GELFAND–TSETLIN BASES FOR REPRESENTATIONS OF FINITE W-ALGEBRAS AND SHIFTED YANGIANS

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Abstract. Remarkable subalgebras of the Yangian for $\mathfrak{gl}_n$ called the shifted Yangians were introduced in a recent work by Brundan and Kleshchev in relation to their study of finite W-algebras. In particular, in that work a classification of finite-dimensional irreducible representations of the shifted Yangians and the associated finite W-algebras was given. We construct a class of these representations in an explicit form via bases of Gelfand–Tsetlin type.

1. Introduction

A striking relationship between the Yangians and finite W-algebras was first discovered by Ragoucy and Sorba [14]; see also Briot and Ragoucy [1]. This relationship was developed in full generality by Brundan and Kleshchev [4]. The finite W-algebras associated to nilpotent orbits in the Lie algebra $\mathfrak{gl}_N$ turned out to be isomorphic to quotients of certain subalgebras of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. These subalgebras, called the *shifted Yangians* in [4], admit a description in terms of generators and relations. This leads to respective presentations of the finite W-algebras and thus provides new tools to study their structure and representations. The representation theory of the shifted Yangians and associated W-algebras was developed in a subsequent paper by Brundan and Kleshchev [5] where many deep and remarkable connections of the shifted Yangian representation theory were explored. In particular, a classification of the finite-dimensional irreducible representations of the shifted Yangians and the finite W-algebras was given in terms of their highest weights. Moreover, in the case of the shifted Yangian associated to $\mathfrak{gl}_2$ all such representations were explicitly constructed.

Our aim in this paper is to construct in an explicit form a family of representations of the shifted Yangians and finite W-algebras via bases of Gelfand–Tsetlin type. Such bases for certain classes of representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ were constructed in different ways by Nazarov and Tarasov [12, 13] and Molev [10]. We mainly employ the approach of [12, 13] which turns out to be more suitable for the generalization to the case of the shifted Yangians. In more detail, following

[4], consider an n-tuple of positive integers $\pi = (p_1, \dots, p_n)$ such that $p_1 \leqslant \dots \leqslant p_n$. We can visualize π as a *pyramid* of left-justified rows of bricks, where the top row contains p_1 bricks, the second row contains p_2 bricks, etc. Such a pyramid determines a finite W-algebra which we denote by $W(\pi)$. For each $k \in \{1, \dots, n\}$ we let π_k denote the pyramid with the rows $(p_1, \dots, p_k)$. Our basis is consistent with the chain of subalgebras

$$W(\pi_1) \subset W(\pi_2) \subset \dots \subset W(\pi_n). \tag{1}$$

In the case of the one-column pyramid $(1, \dots, 1)$ of height n we recover the classical Gelfand–Tsetlin basis for representations of the Lie algebra $\mathfrak{gl}_n$. For any π, the formulas for the action of the Drinfeld generators of $W(\pi)$ in the basis turn out to be quite similar to the Yangian case. These explicit constructions of representations of $W(\pi)$ proved to be useful for a description of the Harish-Chandra modules over finite W-algebras and a proof of the associated Gelfand–Kirillov conjecture based on recent results of Futorny and Ovsienko [8, 9]; see our forthcoming paper [7].

2. Shifted Yangians and W-algebras

As in [4], given a pyramid $\pi = (p_1, \dots, p_n)$ with $p_1 \leqslant \dots \leqslant p_n$, introduce the corresponding *shifted Yangian* $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ as the associative algebra defined by generators

$$\begin{aligned} &d_i^{(r)}, \quad i = 1, \dots, n, && r \geqslant 1, \\ &f_i^{(r)}, \quad i = 1, \dots, n-1, && r \geqslant 1, \\ &e_i^{(r)}, \quad i = 1, \dots, n-1, && r \geqslant p_{i+1} - p_i + 1, \end{aligned} \tag{2}$$

subject to the following relations:

$$\begin{aligned} &[d_i^{(r)}, d_j^{(s)}] = 0, \\ &[e_i^{(r)}, f_j^{(s)}] = -\,\delta_{ij} \sum_{t=0}^{r+s-1} d_i'^{\,(t)}\, d_{i+1}^{(r+s-t-1)}, \\ &[d_i^{(r)}, e_j^{(s)}] = (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)}\, e_j^{(r+s-t-1)}, \\ &[d_i^{(r)}, f_j^{(s)}] = (\delta_{i,j+1} - \delta_{ij}) \sum_{t=0}^{r-1} f_j^{(r+s-t-1)}\, d_i^{(t)}, \end{aligned}$$

