unless $\nu \preccurlyeq \lambda + \mu$. In particular we have $c_{\lambda,\mu;\nu} = 0$ when $\tau(\nu) \neq \tau(\lambda + \mu)$. It follows that $\sigma_n = \sigma_l \circ \sigma_m$, and so $g_n = g_l g_m$ (see Proposition 4.12). We will use this fact later.

By Corollary 6.10 and Lemma 6.13, for any $\lambda \in P^+$ we have

$$P_\lambda \mathbb{1}_0 = \mathbb{1}_0 T_{t_\lambda} \mathbb{1}_0 = \frac{W_{0\lambda}(q)}{W_0^2(q) q_{w_\lambda}} \sum_{w \in W_0 t_\lambda W_0} q_w T_w,$$

and so if $i \in I_P$, $\tau(\lambda) = l$ and $j = \sigma_i(l)$ we have (see Lemma 6.12)

$$T_{g_i} P_\lambda \mathbb{1}_0 T_{g_j^{-1}} = \frac{W_{0\lambda}(q)}{W_0^2(q) q_{w_\lambda}} \sum_{w \in W_i \sigma_i(t_\lambda') W_j} q_w T_w. \tag{6.9}$$

We can replace the t_λ' by w_λ in the above because $W_i \sigma_i(t_\lambda') W_j = W_i \sigma_i(w_\lambda) W_j$ by Proposition 4.15(i) and the fact that $\sigma_i(W_l) = W_j$.

Using the fact that $g_n = g_l g_m$ if $c_{\lambda,\mu;\nu} \neq 0$ we have, by (6.9)

$$\begin{aligned}
P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}} &= (P_\lambda \mathbb{1}_0 T_{g_l^{-1}})(T_{g_l} P_\mu \mathbb{1}_0 T_{g_n^{-1}}) \\
&= \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_0^4(q) q_{w_\lambda} q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} T_{w_1} T_{w_2} \\
&= \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_0^4(q) q_{w_\lambda} q_{w_\mu}} \sum_{w_3 \in W} \left(\sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} d_{w_1, w_2; w_3} T_{w_3} \right).
\end{aligned}$$

So the coefficient of T_{w_ν} in the expansion of $P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}}$ in terms of the T_w's is

$$\frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_0^4(q) q_{w_\lambda} q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} d_{w_1, w_2; w_\nu}. \tag{6.10}$$

On the other hand, by Theorem 6.11 we have

$$\begin{aligned}
P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}} &= \sum_{\eta \preccurlyeq \lambda + \mu} c_{\lambda,\mu;\eta} P_\eta \mathbb{1}_0 T_{g_n^{-1}} \\
&= \sum_{\eta \preccurlyeq \lambda + \mu} \left(\frac{W_{0\eta}(q)}{W_0^2(q) q_{w_\eta}} c_{\lambda,\mu;\eta} \sum_{w \in W_0 w_\eta W_n} q_w T_w \right).
\end{aligned}$$

Since the double cosets $W_0 w_\eta W_n$ are disjoint over $\{\eta \in P^+ \mid \eta \preccurlyeq \lambda + \mu\}$ we see that the coefficient of T_{w_ν} is

$$\frac{W_{0\nu}(q)}{W_0^2(q)} c_{\lambda,\mu;\nu}. \tag{6.11}$$

The theorem now follows by equating (6.10) and (6.11). $\quad\square$

Corollary 6.15. *Suppose that an irreducible affine building exists with parameter system $\{q_s\}_{s \in S}$. Then for all $\lambda, \mu, \nu \in P^+$ we have $a_{\lambda, \mu; \nu} = c_{\lambda, \mu; \nu}$.*

Proof. This follows from Theorem 6.14 and Propositions 5.26 and 6.4. $\quad\square$

Theorem 6.16. *Suppose that an irreducible affine building exists with parameters $\{q_s\}_{s \in S}$. Then the map $P_\lambda(x) \mapsto A_\lambda$ determines an algebra isomorphism, and so $\mathcal{A} \cong Z(\tilde{\mathcal{H}})$.*

Proof. Since $\{P_\lambda(x) \mid \lambda \in P^+\}$ is a basis of $\mathbb{C}[P]^{W_0}$ and $\{A_\lambda \mid \lambda \in P^+\}$ is a basis of $\mathcal{A}$, there exists a unique vector space isomorphism $\Phi : Z(\tilde{\mathcal{H}}) \to \mathcal{A}$ with $\Phi(P_\lambda) = A_\lambda$ for all $\lambda \in P^+$. Since $a_{\lambda, \mu; \nu} = c_{\lambda, \mu; \nu}$ by Corollary 6.15, we see that Φ is an algebra isomorphism. $\quad\square$

