Partial Differential Equations

On trichotomy of positive singular solutions associated with the Hardy–Sobolev operator☆

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Abstract

In this Note, we present a complete classification of singularities of positive solutions of the equation $\Delta u + \frac{\mu}{|x|^2} u = h(u)$ in $\Omega \setminus \{0\}$, where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 3$, $0 \in \Omega$, and $0 < \mu < \frac{(N-2)^2}{4}$. The case $\mu = 0$ with $h(t) = t^q$, $q > 1$ were treated by Brezis and Véron.

Résumé

Sur la trichotomie des solutions positives singulières associées à l'opérateur de Hardy–Sobolev. Dans cette Note, nous présentons une classification complète des singularités de solutions positives de l'équation $\Delta u + \frac{\mu}{|x|^2} u = h(u)$ dans $\Omega \setminus \{0\}$, où Ω est un domaine borné de $\mathbb{R}^N$, $N \geq 3$, $0 \in \Omega$, et où $0 < \mu < \frac{(N-2)^2}{4}$. Le cas $\mu = 0$ avec $h(t) = t^q$, $q > 1$ a été traité par Brezis et Véron.

Version française abrégée

Soit Ω un domaine borné de $\mathbb{R}^N$ $(N \geq 3)$ et $0 \in \Omega$. Pour tout $\mu > 0$, soit L_μ l'opérateur de Hardy–Sobolev défini par $L_\mu := -\left(\Delta + \frac{\mu}{|x|^2}\right)$. Grâce à l'inégalité de Hardy (voir, par exemple, [3] et [1]), l'opérateur $L_\mu^{-1} : L^2(\Omega) \to L^2(\Omega)$ est positif, compact et auto-adjoint pour tout $\mu \in (0, \mu^*)$, où $\mu^* := (N-2)^2/4$ est la meilleure constante dans l'inégalité de Hardy. Soit $h : \mathbb{R} \to \mathbb{R}$ une fonction localement lipschitzienne tels que $h > 0$ sur $(0, \infty)$ et $h(0) = 0$.

Pour tout $\mu \in (0, \mu^*)$, on considère le problème semilinéaire $L_\mu u + h(u) = 0$ dans $\Omega^* := \Omega \setminus \{0\}$ (c'est-à-dire (1)). On dit que $u \in C^1(\Omega^*)$ est une solution faible du problème (1) si u vérifie (1) au sens des distributions dans $\mathcal{D}'(\Omega^*)$. Si l'on suppose que h est régulière alors les estimations elliptiques standard impliquent que les solutions faibles du problème (1) sont dans $C^\infty(\Omega^*)$. En utilisant le principe du maximum fort (voir [10, Théorème 1.1]) on obtient que toute solution non négative et non identiquement nulle est alors positive dans Ω^*. De plus, on montre que toute solution positive $u(x)$ du problème (1) tend vers l'infini quand $|x|$ tend vers zéro (voir [5]). Notons que l'équation (1) peut avoir des solutions classiques dans Ω si la condition de Lipschitz locale sur h n'est pas vérifiée. Par exemple, $u(x) := |x|^\lambda$, $\lambda > 2$ est une telle solution pour l'équation $L_\mu u + (\lambda^2 + (N-2)\lambda + \mu)u^{1-2/\lambda} = 0$ dans Ω.

On désigne par $\Phi_\mu^\pm$ *les solutions fondamentales* de l'équation $L_\mu v = 0$ dans Ω^* (voir (2)). Guerch and Véron dans [9, Théorème 3.1] ont donné une condition nécessaire et suffisante sur h pour l'existence des solutions faibles du problème (1) qui vérifient $\lim_{|x|\to 0} u(x)/\Phi_\mu^+(x) \in \mathbb{R}$. Théorème 1.1 dans [9] fournit une condition suffisante sur h pour avoir une solution du problème (1) qui peut être prolongée comme une solution de la même équation dans $\mathcal{D}'(\Omega)$. Une question naturelle se pose : comment les solutions faibles du problème (1) peuvent-elles se comporter au voisinage

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de zéro ? Le théorème suivant fournit la réponse sous une hypothèse de *variation régulière d'indice q* ($q > 1$) posée sur la fonction h, ce qui signifie que $\lim_{t\to\infty} h(\lambda t)/h(t) = \lambda^q$ pour chaque $\lambda > 0$, voir [11]. Soit $H(t) := \int_0^t h(s)\,ds$ pour $t > 0$. On donne une trichotomie des solutions positives du problème (1) dans le cas $q < q^*$, où q^* est défini par (3).