$$
\begin{aligned}
&[e_i^{(r)}, e_i^{(s+1)}] - [e_i^{(r+1)}, e_i^{(s)}] = e_i^{(r)} e_i^{(s)} + e_i^{(s)} e_i^{(r)},\\
&[f_i^{(r+1)}, f_i^{(s)}] - [f_i^{(r)}, f_i^{(s+1)}] = f_i^{(r)} f_i^{(s)} + f_i^{(s)} f_i^{(r)},\\
&[e_i^{(r)}, e_{i+1}^{(s+1)}] - [e_i^{(r+1)}, e_{i+1}^{(s)}] = -e_i^{(r)} e_{i+1}^{(s)},\\
&[f_i^{(r+1)}, f_{i+1}^{(s)}] - [f_i^{(r)}, f_{i+1}^{(s+1)}] = -f_{i+1}^{(s)} f_i^{(r)},
\end{aligned}
$$

$$
\begin{aligned}
&[e_i^{(r)}, e_j^{(s)}] = 0 && \text{if} \quad |i-j|>1,\\
&[f_i^{(r)}, f_j^{(s)}] = 0 && \text{if} \quad |i-j|>1,\\
&[e_i^{(r)}, [e_i^{(s)}, e_j^{(t)}]] + [e_i^{(s)}, [e_i^{(r)}, e_j^{(t)}]] = 0 && \text{if} \quad |i-j|=1,\\
&[f_i^{(r)}, [f_i^{(s)}, f_j^{(t)}]] + [f_i^{(s)}, [f_i^{(r)}, f_j^{(t)}]] = 0 && \text{if} \quad |i-j|=1,
\end{aligned}
$$

for all admissible i, j, r, s, t, where $d_i^{(0)} = 1$ and the elements $d_i^{\prime\,(r)}$ are found from the relations

$$
\sum_{t=0}^{r} d_i^{(t)}\, d_i^{\prime\,(r-t)} = \delta_{r0}, \qquad r = 0, 1, \dots .
$$

Note that the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ depends only on the differences $p_{i+1} - p_i$. In the particular case of a rectangular pyramid π with $p_1 = \dots = p_n$, the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ is isomorphic to the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$; see e.g. [11] for the description of its structure and representations. The isomorphism with the *RTT* presentation of $\mathrm{Y}(\mathfrak{gl}_n)$ was constructed in [3] providing a proof of the original result of Drinfeld [6]. Moreover, for an arbitrary pyramid π, the shifted Yangian $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ can be regarded as a natural subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$. Note also that the shifted Yangians can be defined for more general types of pyramids. However, in accordance to [4], each of these algebras is isomorphic to $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ for an appropriate left-justified pyramid π.

Introduce formal generating series in u^{-1} by

$$
d_i(u) = 1 + \sum_{r=1}^{\infty} d_i^{(r)}\, u^{-r}, \qquad f_i(u) = \sum_{r=1}^{\infty} f_i^{(r)}\, u^{-r},
$$

$$
e_i(u) = \sum_{r=p_{i+1}-p_i+1}^{\infty} e_i^{(r)}\, u^{-r}
$$

and set

$$
a_i(u) = d_1(u)\, d_2(u-1) \dots d_i(u-i+1)
$$

for $i = 1, \dots, n$, and

$$
b_i(u) = a_i(u)\, e_i(u-i+1), \qquad c_i(u) = f_i(u-i+1)\, a_i(u)
$$

for $i = 1, \dots, n-1$. It is clear that the coefficients of the series $a_i(u)$, $b_i(u)$ and $c_i(u)$ generate the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$. It is not difficult to rewrite the defining relations in terms of these coefficients. We point out a few of these relations here which will be frequently used later on; see also [3]. We have

$$[a_i(u), c_j(v)] = 0, \qquad [b_i(u), c_j(v)] = 0, \qquad \text{if} \quad i \neq j, \tag{3}$$

$$[c_i(u), c_j(v)] = 0, \qquad \text{if} \quad |i-j| \neq 1, \tag{4}$$

$$(u-v)\,[a_i(u), c_i(v)] = c_i(u)\,a_i(v) - c_i(v)\,a_i(u). \tag{5}$$

Let N be the number of bricks in the pyramid π. Due to the main result of [4], the *finite W-algebra* $W(\pi)$, associated to $\mathfrak{gl}_N$ and the pyramid π, can be defined as the quotient of $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ by the two-sided ideal generated by all elements $d_1^{(r)}$ with $r \geqslant p_1 + 1$. We refer the reader to [4, 5] for a discussion of the origins of the finite W-algebras and more references. Note that in the case of a rectangular pyramid of height p, the algebra $W(\pi)$ is isomorphic to the *Yangian of level* p; this relationship was originally observed in [1] and [14].