Theorem 6.17. *The algebra $Z(\tilde{\mathcal{H}})$ is generated by $\{P_{\lambda_i}(x) \mid i \in I_0\}$, and so $\mathcal{A}$ is generated by $\{A_{\lambda_i} \mid i \in I_0\}$.*

Proof. First we define a less restrictive partial order on P^+ than $\preccurlyeq$. For $\lambda, \mu \in P^+$ we define $\mu < \lambda$ if and only if $\lambda - \mu$ is an $\mathbb{R}^+$-linear combination of $(R^\vee)^+$ and $\lambda \neq \mu$. Clearly if $\mu \prec \lambda$ then $\mu < \lambda$. Observe also that $\lambda_i > 0$ for all $i \in I_0$ (see Exercises 7 and 8 on p. 72 of [13]). Thus if $\lambda = \lambda' + \lambda_i$ for some $\lambda' \in P^+$ and $i \in I_0$, we have $\lambda - \lambda' = \lambda_i > 0$ and so $\lambda' < \lambda$.

Let $\mathcal{P}(\lambda)$ be the statement that P_λ is a polynomial in $P_{\lambda_1}, \dots, P_{\lambda_n}$ (and $P_0 = 1$). Suppose that $\mathcal{P}(\lambda)$ fails for some $\lambda \in P^+$. Since $\{\mu \in P^+ \mid \mu \leqslant \lambda\}$ is finite (by the proof of [13, Lemma 13.2B]) we can pick $\lambda \in P^+$ minimal with respect to $\leqslant$ such that $\mathcal{P}(\lambda)$ fails. There is an i such that $\lambda - \lambda_i = \lambda'$ is in P^+. Then $\lambda' < \lambda$ and $P_\lambda = c P_{\lambda'} P_{\lambda_i} +$ a linear combination of P_μ's, where $\mu < \lambda$, $\mu \neq \lambda$. Then $\mathcal{P}(\lambda')$ holds, as does $\mathcal{P}(\mu)$ for all these μ's. So $\mathcal{P}(\lambda)$ holds, a contradiction. $\quad\square$

7. A positivity result and hypergroups

Here we show that the structure constants $c_{\lambda, \mu; \nu}$ of the algebra $\mathbb{C}[P]^{W_0}$ are, up to positive normalisation factors, polynomials with nonnegative integer coefficients in the variables $\{q_s - 1 \mid s \in S\}$. This result has independently been obtained by Schwer in [31], where a formula for $c_{\lambda, \mu; \nu}$ is given (in the case $q_s = q$ for all $s \in S$).

Thus if $q_s \geqslant 1$ for all $s \in S$, then $c_{\lambda, \mu; \nu} \geqslant 0$ for all $\lambda, \mu, \nu \in P^+$. This result was proved for root systems of type A_n by Miller Malley in [24], where the numbers $c_{\lambda, \mu; \nu}$ are Hall polynomials (up to positive normalisation factors). Note that it is clear from (5.4) and Corollary 6.15 that $c_{\lambda, \mu; \nu} \geqslant 0$ when there exists a building with parameters $\{q_s\}_{s \in S}$.

In a recent series of papers [12,27,31,33] the numbers $a_{\lambda, \mu}$ appearing in $P_\lambda(x) = \sum_\mu a_{\lambda, \mu} m_\mu$ are studied. Here m_μ is the monomial symmetric function $\sum_{\gamma \in W_0 \mu} x^\gamma$, where $W_0 \mu$ is the orbit $\{w\mu \mid w \in W_0\}$. We will provide a connection with the results we prove here and the numbers $a_{\lambda, \mu}$ in [26, Theorem 6.2]. In particular, for $\lambda \in P^+$, let $\Pi_\lambda \subset P$

denote the saturated set (see [4, VI, §1, Exercise 23]) with highest coweight λ. If $\mu \notin \Pi_\lambda$ then $a_{\lambda,\mu} = 0$, and for all $\mu \in \Pi_\lambda$,

$$a_{\lambda,\mu} = \sqrt{\frac{N_{\nu-\mu}}{N_\nu}} c_{\lambda,\mu;\nu},$$

where ν is any dominant coweight with each $\langle \nu, \alpha_i \rangle$ 'sufficiently large'.

The results of this section show how to construct a (commutative) polynomial hypergroup, in the sense of [3] (see also [17] where the A_2 case is discussed).

For each $w_1, w_2, w_3 \in W$, let $d'_{w_1,w_2;w_3} = q_{w_1} q_{w_2} q_{w_3}^{-1} d_{w_1,w_2;w_3}$.

