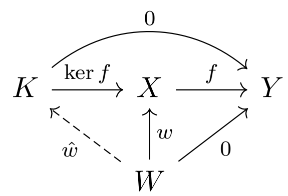
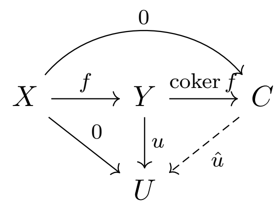
Additive and Abelian Categories

- A *zero object* $Z \in \mathcal{C}$ is an object which is both initial and terminal.
- A *subobject* of an object A in a category $\mathcal{C}$ is a monomorphism $u : B \to A$. If $v : C \to A$ is another monomorphism, say that u and v are *equivalent as subobjects of* A if there is an isomorphism $\phi : B \to C$ such that $u = v\phi$.
- A *pointed category* is a category admitting a zero object.
- In a pointed category, a *biproduct* of X, Y is the data (B, p_X, p_Y, i_X, i_Y) such that (B, p_X, p_Y) is a categorical product of X and Y, (B, i_X, i_Y) is a categorical coproduct of X and Y, and this data is *compatible* in the sense that $p_X \circ i_X = \mathrm{id}_X$, $p_Y \circ i_Y = \mathrm{id}_Y$, $p_X \circ i_Y = 0_{YX}$, $p_Y \circ i_X = 0_{XY}$.
- The category $\mathcal{A}$ is *additive* if it satisfies the following conditions:
  1. There is a zero object in $\mathcal{A}$.
  2. For any $X, Y \in \mathcal{A}$, a categorical product $X \times Y \in \mathcal{A}$ exists.
  3. Each hom-set $\mathcal{A}(X, Y)$ is an abelian group, and composition of morphisms is bilinear, i.e. the maps $\mathcal{A}(Y, Z) \times \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)$, $(f, g) \mapsto f \circ g$ are $\mathbb{Z}$-bilinear.
- In an additive category, if a product $(X \times Y, p_X, p_Y)$ exists, it extends uniquely to a biproduct $(X \times Y, p_X, p_Y, i_X, i_Y)$. Similarly for a coproduct.
- A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is an *additive functor* if $F(f + g) = Ff + Fg$.
- Let $(X \xrightarrow{f} Y) \in \mathcal{C}$, a pointed category. A *kernel* of f is an object $K \in \mathcal{C}$, along with a map $(K \xrightarrow{\ker f} X)$, such that $f \circ (\ker f) = 0$, and whenever $(W \xrightarrow{w} X)$ satisfies $f \circ w = 0$, there exists a unique $\hat{w} : W \to K$ such that



- Let $(X \xrightarrow{f} Y) \in \mathcal{C}$, a pointed category. A *cokernel* of f is an object $C \in \mathcal{C}$, along with a map $(Y \xrightarrow{\mathrm{coker}\, f} C)$, such that $(\mathrm{coker}\, f) \circ f = 0$, and whenever $(Y \xrightarrow{u} U)$ satisfies $u \circ f = 0$, there exists a unique $(C \xrightarrow{\hat{u}} U)$ such that



- The *image* of $f : X \to Y$ is $\mathrm{im}\, f = \ker(\mathrm{coker}\, f)$, whenever it exists.
- The *coimage* of $f : X \to Y$ is $\mathrm{coim}\, f = \mathrm{coker}(\ker f)$, whenever it exists.
- The additive category $\mathcal{A}$ is called an *abelian category* if all morphisms admit kernels and cokernels, and furthermore that every monomorphism arises as a kernel, and every epimorphism arises as a cokernel.
- $\mathcal{A}$ abelian. The sequence $(A \xrightarrow{f} B \xrightarrow{g} C) \in \mathcal{A}$ is *exact at* B if $\mathrm{im}\, f \cong \ker g$ as subobjects of B.

Easy exercises about biproducts and additive categories:

1. If $Z, Z' \in \mathcal{C}$ are zero objects, there is a unique isomorphism $Z \xrightarrow{\sim} Z'$.

2. Let $\mathcal{C}$ be a pointed category, and $X, Y \in \mathcal{C}$ objects. Define the zero map $X \xrightarrow{0_{XY}} Y$. If $\mathcal{C}$ happens to be additive, show 0_{XY} is necessarily the identity in the abelian group $\mathcal{C}(X, Y)$.

3. If X is an object in an additive category $\mathcal{A}$, then $\mathcal{A}(X, X)$ is naturally a unital ring.

4. Let $\mathcal{C}$ be a category with binary products and coproducts. Given an object $A \in \mathcal{C}$, define the *diagonal* map $\Delta_A : A \to A \times A$ and the *codiagonal* map $\nabla_A : A \amalg A \to A$.

5. Let $\mathcal{C}$ be a pointed category with binary biproducts. Show that each hom-set is naturally a commutative monoid, with $\mathbb{N}$-bilinear composition. If $\mathcal{C}$ is additive, does this commutative monoid structure necessarily agree with the abelian group structure?