We will use the same notation for the images of the elements of $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ in the quotient algebra $W(\pi)$. Set

$$A_i(u) = u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+1)^{p_i}\,a_i(u)$$

for $i = 1, \dots, n$, and

$$B_i(u) = u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+2)^{p_{i-1}}\,(u-i+1)^{p_{i+1}}\,b_i(u),$$
$$C_i(u) = u^{p_1}\,(u-1)^{p_2}\,\dots(u-i+1)^{p_i}\,c_i(u)$$

for $i = 1, \dots, n-1$. The following lemma is immediate from the results of Brown and Brundan [2]. Here we regard $A_i(u)$, $B_i(u)$, and $C_i(u)$ as series with coefficients in $W(\pi)$.

Lemma 2.1. *All series $A_i(u)$, $B_i(u)$, and $C_i(u)$ are polynomials in u.*

Proof. In terms of the *RTT* presentation of the Yangian, each of the series $a_i(u) \in \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]]$ coincides with a quantum minor of the matrix of the generators; see [3, Theorem 8.6]. Therefore the statement for the $A_i(u)$ follows from the results of [2, Section 3]. Note that the polynomial $A_i(u)$ in u is monic of degree $p_1 + \dots + p_i$. Furthermore, the defining relations of $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ imply $[f_i^{(1)}, a_i(u)] = c_i(u)$, and so $C_i(u) = [f_i^{(1)}, A_i(u)]$ is a polynomial in u of degree $p_1 + \dots + p_i - 1$. Similarly,

$$b_i(u)\,(u-i+1)^{p_{i+1}-p_i} = [a_i(u), e_i^{(p_{i+1}-p_i+1)}],$$

which gives

$$B_i(u) = [A_i(u), e_i^{(p_{i+1}-p_i+1)}],$$

so that $B_i(u)$ is a polynomial in u of degree $p_1+\cdots+p_i-1$. $\square$

Note that by [5, Theorem 6.10], all coefficients of the polynomial $A_n(u)$ belong to the center of $W(\pi)$ and these coefficients (excluding the leading one) are algebraically independent generators of the center.

For $i=1,\dots,n-1$ define the elements $h_i^{(r)}\in \mathrm{Y}_\pi(\mathfrak{gl}_n)$ by the expansion

$$1+\sum_{r=1}^{\infty} h_i^{(r)}\,u^{-r}=d_i(u)^{-1}\,d_{i+1}(u)$$

and set

$$H_i^{(r)}(u)=u^r+u^{r-1}\,h_i^{(1)}+\cdots+h_i^{(r)}.$$

Lemma 2.2. *For $i=1,\dots,n-1$ in the algebra $W(\pi)$ we have*

$$(u-v)\,[B_i(u),C_i(v)]=A'_{i+1}(u)\,A_i(v)-A'_{i+1}(v)\,A_i(u),$$

where $A'_{i+1}(u)$ is the polynomial in u with coefficients in $W(\pi)$ given by

$$\begin{aligned}A'_{i+1}(u)&=u^{p_1}\,(u-1)^{p_2}\dots(u-i+2)^{p_{i-1}}\,(u-i+1)^{p_{i+1}}\\&\quad\times a_i(u+1)^{-1}\big(a_{i+1}(u+1)\,a_{i-1}(u)+c_i(u+1)\,b_i(u)\big)\\&\quad-H_i^{(p_{i+1}-p_i)}(u-i+1)\,A_i(u).\end{aligned}$$

Moreover,

$$\begin{aligned}B_i(u)\,C_i(u-1)&=A'_{i+1}(u)\,A_i(u-1)-A_{i+1}(u)\,A_{i-1}(u-1)\\&\quad+H_i^{(p_{i+1}-p_i)}(u-i)\,A_i(u)\,A_i(u-1).\end{aligned}$$

Proof. Observe that for any fixed $i\in\{1,\dots,n-1\}$ the elements $d_i^{(r)}$, $d_{i+1}^{(r)}$, $e_i^{(r)}$ and $f_i^{(r)}$ of $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ satisfy the defining relations of the shifted Yangian $\mathrm{Y}_{\pi_i}(\mathfrak{gl}_2)$, where $\pi_i=(p_i,p_{i+1})$. Therefore, it suffices to prove the first relation in the case $i=1$; the proof for the remaining values of i will then easily follow. Working in the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$, we can derive the relation

$$(u-v-1)\,[d_1(u),e_1(v)]=\big(e_1(v)-e_1(u)\big)\,d_1(u);$$

see e.g. [3]. This allows us to calculate the commutators $[d_1(u),e_1^{(r)}]$ and leads to an equivalent expression for $b_1(u)$ in the subalgebra $\mathrm{Y}_\pi(\mathfrak{gl}_2)$:

$$b_1(u)=d_1(u)\,e_1(u)=(1-u^{-1})^{p_2-p_1}\,e_1(u-1)\,d_1(u).$$

Furthermore, starting from the relations

$$[e_1^{(r)},f_1^{(s)}]=-\sum_{t=0}^{r+s-1}d_1'^{\,(t)}\,d_2^{(r+s-t-1)}$$

in $\mathrm{Y}_\pi(\mathfrak{gl}_2)$, it is now straightforward to derive that

$$\begin{aligned}(u-v)\,[b_1(u),c_1(v)] &= a_1(u+1)^{-1}\big(a_2(u+1)+c_1(u+1)\,b_1(u)\big)\,a_1(v)\\ &\quad -(u^{-1}v)^{p_2-p_1}a_1(v+1)^{-1}\big(a_2(v+1)+c_1(v+1)\,b_1(v)\big)\,a_1(u)\\ &\quad - u^{p_1-p_2}\big(H_1^{(p_2-p_1)}(u)-H_1^{(p_2-p_1)}(v)\big)\,a_1(u)\,a_1(v).\end{aligned}$$

The desired relation in $W(\pi)$ is then obtained by multiplying both sides by the product $u^{p_2}\,v^{p_1}$. Furthermore, by the defining relations,

$$\begin{aligned}u^{p_2}\,a_1(u+1)^{-1}&\big(a_2(u+1)+c_1(u+1)\,b_1(u)\big)\\ &= u^{p_2}\,\big(d_2(u)+f_1(u)\,d_1(u)\,e_1(u)\big).\end{aligned}$$

This is a polynomial in u due to [5, Theorem 3.5]. Hence, by Lemma 2.1, $A_2'(u)$ is a polynomial in u too.

The second part of the lemma is implied by the first by taking into account the relations in the shifted Yangian $\mathrm{Y}_\pi(\mathfrak{gl}_n)$,

$$a_i(u)^{-1}c_i(u) = c_i(u-1)\,a_i(u-1)^{-1}$$

and

$$(u-i)^{p_{i+1}-p_i}\,a_i(u-1)^{-1}\,b_i(u-1) = (u-i+1)^{p_{i+1}-p_i}\,b_i(u)\,a_i(u)^{-1},$$

which are implied by the defining relations. □

3. Construction of basis vectors

Using the canonical homomorphism $\mathrm{Y}_\pi(\mathfrak{gl}_n)\to W(\pi)$ we can extend every representation of the finite W-algebra $W(\pi)$ to the shifted Yangian $\mathrm{Y}_\pi(\mathfrak{gl}_n)$. In what follows we work with representations of $W(\pi)$, and the results can be easily interpreted in the shifted Yangian context.

Let us recall some definitions and results from [5] regarding representations of $W(\pi)$. Fix an n-tuple $\lambda(u)=\big(\lambda_1(u),\dots,\lambda_n(u)\big)$ of monic polynomials in u with coefficients in $\mathbb{C}$, where $\lambda_i(u)$ has degree p_i. We let $L(\lambda(u))$ denote the irreducible highest weight representation of $W(\pi)$ with the highest weight $\lambda(u)$. Then $L(\lambda(u))$ is generated by a nonzero vector ζ (the highest vector) such that

$$\begin{aligned}B_i(u)\,\zeta &= 0 &&\text{for}\quad i=1,\dots,n-1, \qquad\text{and}\\ u^{p_i}\,d_i(u)\,\zeta &= \lambda_i(u)\,\zeta &&\text{for}\quad i=1,\dots,n.\end{aligned}$$

Write

$$\lambda_i(u)=(u+\lambda_i^{(1)})\,(u+\lambda_i^{(2)})\dots(u+\lambda_i^{(p_i)}),\qquad i=1,\dots,n.$$

We will assume that the parameters $\lambda_i^{(k)}$ satisfy the conditions: for any value $k \in \{1, \dots, p_i\}$ we have

$$\lambda_i^{(k)} - \lambda_{i+1}^{(k)} \in \mathbb{Z}_+, \qquad i = 1, \dots, n-1,$$

where $\mathbb{Z}_+$ denotes the set of nonnegative integers. In this case the representation $L(\lambda(u))$ of $W(\pi)$ is finite-dimensional.