Lemma 7.1. *For all $w_1, w_2, w_3 \in W$, $d'_{w_1,w_2;w_3}$ is a polynomial with nonnegative integer coefficients in the variables $q_s - 1$, $s \in S$.*

Proof. We prove the result by induction on $\ell(w_2)$. When $\ell(w_2) = 1$, so $w_2 = s$ for some $s \in S$, we have

$$d'_{w_1,s;w_3} = \begin{cases} 1 & \text{if } \ell(w_1 s) = \ell(w_1) + 1 \text{ and } w_3 = w_1 s, \\ q_s & \text{if } \ell(w_1 s) = \ell(w_1) - 1 \text{ and } w_3 = w_1 s, \\ q_s - 1 & \text{if } \ell(w_1 s) = \ell(w_1) - 1 \text{ and } w_3 = w_1, \\ 0 & \text{otherwise,} \end{cases}$$

proving the result in this case.

Suppose that $n \geqslant 2$ and that the result is true for $\ell(w_2) < n$. Then if $\ell(w_2) = n$, write $w_2 = ws$ with $\ell(w) = n - 1$. Thus

$$T_{w_1} T_{w_2} = (T_{w_1} T_w) T_s = \sum_{w' \in W} d_{w_1,w;w'} T_{w'} T_s = \sum_{w_3 \in W} \left(\sum_{w' \in W} d_{w_1,w;w'} d_{w',s;w_3} \right) T_{w_3},$$

which implies that

$$d'_{w_1,w_2;w_3} = \sum_{w' \in W} d'_{w_1,w;w'} d'_{w',s;w_3}.$$

The result follows since $\ell(w) < n$ and $\ell(s) = 1$. $\quad\square$

For each $\lambda, \mu, \nu \in P^+$, let

$$c'_{\lambda,\mu;\nu} = \frac{W_0(q) W_{0\nu}(q)}{W_{0\lambda}(q) W_{0\mu}(q)} \frac{q_{w_\lambda} q_{w_\mu}}{q_{w_\nu}} c_{\lambda,\mu;\nu}. \tag{7.1}$$

Theorem 7.2. *For all $\lambda, \mu, \nu \in P^+$, the structure constants $c'_{\lambda,\mu;\nu}$ are polynomials with nonnegative integer coefficients in the variables $q_s - 1$, $s \in S$.*

Proof. We will use the same notation as in Theorem 6.14, so let $\tau(\lambda) = l$, $\tau(\mu) = m$ and $\tau(\nu) = n$. By Theorem 6.14 we have

$$c'_{\lambda,\mu;\nu} = \frac{1}{W_0(q)} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} d'_{w_1,w_2;w_\nu},$$

and so it immediately follows from Lemma 7.1 that $W_0(q) c'_{\lambda,\mu;\nu}$ is a polynomial in the variables $q_s - 1$, $s \in S$, with nonnegative integer coefficients. The result stated in the theorem is stronger than this, and so we need to sharpen the methods used in the proof of Theorem 6.14.

We make the following observations. See Proposition 4.15 for proofs of similar facts (we use the notations of Proposition 4.15 here). Firstly, each $w_1 \in W_0 w_\lambda W_l$ can be written uniquely as $w_1 = u_1 w_\lambda w_l$ for some $u_1 \in W_0^\lambda$ and $w_l \in W_l$, and moreover $\ell(w_1) = \ell(u_1) + \ell(w_\lambda) + \ell(w_l)$. Similarly, each $w_2 \in W_l \sigma_l(w_\mu) W_n$ can be written uniquely as $w_2 = w_l' \sigma_l(w_\mu) u_2$ for some $u_2 \in W_n^\mu$ and $w_l' \in W_l$, and moreover $\ell(w_2) = \ell(w_l') + \ell(\sigma_l(w_\mu)) + \ell(u_2)$.

Secondly, each $w \in W_0 w_\lambda$ can be written uniquely as $w = u w_\lambda$ for some $u \in W_0^\lambda$, and moreover $\ell(w) = \ell(u) + \ell(w_\lambda)$. Similarly, each $w' \in \sigma_l(w_\mu) W_n$ can be written uniquely as $w' = \sigma_l(w_\mu) u'$ for some $u' \in W_n^\mu$, and, moreover, $\ell(w') = \ell(\sigma_l(w_\mu)) + \ell(u')$.

Using these facts, along with the facts that $\mathbb{1}_l^2 = \mathbb{1}_l$ and $W_l(q) = W_0(q)$, we have (compare with the proof of Theorem 6.14)

$$P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}} = \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_0^4(q) q_{w_\lambda} q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda W_l \\ w_2 \in W_l \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} T_{w_1} T_{w_2}$$

$$= \frac{W_{0\lambda}(q) W_{0\mu}(q) W_l^2(q)}{W_0^4(q) q_{w_\lambda} q_{w_\mu}} \left(\sum_{u_1 \in W_0^\lambda} q_{u_1 w_\lambda} T_{u_1 w_\lambda} \right) \mathbb{1}_l^2$$

$$\times \left(\sum_{u_2 \in W_n^\mu} q_{\sigma_l(w_\mu) u_2} T_{\sigma_l(w_\mu) u_2} \right)$$

$$= \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_0^2(q) q_{w_\lambda} q_{w_\mu}} \left(\sum_{w \in W_0 w_\lambda} q_w T_w \right) \mathbb{1}_l \left(\sum_{w' \in \sigma_l(w_\mu) W_n} q_{w'} T_{w'} \right)$$

$$= \frac{W_{0\lambda}(q) W_{0\mu}(q)}{W_0^3(q) q_{w_\lambda} q_{w_\mu}} \sum_{\substack{w_1 \in W_0 w_\lambda, \, w_2 \in W_l \\ w_3 \in \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} q_{w_3} T_{w_1} T_{w_2} T_{w_3}.$$