Théorème 0.1. *Soient $N \geq 3$ et $\mu \in (0, \mu^*)$, où $\mu^* = \frac{(N-2)^2}{4}$. On suppose que h est une fonction à variation régulière d'indice $q \in (1, q^*)$. Soit $u \in C^1(\Omega^*)$ une solution faible positive du problème* (1). *Alors, quand $|x| \to 0$ on a :*

(A) *soit $u(x)/\Phi_\mu^-(x)$ converge vers un nombre positif ;*
(B) *ou $u(x)/\Phi_\mu^+(x)$ converge vers un nombre positif ;*
(C) *ou $u(x)/\Phi_\mu^+(x)$ tend vers l'infini. Dans ce cas, la solution u vérifie de plus* (4).

On note que (i) seules les solutions de la catégorie **A** sont dans $W^{1,2}(\Omega)$; (ii) q^* est l'exponent *critique* pour le Théorème 0.1 : Si $q > q^*$, alors pour toute solution positive u on montre que $\lim_{|x|\to 0} u(x)/\Phi_\mu^-(x) \in [0, \infty)$. Cette affirmation est aussi vrai pour $q = q^*$ si $h(t) = t^q$ (voir [5]) ; (iii) le cas $N = 2$ pour l'opérateur de Hardy L_μ défini par $-\Delta - \mu \left(|x| \log \frac{1}{|x|}\right)^{-2}$ avec $\mu \in (0, \frac{1}{4})$ est abordé dans [5], où on établit une version de Théorème 0.1 pour tout $q \in (1, \infty)$.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 3$ and $0 \in \Omega$. For any parameter $\mu > 0$, let $L_\mu := -\left(\Delta + \frac{\mu}{|x|^2}\right)$ be the *Hardy–Sobolev operator*. Owing to the classical *Hardy inequality* (see, for example, [3] and [1]), the operator $L_\mu^{-1} : L^2(\Omega) \to L^2(\Omega)$ is positive-definite, compact and self-adjoint, for any μ in $(0, \mu^*)$, where $\mu^* := (N-2)^2/4$ is the best constant in the Hardy inequality. Let $h : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz such that $h > 0$ on $(0, \infty)$ and $h(0) = 0$.

Let $\mu \in (0, \mu^*)$ and consider the semilinear equation

$$L_\mu u + h(u) = 0 \quad \text{in } \ \Omega^* := \Omega \setminus \{0\}. \tag{1}$$

We say that $u \in C^1(\Omega^*)$ is a *weak solution* of (1) if u satisfies (1) in the sense of distributions in $\mathcal{D}'(\Omega^*)$. If h is smooth, by standard elliptic estimates, weak solutions of (1) are $C^\infty(\Omega^*)$. By the strong maximum principle (Theorem 1.1 in [10]), any non-negative and non-trivial weak solution u of (1) is positive in Ω^* and $\liminf_{|x|\to 0} u(x) > 0$. Moreover, by careful use of the radial solutions of (1) and the comparison principle (Lemma 2.1), we infer that any positive solution of (1) blows-up at zero (see [5]). However, the equation (1) may admit classical solutions in Ω if the locally Lipschitz condition on h fails. For example, $u(x) := |x|^\lambda$, $\lambda > 2$ is a $C^2(\Omega)$-solution of $L_\mu u + (\lambda^2 + (N-2)\lambda + \mu)u^{1-2/\lambda} = 0$ in Ω.