Denote by q_k the number of bricks in the column k of the pyramid π. We have $q_1 \geqslant \dots \geqslant q_l > 0$, where $l = p_n$ is the number of the columns in π. If $p_{i-1} < k \leqslant p_i$ for some $i \in \{1, \dots, n\}$ (taking $p_0 = 0$), then we set $\lambda^{(k)} = (\lambda_i^{(k)}, \dots, \lambda_n^{(k)})$. Then $q_k = n - i + 1$. Let $L(\lambda^{(k)})$ denote the finite-dimensional irreducible representation of the Lie algebra $\mathfrak{gl}_{q_k}$ with the highest weight $\lambda^{(k)}$. The vector space

$$L(\lambda^{(1)}) \otimes \dots \otimes L(\lambda^{(l)}) \tag{6}$$

can be equipped with an action of the algebra $W(\pi)$, and $L(\lambda(u))$ is isomorphic to a subquotient of the module (6). In particular,

$$\dim L(\lambda(u)) \leqslant \prod_{k=1}^{l} \dim L(\lambda^{(k)}). \tag{7}$$

In what follows we will only consider a certain family of representations of $W(\pi)$ by imposing a *generality condition* on the highest weights of the representations $L(\lambda(u))$. We will assume that

$$\lambda_i^{(k)} - \lambda_j^{(m)} \notin \mathbb{Z}, \qquad \text{for all} \quad i, j \quad \text{and all} \quad k \neq m.$$

The *Gelfand–Tsetlin pattern* $\Lambda(u)$ (associated with the highest weight $\lambda(u)$) is an array of monic polynomials in u of the form

$$\begin{array}{ccccccc}
\lambda_{n1}(u) & & \lambda_{n2}(u) & & \dots & & \lambda_{nn}(u) \\
& \lambda_{n-1,1}(u) & & \dots & & \lambda_{n-1,n-1}(u) & \\
& & \dots & & \dots & & \\
& & \lambda_{21}(u) & & \lambda_{22}(u) & & \\
& & & \lambda_{11}(u) & & &
\end{array}$$

where

$$\lambda_{ri}(u) = (u + \lambda_{ri}^{(1)}) \dots (u + \lambda_{ri}^{(p_i)}), \qquad 1 \leqslant i \leqslant r \leqslant n,$$

with $\lambda_{ni}^{(k)} = \lambda_i^{(k)}$ and the following conditions hold

$$\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}_+ \qquad \text{and} \qquad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}_+$$

for $k = 1, \dots, p_i$ and $1 \leqslant i \leqslant r \leqslant n-1$. We have $\lambda_{ni}(u) = \lambda_i(u)$ for $i = 1, \dots, n$, so that the top row coincides with $\lambda(u)$.

Most arguments in the rest of the paper will not be essentially different from [13, Section 3], so we only sketch the main steps in the construction of the basis. Given a pattern $\Lambda(u)$, introduce the corresponding element ζ_Λ of $L(\lambda(u))$ by the formula

$$\zeta_\Lambda = \overrightarrow{\prod_{i=1,\dots,n-1}} \Big\{ \prod_{k=1}^{p_i} \Big(C_{n-1}(-l_{n-1,i}^{(k)} - 1) \dots C_{n-1}(-l_i^{(k)}) \Big)$$
$$\times \prod_{k=1}^{p_i} \Big(C_{n-2}(-l_{n-2,i}^{(k)} - 1) \dots C_{n-2}(-l_i^{(k)} + 1)\, C_{n-2}(-l_i^{(k)}) \Big)$$
$$\times \dots \times \prod_{k=1}^{p_i} \Big(C_i(-l_{ii}^{(k)} - 1) \dots C_i(-l_i^{(k)} + 1)\, C_i(-l_i^{(k)}) \Big) \Big\} \zeta,$$

where we have used the notation

$$l_i^{(k)} = \lambda_i^{(k)} - i + 1 \qquad \text{and} \qquad l_{ri}^{(k)} = \lambda_{ri}^{(k)} - i + 1.$$

Note that by (4) we have $[C_i(u), C_i(v)] = 0$, so that the order of the factors in the products over k is irrelevant.

Lemma 3.1. *We have*

$$A_r(u)\, \zeta_\Lambda = \lambda_{r1}(u) \dots \lambda_{rr}(u - r + 1)\, \zeta_\Lambda,$$

for $r = 1, \dots, n$.

Proof. When applying $A_r(u)$ to ζ_Λ, separating the first factor, we need to calculate $A_r(u)\, C_s(v)\, \eta$ for the respective value of v. By (3), the operator $A_r(u)$ commutes with $C_s(v)$ for $s \neq r$. Furthermore, by (5),

$$A_r(u)\, C_r(v)\, \eta = \frac{1}{u - v}\, C_r(u)\, A_r(v)\, \eta + \frac{u - v - 1}{u - v}\, C_r(v)\, A_r(u)\, \eta.$$

The calculation is completed by induction on the number of factors $C_i(v)$ in the expression for ζ_Λ, taking into account that $A_r(v)\, \eta = 0$. □