It is simple to see that

$$\sum_{\substack{w_1 \in W_0 w_\lambda, \, w_2 \in W_l \\ w_3 \in \sigma_l(w_\mu) W_n}} q_{w_1} q_{w_2} q_{w_3} T_{w_1} T_{w_2} T_{w_3} = \sum_{w \in W} d_w(\lambda, \mu) q_w T_w$$

where $d_w(\lambda, \mu)$ is a linear combination of products of $d'_{w_1,w_2;w_3}$'s with nonnegative integer coefficients, and so

$$P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}} = \frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_0^3(q)q_{w_\lambda}q_{w_\mu}} \sum_{w \in W} d_w(\lambda, \mu)q_w T_w.$$

So the coefficient of T_{w_ν} when $P_\lambda \mathbb{1}_0 P_\mu \mathbb{1}_0 T_{g_n^{-1}}$ is expanded in terms of the T_w's is

$$\frac{W_{0\lambda}(q)W_{0\mu}(q)}{W_0^3(q)} \frac{q_{w_\nu}}{q_{w_\lambda}q_{w_\mu}} d_{w_\nu}(\lambda, \mu). \tag{7.2}$$

Comparing (7.2) with (6.11) we see that $c'_{\lambda,\mu;\nu} = d_{w_\nu}(\lambda, \mu)$, and so the result follows from Lemma 7.1 and the fact that $d_{w_\nu}(\lambda, \mu)$ is a linear combination of products of $d'_{w_1,w_2;w_3}$'s with nonnegative integer coefficients. $\quad\square$

Acknowledgments

Appendix A. Parameter systems of regular affine buildings

For an $\tilde{X}_n$ building there $n + 1$ vertices in the Coxeter graph. The special vertices are marked with an s. If all of the parameters are equal we write $q_i = q$.

$\tilde{E}_6$:

$\tilde{E}_7$:

$\tilde{E}_8$:

$\tilde{F}_4$:

$\tilde{G}_2$:
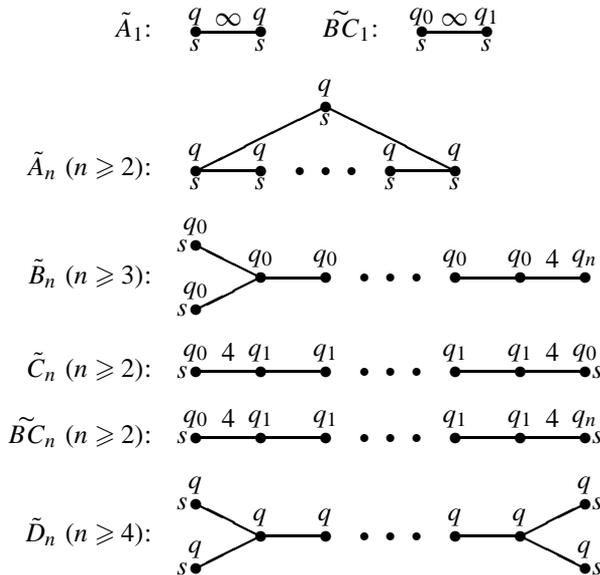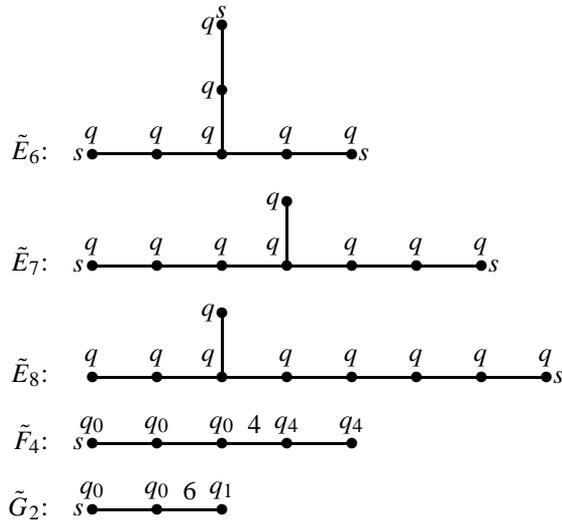
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