Throughout this Note, $\Phi_\mu^\pm$ denote the *fundamental solutions* of the equation $L_\mu v = 0$ in Ω^*, namely

$$\Phi_\mu^\pm(x) := |x|^{-\left(\frac{N-2}{2} \pm \sqrt{\mu^* - \mu}\right)} \quad \text{for } \ x \neq 0, \quad \mu \in (0, \mu^*). \tag{2}$$

Guerch and Véron [9, Theorem 3.1] provide a necessary and sufficient condition on h for the existence of weak solutions of (1) satisfying $\lim_{|x|\to 0} u(x)/\Phi_\mu^+(x) \in \mathbb{R}$. Among other results, Theorem 1.1 in [9] gives a sufficient condition on h for which a solution of (1) can be extended as a solution of the same equation in $\mathcal{D}'(\Omega)$. These results raise the issue of classifying the asymptotic behavior of weak solutions of (1) near zero. We answer this question under the assumption that h is *regularly varying at infinity of index q with* $q > 1$ (in short, $h \in RV_q$), which means that $\lim_{t\to\infty} h(\lambda t)/h(t) = \lambda^q$ for any $\lambda > 0$, see [11]. Set $H(t) := \int_0^t h(s)\,ds$ for $t > 0$.

We reveal below a trichotomy of positive singular solutions of (1) in the *subcritical* case $q < q^*$, where

$$q^* := \frac{N + 2 + 2\sqrt{\mu^* - \mu}}{N - 2 + 2\sqrt{\mu^* - \mu}}. \tag{3}$$

Theorem 1.1. *Let $N \geq 3$ and $\mu \in (0, \mu^*)$, where μ^* is the Hardy constant. We assume that h is regularly varying at infinity of index $q \in (1, q^*)$. Let $u \in C^1(\Omega^*)$ be a positive weak solution of* (1). *Then as $|x| \to 0$, we have:*

(A) *either $u(x)/\Phi_\mu^-(x)$ converges to a positive number;*
(B) *or $u(x)/\Phi_\mu^+(x)$ converges to a positive number;*

(C) *or* $u(x)/\Phi_\mu^+(x)$ *tends to* ∞*, in which case*

$$\lim_{|x|\to 0}\frac{1}{|x|}\int_{u(x)}^{\infty}\frac{ds}{\sqrt{H(s)}}=M,\quad M=M(\mu,q,N):=\left(\frac{2(q+1)}{N-(N-2)q+\mu(q-1)^2/2}\right)^{1/2}. \tag{4}$$

Remarks.

(i) By the usual translation of the form $v(y):=u(x+ry)$ for $|y|<1$, where $r:=|x|/2$, $x\in\Omega^*$, together with the standard elliptic estimates for v, it follows that if $u(x)\le |x|^{-\alpha}$ for some $\alpha>0$, then $|\nabla u|\le C|x|^{-(\alpha+1)}$ for some positive constant C independent of x. This asserts that only the Category **A** solutions are in $W^{1,2}(\Omega)$.

(ii) The exponent q^* in (3) is *critical* for Theorem 1.1: If $q>q^*$, then for any positive solution u of (1) we have $\lim_{|x|\to 0}u(x)/\Phi_\mu^-(x)\in[0,\infty)$, hence $u\in W^{1,2}(\Omega)$. This assertion is true for $q=q^*$ if $h(t)=t^q$ (to appear in [5]).

(iii) The 2-dimensional Hardy operator $L_\mu:=-\Delta-\mu\left(|x|\log\frac{1}{|x|}\right)^{-2}$ with $\mu\in(0,\frac14)$ is considered in [5], where we show that an appropriate version of the Theorem 1.1 is valid for any $q\in(1,\infty)$.

The analysis of weak solutions of (1) for the case $\mu=0$ has been pioneered by Brezis and Véron [4] and subsequently studied by many other authors. Given $q\ge N/(N-2)$ and the equation

$$-\Delta u+u^q=0\quad\text{in}\quad\Omega^*, \tag{5}$$

it is known from [4] that any non-negative solution can be extended as a classical solution of (5) in Ω.

For $1<q<N/(N-2)$, Véron [12], [13] gives a complete classification of isolated singularities of non-negative weak solutions of (5). More precisely, as $|x|\to 0$ any non-negative solution u of (5) satisfies one of the following: (i) either $u(x)$ admits a finite limit and u can be extended as a C^2-solution of (5) in Ω; (ii) or $|x|^{N-2}u(x)$ converges to some positive constant; (iii) or $|x|^{2/(q-1)}u(x)$ converges to $M(0,q,N)$. A simpler proof were obtained by Brezis and Oswald [2]. Recently, the above result of Véron were extended by Cîrstea and Du [6, Theorem 1.1] to equations of the form $-\Delta u+h(u)=0$ in Ω^* for $h\in RV_q$ and $1<q<N/(N-2)$.