Lemma 3.2. *For any* $1 \leqslant i \leqslant r \leqslant n-1$ *and* $k = 1, \dots, p_i$ *we have*

$$B_r(-l_{ri}^{(k)})\, \zeta_\Lambda = -\lambda_1(-l_{ri}^{(k)}) \dots \lambda_i(-l_{ri}^{(k)} - i + 1)$$
$$\times \lambda_{r+1,i+1}(-l_{ri}^{(k)} - i) \dots \lambda_{r+1,r+1}(-l_{ri}^{(k)} - r)$$
$$\times \lambda_1(-l_{ri}^{(k)} - 1) \dots \lambda_{i-1}(-l_{ri}^{(k)} - i + 1)$$
$$\times \lambda_{r-1,i}(-l_{ri}^{(k)} - i) \dots \lambda_{r-1,r-1}(-l_{ri}^{(k)} - r + 1)\, \zeta_{\Lambda + \delta_{ri}^{(k)}},$$

where $\zeta_{\Lambda+\delta^{(k)}_{ri}}$ *corresponds to the pattern obtained from* $\Lambda(u)$ *by replacing* $\lambda^{(k)}_{ri}$ *by* $\lambda^{(k)}_{ri}+1$, *and the vector* ζ_Λ *is considered to be zero, if* $\Lambda(u)$ *is not a pattern.*

Proof. The argument is based on Lemma 2.2. As in the proof of Lemma 3.1, separating the first factor, we need to calculate $B_r(-l^{(k)}_{ri})\, C_s(v)\, \eta$ for the respective value of v. By (3), the operator $B_r(u)$ commutes with $C_s(v)$ for $s \neq r$. If $s = r$ then we consider two cases. If $-l^{(k)}_{ri} - v \neq 1$, then applying the first relation of Lemma 2.2 together with Lemma 3.1, we find that

$$B_r(-l^{(k)}_{ri})\, C_r(v)\, \eta = C_r(v)\, B_r(-l^{(k)}_{ri})\, \eta$$

and proceed by induction. If $v = -l^{(k)}_{ri} - 1$, then we apply the second relation of Lemma 2.2 together with Lemma 3.1 to get

$$B_r(-l^{(k)}_{ri})\, C_r(-l^{(k)}_{ri} - 1)\, \eta = -A_{r+1}(-l^{(k)}_{ri})\, A_{r-1}(-l^{(k)}_{ri} - 1)\, \eta.$$

One more application of Lemma 3.1 leads to the desired formula. □

The following theorem provides a basis of the Gelfand–Tsetlin type for the representation $L(\lambda(u))$.

Theorem 3.3. *The vectors* ζ_Λ *parameterized by all patterns* $\Lambda(u)$ *associated with the highest weight* $\lambda(u)$, *form a basis of the representation* $L(\lambda(u))$ *of the algebra* $W(\pi)$.

Proof. It is easy to verify that if the array of monic polynomials obtained from $\Lambda(u)$ by increasing the entry $\lambda^{(k)}_{ri}$ by 1 is a pattern, then the coefficient of the vector $\zeta_{\Lambda+\delta^{(k)}_{ri}}$ in the formula of Lemma 3.2 is nonzero. This implies that each vector $\zeta_\Lambda \in L(\lambda(u))$ associated with a pattern $\Lambda(u)$ is nonzero.

Furthermore, by Lemma 3.1, ζ_Λ is an eigenvector for all operators $A_r(u)$ with distinct sets of eigenvalues. This shows that the vectors ζ_Λ are linearly independent.

Finally, for each $i \in \{1, \dots, n\}$ and $p_{i-1} < k \leqslant p_i$ the set of parameters $\big(\lambda^{(k)}_{rj}\big)$ with $i \leqslant j \leqslant r \leqslant n$ forms a Gelfand–Tsetlin pattern associated with the highest weight $\lambda^{(k)}$ of the irreducible representation $L(\lambda^{(k)})$ of the Lie algebra $\mathfrak{gl}_{q_k}$. Hence, the number of patterns $\Lambda(u)$ coincides with the product of dimensions $\dim L(\lambda^{(k)})$ for $k = 1, \dots, l$. Comparing this with (7), we conclude that the number of patterns coincides with $\dim L(\lambda(u))$. □

Note that by Theorem 3.3, we have the equality in (7), and thus we recover a result from [5] that the representation (6) of $W(\pi)$ is irreducible.