6. (*A more efficient definition of biproducts*) Let $\mathcal{C}$ be an additive category, and suppose we have a diagram of the form

$$X \underset{p_X}{\overset{i_X}{\rightleftarrows}} B \underset{p_Y}{\overset{i_Y}{\leftrightarrows}} Y$$

satisfying the three equations

$$p_X i_X = \mathrm{id}_X, \quad p_Y i_Y = \mathrm{id}_Y, \quad i_X p_X + i_Y p_Y = \mathrm{id}_B \,.$$

(This diagram and set of equations is a *binary biproduct diagram*). Show that these maps then equip B with the structure of a biproduct of X and Y. Conversely, show that the equation $i_X p_X + i_Y p_Y = \mathrm{id}_B$ holds for any biproduct.

7. Let $e_1, \ldots, e_n$ be a basis of the k-vector space V. This basis determines injections $i_j : k \to V$, $\lambda \mapsto \lambda e_j$ equipping V with the structure of a coproduct of n copies of k. There is a unique compatible product structure making V into a biproduct $k^{\oplus n}$: what is it?

8. **Important exercise**. Suppose $\mathcal{C}$ is a pointed category with binary biproducts. Explain how a map $A \oplus B \xrightarrow{f} C \oplus D$ may be represented as a matrix of maps

$$[f] = \begin{pmatrix} f_{AC} & f_{BC} \\ f_{AD} & f_{BD} \end{pmatrix} = \begin{pmatrix} f_{AC} : A \to C & f_{BC} : B \to C \\ f_{AD} : A \to D & f_{BD} : B \to D \end{pmatrix}$$

Write down formulas for each of the maps in the matrix. Show that the maps in the matrix uniquely determine f. Show that $[f \circ g] = [f][g]$, i.e. that composition is matrix multiplication.

9. Write down the maps i_X, i_Y, p_X, p_Y in the biproduct $A \oplus B$ in matrix form. Write down the diagonal $A \to A \oplus A$ and the codiagonal $A \oplus A \to A$ in matrix form.

10. (*Non-essential exercise: a category with addition but not subtraction*) The category Rel has sets as its objects, and relations as its morphisms: A morphism $R : A \to B$ is a subset of $B \times A$, with notation bRa meaning $(b, a) \in R$. The composition rule for $R : A \to B$ and $S : B \to C$ is

$$S \circ R : A \to C, \quad c(S \circ R)a \iff \exists b \in B \text{ such that } cSb \text{ and } bRa.$$

   a) Show this is a pointed category (identity morphism, composition is associative, zero object).
   b) Show that the disjoint union of sets can be equipped with a biproduct structure. (5) now implies that morphisms can be added. Can they always be subtracted?

Some exercises on abelian categories:

1. Show that if $u : B \to A$ and $v : C \to A$ are subobjects of A in Vect_k, that they are equivalent subobjects iff $\operatorname{im} u = \operatorname{im} v$ (where the image is a vector space image, not a categorical one).
2. Let $\mathcal{C}$ be a pointed category (so that kernels and cokernels are defined). Show the following:
   a) Kernels are always monic.
   b) Cokernels are always epic.
3. In an abelian category, a morphism which is both monic and epic is an isomorphism.
4. In an abelian category, every arrow f factors as $f = me$, where m is monic and e is epic. (*Full disclosure: I have no idea how annoying this proof really is but it looks kinda annoying.*)
5. Show that the category of quiver representations $\mathsf{Rep}_k Q$ is abelian. (Either show it directly, or show $\mathsf{Rep}_k Q$ is isomorphic to the category $kQ{-}\mathsf{mod}$, where kQ is the path algebra).
6. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a sequence in an abelian category. Verify the usual stuff:
   a) Exactness at A iff f is monic.
   b) Exactness at C iff g is epic.
   c) Exactness at A, B, and C iff $f = \ker g$ and $g = \operatorname{coker} f$.

Determine why each of the following categories fails to be additive/abelian:

1. The category of groups and group homomorphisms.
2. The full subcategory of k-vector spaces whose dimensions are powers of 2.
3. The full subcategory of even-dimensional k-vector spaces.
4. The full subcategory of $\mathbb{Z}$-modules admitting a finite basis.
5. $K^+(\mathbb{Z}{-}\mathsf{mod})$, the homotopy category of bounded-below complexes of $\mathbb{Z}$-modules. Hint: start with the nontrivial morphism $(\cdots \to 0 \to \mathbb{Z} \to 0 \to \cdots) \to (\cdots \to 0 \to \mathbb{Z}/(2) \to 0 \to \cdots)$. Since $K^+(\mathcal{A})$ may not be abelian, we care about its triangulated structure instead, where distinguished triangles would take the place of short exact sequences.