2. Proof of Theorem 1.1

For a clear exposition and the purpose of this presentation, we outline a proof for the power nonlinearity $h(t):=t^q$. The complete proof for general nonlinearity h will appear in [5]. A function $v\in C^2(\Omega^*)$ is called a *sub-solution* (*super-solution*) of (1) if $L_\mu v+h(v)\le(\ge)\,0$ in Ω^*. Throughout the proof we use the following comparison principle, which follows from Lemma 2.1 in [7].

Lemma 2.1 (Comparison principle). *Let* $N\ge 3$ *and* U *be a smooth bounded domain in* $\mathbb{R}^N$ *with* $\overline{U}\subset\mathbb{R}^N\setminus\{0\}$*. Let* g *be continuous on* $(0,\infty)$ *and* $g(t)/t$ *be increasing in* $(0,\infty)$*. If* $v_1,v_2\in C^2(U)$ *are positive functions such that*

$$\begin{cases} L_\mu v_1+g(v_1)\le 0\le L_\mu v_2+g(v_2)\ \text{ in } U,\\ \limsup\limits_{x\to\partial U}[v_1(x)-v_2(x)]\le 0,\end{cases} \tag{6}$$

then $v_1\le v_2$ *in* U.

Let u be a positive weak solution of $L_\mu v+v^q=0$ in Ω^* with $q\in(1,q^*)$. We have $u\in C^2(\Omega^*)$ and $\lim_{|x|\to 0}u(x)=\infty$. Without loss of generality, we can assume that the closed unit ball is strictly contained in Ω. Set

$$f^{\pm}(x):=\frac{u(x)}{\Phi_\mu^{\pm}(x)}\quad\text{for } x\in B_1^*(0):=B_1(0)\setminus\{0\}.$$

The above functions play a crucial role in our analysis. If $\limsup_{|x|\to 0}f^+(x)=c\in(0,\infty)$, from Guerch–Véron [9, Theorem 2.1] it follows that $f^+(x)$ converges to c as $|x|\to 0$. Hence u is of Category **B** in Theorem 1.1. We next prove that the Category **A** and **C** in Theorem 1.1 correspond to the remaining two cases, respectively:

$$\textbf{I}.\ \limsup_{|x|\to 0}f^+(x)=0;\qquad\qquad \textbf{II}.\ \limsup_{|x|\to 0}f^+(x)=\infty.$$

This will be achieved via several steps. We first obtain a sharp upper-bound for $|x|^{2/(q-1)}u(x)$ by devising a family of super-solutions of (1) and using Lemma 2.1. Then we provide a positive radially symmetric solution w_∞ of $L_\mu v + v^q = 0$ in $B^*_{1/2}(0)$ such that $cu \le w_\infty \le u$ in $B^*_{1/2}(0)$ for some constant $c > 0$. Step 3–Step 5 are concerned with positive radial solutions. In Step 3 we show that $\lim_{r\to 0} f^\pm(r)$ exists in $[0, \infty]$. We prove that solutions of Type **I** and **II** above are of Category **A** and **C**, respectively: We argue with radial solutions in Steps 4 and 5, then in the general case we use a reduction to radial symmetry (see Steps 6 and 7). The reduction procedure relies on Step 3 and the construction of w_∞ in Step 2. We devise the super-solutions (sub-solutions) in Step 1 (Step 5) inspired by the work in [6] for $\mu = 0$.