4. Action of the generators

We will calculate the action of the generators of $W(\pi)$ in a normalized basis of $L(\lambda(u))$. For any pattern $\Lambda(u)$ associated to $\lambda(u)$ set

$$N_\Lambda=\prod_{(r,i)}\prod_{j=1}^{i-1}\prod_{m=1}^{p_j}\prod_{k=1}^{p_i}(l_j^{(m)}-l_i^{(k)})(l_j^{(m)}-l_i^{(k)}+1)\dots(l_j^{(m)}-l_{ri}^{(k)}-1)$$
$$\times\prod_{j=i}^{r-1}\prod_{m=1}^{p_j}\prod_{k=1}^{p_i}(l_{r-1,j}^{(m)}-l_i^{(k)})(l_{r-1,j}^{(m)}-l_i^{(k)}+1)\dots(l_{r-1,j}^{(m)}-l_{ri}^{(k)}-1),$$

where the pairs (r,i) run over the set of indices satisfying $1\leqslant i\leqslant r\leqslant n-1$. This constant is clearly nonzero for any pattern $\Lambda(u)$. Introduce normalized vectors $\xi_\Lambda\in L(\lambda(u))$ by

$$\xi_\Lambda=N_\Lambda^{-1}\,\zeta_\Lambda.$$

By Theorem 3.3, the vectors ξ_Λ form a basis of the representation $L(\lambda(u))$. The algebra $W(\pi)$ is generated by the coefficients of the polynomials $A_r(u)$ with $r=1,\dots,n$ and the coefficients of the polynomials $B_r(u)$ and $C_r(u)$ with $r=1,\dots,n-1$. Since $B_r(u)$ and $C_r(u)$ are polynomials in u of degree less than $p_1+\dots+p_r$, it suffices to find the values of these polynomials at $p_1+\dots+p_r$ different values of u. The polynomial can then be calculated by the Lagrange interpolation formula. For these values we take the numbers $-l_{ri}^{(k)}$ with $i=1,\dots,r$ and $k=1,\dots,p_i$.

Theorem 4.1. *We have*

$$A_r(u)\,\xi_\Lambda=\lambda_{r1}(u)\dots\lambda_{rr}(u-r+1)\,\xi_\Lambda, \tag{8}$$

for $r=1,\dots,n$, *and*

$$\begin{aligned}B_r(-l_{ri}^{(k)})\,\xi_\Lambda&=-\lambda_{r+1,1}(-l_{ri}^{(k)})\dots\lambda_{r+1,r+1}(-l_{ri}^{(k)}-r)\,\xi_{\Lambda+\delta_{ri}^{(k)}},\\ C_r(-l_{ri}^{(k)})\,\xi_\Lambda&=\lambda_{r-1,1}(-l_{ri}^{(k)})\dots\lambda_{r-1,r-1}(-l_{ri}^{(k)}-r+2)\,\xi_{\Lambda-\delta_{ri}^{(k)}},\end{aligned} \tag{9}$$

for $r=1,\dots,n-1$, *where* $\xi_{\Lambda\pm\delta_{ri}^{(k)}}$ *corresponds to the pattern obtained from* $\Lambda(u)$ *by replacing* $\lambda_{ri}^{(k)}$ *by* $\lambda_{ri}^{(k)}\pm1$.

Proof. The formulas for the action of $A_r(u)$ and $B_r(-l_{ri}^{(k)})$ follow respectively from Lemmas 3.1 and 3.2 by taking into account the normalization constant. Now consider the vector $C_r(-l_{ri}^{(k)})\,\xi_\Lambda$. Arguing as in the proof of Lemma 3.1, and using (8), we find that

$$A_s(u)\,C_r(-l_{ri}^{(k)})\,\xi_\Lambda = C_r(-l_{ri}^{(k)})\,A_s(u)\,\xi_\Lambda = \lambda_{s1}(u)\dots\lambda_{ss}(u-s+1)\,C_r(-l_{ri}^{(k)})\,\xi_\Lambda$$

for $s \neq r$, while

$$A_r(u)\,C_r(-l_{ri}^{(k)})\,\xi_\Lambda = \frac{u+l_{ri}^{(k)}-1}{u+l_{ri}^{(k)}}C_r(-l_{ri}^{(k)})\,A_r(u)\,\xi_\Lambda = \frac{u+l_{ri}^{(k)}-1}{u+l_{ri}^{(k)}}\,\lambda_{r1}(u)\dots\lambda_{rr}(u-r+1)\,C_r(-l_{ri}^{(k)})\,\xi_\Lambda.$$

If $\lambda_{ri}^{(k)} = \lambda_{r+1,i+1}^{(k)}$, then the vector $\xi_{\Lambda-\delta_{ri}^{(k)}}$ is zero and we need to show that $C_r(-l_{ri}^{(k)})\,\xi_\Lambda = 0$. Indeed, otherwise the vector $C_r(-l_{ri}^{(k)})\,\xi_\Lambda$ must be proportional to a certain basis vector of $L(\lambda(u))$. However, this is impossible because none of the basis vectors has the same set of eigenvalues as $C_r(-l_{ri}^{(k)})\,\xi_\Lambda$.