Step 1. *Sharp upper-bound for* $|x|^{2/(q-1)}u(x)$*:* Let M be given by (4). We show that

$$\limsup_{|x|\to 0} |x|^{\frac{2}{q-1}}\, u(x) \le \widetilde{M}, \quad \text{where } \ \widetilde{M} = \widetilde{M}(q) := \left(\frac{2\sqrt{q+1}}{M(q-1)} \right)^{2/(q-1)}. \tag{7}$$

By direct calculation, we see that $\psi(x) := \widetilde{M}\, |x|^{-\frac{2}{q-1}}$ for $x \in B^*_1(0)$ satisfies $L_\mu v + v^q = 0$ in $B^*_1(0)$. Since $q < q^*$, we have $\lim_{|x|\to 0} |x|^{2/(q-1)} \Phi^+_\mu(x) = 0$. Thus to conclude (7), it is enough to prove that

$$u(x) \le \psi(x) + C\Phi^+_\mu(x) \quad \text{for } 0 < |x| < 1, \quad \text{where } \ C := \max_{|y|=1} u(y). \tag{8}$$

Since $L_\mu \Phi^+_\mu = 0$ in $B^*_1(0)$, the function $\psi(x) + C\Phi^+_\mu(x)$ is a super-solution of $L_\mu v + v^q = 0$ in $B^*_1(0)$. Fix $\lambda > 0$ sufficiently large. Let $\varepsilon \in (0, 1)$ be small enough and define $\psi_\varepsilon : (\varepsilon, 1) \to (0, \infty)$ by

$$\psi_\varepsilon(r) := \widetilde{M}_\varepsilon\, (r-\varepsilon)^{-\frac{2}{q-1}\left(1+\frac{\lambda}{\log(1/\varepsilon)}\right)} \quad \text{for } \varepsilon < r < 1, \ \ \widetilde{M}_\varepsilon > 0.$$

By careful computations, there exists $\widetilde{M}_\varepsilon > 0$ such that $\widetilde{M}_\varepsilon \nearrow \widetilde{M}$ as $\varepsilon \to 0$ and $L_\mu \psi_\varepsilon + (\psi_\varepsilon)^q \ge 0$ for $\varepsilon < |x| < 1$. Since $\lim_{r\searrow\varepsilon} \psi_\varepsilon(r) = \infty$, by the comparison principle in Lemma 2.1, we infer that $u(x) \le \psi_\varepsilon(|x|) + C\Phi^+_\mu(x)$ for $\varepsilon < |x| < 1$. By letting $\varepsilon \to 0$, we obtain (8) and conclude the proof of (7).

Step 2. *Construction of* w_∞*:* Using essentially the Harnack inequality [8, Theorem 8.20] and Step 1, it follows that there exists a constant $K > 1$, which is independent of u, such that

$$\max_{|x|=r} u(x) \le K \min_{|x|=r} u(x) \quad \text{for every } \ 0 < r < 1/2. \tag{9}$$

We construct below a positive radial solution w_∞ of $L_\mu v + v^q = 0$ in $B^*_{1/2}(0)$ such that

$$u/K \le w_\infty \le u \quad \text{in } \ B^*_{1/2}(0). \tag{10}$$

By the sub/super-solutions method, for every integer $n \ge 3$ there exists a positive solution w_n of

$$\begin{cases} L_\mu v + v^q = 0 & \text{in } A_n := \{x \in \mathbb{R}^N : \ 1/n < |x| < 1/2\}, \\ v(x) = \min_{|y|=|x|} u(y) & \text{for } x \in \partial A_n. \end{cases} \tag{11}$$

By Lemma 2.1, w_n is a unique solution to (11). Owing to the rotation symmetry of L_μ and the boundary condition, w_n is radially symmetric. By (9) we have $u/K \le w_n$ on ∂A_n for every $n \ge 3$. Since u/K is a sub-solution of (11), it follows from the comparison principle that $u/K \le w_n \le u$ and $w_m \le w_n$ in A_n for any $m \ge n \ge 3$. Thus, up to a subsequence, w_n converges to some w_∞ in $C^2_{\rm loc}(B^*_{1/2}(0))$ as $n \to \infty$. This w_∞ satisfies the above-mentioned properties.

In Step 3–Step 5 we assume that u is a positive radial solution of $L_\mu v + v^q = 0$ in $B^*_1(0)$.