If $\lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \geqslant 1$, then by the same argument we have

$$C_r(-l_{ri}^{(k)})\,\xi_\Lambda = \alpha\,\xi_{\Lambda-\delta_{ri}^{(k)}}$$

for a certain constant α. Its value is found by the application of the operator $B_r(-l_{ri}^{(k)}+1)$ to the vectors on both sides with the use of (8), (9) and the second relation in Lemma 2.2. □

Note that in the particular case of a rectangular pyramid π the normalized basis $\{\xi_\Lambda\}$ coincides with the basis of [10] constructed in a different way.

Let us denote by π' the pyramid with the rows $p_1,\dots,p_{n-1}$. Then the finite W-algebra $W(\pi')$ may be identified with the subalgebra of $W(\pi)$ generated by the elements (2), excluding all $h_n^{(r)}$, $e_{n-1}^{(r)}$ and $f_{n-1}^{(r)}$. Theorem 4.1 implies the following *branching rule* for the reduction $W(\pi)\downarrow W(\pi')$ and thus shows that the basis $\{\xi_\Lambda\}$ is consistent with the chain of subalgebras (1).

Corollary 4.2. *The restriction of the $W(\pi)$-module $L(\lambda(u))$ to the subalgebra $W(\pi')$ is isomorphic to the direct sum of irreducible highest weight $W(\pi')$-modules $L'(\mu(u))$,*

$$L(\lambda(u))|_{W(\pi')} \cong \bigoplus_{\mu(u)} L'(\mu(u)),$$

where $\mu(u)$ runs over all $(n-1)$-tuples of monic polynomials in u of the form $\mu(u) = \big(\mu_1(u), \dots, \mu_{n-1}(u)\big)$, such that

$$\mu_i(u) = (u + \mu_i^{(1)})\,(u + \mu_i^{(2)}) \dots (u + \mu_i^{(p_i)}), \qquad i = 1, \dots, n-1,$$

and the following conditions hold:

$$\lambda_i^{(k)} - \mu_i^{(k)} \in \mathbb{Z}_+ \qquad \text{and} \qquad \mu_i^{(k)} - \lambda_{i+1}^{(k)} \in \mathbb{Z}_+$$

for $k = 1, \dots, p_i$ and $1 \leqslant i \leqslant r \leqslant n-1$. □

For each $i = 1, \dots, n-1$ introduce the polynomials $\tau_{ni}(u)$ and $\tau_{in}(u)$ with coefficients in $W(\pi)$ by the formulas

$$\begin{aligned} \tau_{ni}(u) &= C_{n-1}(u)\, C_{n-2}(u) \dots C_i(u), \\ \tau_{in}(u) &= B_i(u)\, B_{i+1}(u) \dots B_{n-1}(u). \end{aligned} \tag{10}$$

Define the vector $\zeta_\mu \in L(\lambda(u))$ corresponding to the $(n-1)$-tuple of polynomials $\mu(u) = (\mu_1(u), \dots, \mu_{n-1}(u))$ by the formula

$$\zeta_\mu = \prod_{i=1}^{n-1} \prod_{k=1}^{p_i} \Big(\tau_{ni}(-m_i^{(k)} - 1) \dots \tau_{ni}(-l_i^{(k)} + 1)\, \tau_{ni}(-l_i^{(k)}) \Big)\, \zeta,$$

where the ordering of the factors corresponds to increasing indices i and k, and we used the notation

$$m_i^{(k)} = \mu_i^{(k)} - i + 1 \qquad \text{and} \qquad l_i^{(k)} = \lambda_i^{(k)} - i + 1.$$

By Theorem 4.1, each vector ζ_μ generates a $W(\pi')$-submodule of $L(\lambda(u))$, isomorphic to $L'(\mu(u))$. Moreover, the operators $\tau_{ni}(-m_i^{(k)})$ and $\tau_{in}(-m_i^{(k)})$ take ζ_μ to the vectors proportional to $\zeta_{\mu-\delta_i^{(k)}}$ and $\zeta_{\mu+\delta_i^{(k)}}$, respectively. So, the polynomials (10) valued at appropriate points can be regarded as the *lowering* and *raising* operators for the reduction $W(\pi) \downarrow W(\pi')$; cf. [11, Chapter 5].

Acknowledgments

The authors acknowledge the support of the Australian Research Council. The first author is supported in part by the CNPq grant (processo 307812/2004-9) and by the Fapesp grant (processo 2005/60337-2). He is grateful to the University of Sydney for the warm hospitality during his visit.

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