Step 3. *Existence of* $\lim_{r\to 0} f^\pm(r) \in [0, \infty]$*:* If we assume the contrary, then $\limsup_{r\to 0} f^\pm(r) > 0$ and there exists $c > 0$ such that $0 \le \liminf_{r\to 0} f^\pm(r) < c < \limsup_{r\to 0} f^\pm(r)$. Let $(r_n)_{n\ge 1}$ be a sequence that decreases to 0 as $n \to \infty$ and satisfies $\lim_{n\to\infty} f^\pm(r_n) = \liminf_{r\to 0} f^\pm(r)$. Then for sufficiently large $n_0 \in \mathbb{N}$, we have $u(r_n) \le c\, \Phi^\pm_\mu(r_n)$ for all $n \ge n_0$. Observe that for $n > n_0$ we have $L_\mu u + u^q \le 0 \le L_\mu \Phi^\pm_\mu + (\Phi^\pm_\mu)^q$ in $r_n < |x| < r_{n_0}$. Thus by the comparison principle, $u(r) \le c\, \Phi^\pm_\mu(r)$ for all $r \in (0, r_{n_0})$. This being a contradiction with the choice of c, we conclude Step 3.

Step 4. *Radial solutions of Type* **I** *are of Category* **A***:* Let u be a positive radial solution of $L_\mu v + v^q = 0$ in $B^*_1(0)$ such that $\lim_{r\to 0} f^+(r) = 0$. We conclude that $\lim_{r\to 0} f^-(r) \in (0, \infty)$ by showing the following: (i) $\frac{d}{dr}(f^-(r))$ is positive on $(0, 1)$; (ii) the assumption $\lim_{r\to 0} f^-(r) = 0$ would lead to $\lim_{r\to 0} u(r) = 0$, which would contradict $\lim_{r\to 0} u(r) = \infty$.

To this end, we set $\gamma := N/2 + \sqrt{\mu^* - \mu}$ and $g(r) := r^{2\gamma+1-N} \frac{d}{dr}(f^-(r))$ for $r \in (0,1)$. Since $\lim_{r\to 0} f^+(r) = 0$ and $\lim_{r\to 0} u(r) = \infty$, we conclude that $\lim_{r\to 0} g(r) = 0$. Moreover, u satisfies $g'(r) = r^\gamma u^q$ in $(0,1)$. By integrating this equation and multiplying it by $r^{N-1-2\gamma}$, we get

$$\frac{d}{dr}(f^-(r)) = b(r) > 0, \quad \text{where } b(r) := r^{N-1-2\gamma} \int_0^r s^\gamma u^q(s)\, ds \quad \text{for } r \in (0,1). \tag{12}$$

Hence $\lim_{r\to 0} f^-(r)$ exists in $[0,\infty)$. Assuming that $\lim_{r\to 0} f^-(r) = 0$, then we have: (a) for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that $u \le \varepsilon \Phi_\mu^-$ in $(0, r_\varepsilon]$; (b) integrating (12) yields $u(r) = \Phi_\mu^-(r) \int_0^r b(s)\, ds$ for every $r \in (0,1)$.

Set $m_0 := \sqrt{\mu^* - \mu} - \frac{N-2}{2} < 0$. Using (a) and (b), we find a constant $C > 0$ independent of ε such that

$$u(r) \le C\, \varepsilon^q\, r^{2+qm_0} \quad \text{for every } r \in (0, r_\varepsilon). \tag{13}$$

Let $m_k := 2 + q\, m_{k-1}$ for any integer $k \ge 1$ and $\widetilde{q} := 2/(q-1)$. We define $q^\#$ as follows

$$q^\# := \frac{N + 2 - 2\sqrt{\mu^* - \mu}}{N - 2 - 2\sqrt{\mu^* - \mu}}. \tag{14}$$

Note that $q^* < q^\#$. Using $q < q^\#$, we deduce that $m_k > m_{k-1}$ and $m_k = -\widetilde{q} + (\widetilde{q} + m_0)\, q^k$ for every integer $k \ge 1$. Since $q > 1$ and the coefficient of q^k is positive, it follows that $\lim_{k\to\infty} m_k = \infty$. Therefore, $m_j > 0$ for sufficiently large j. Since $\varepsilon > 0$ is arbitrary, (13) yields that $\lim_{r\to 0} u(r)/r^{m_1} = 0$. If $m_1 \ge 0$, it follows that $\lim_{r\to 0} u(r) = 0$. If $m_1 < 0$, by using (13) in (b) we iterate the above arguments and find that $\lim_{r\to 0} u(r)/r^{m_j} = 0$ for some $m_j > 0$. Thus both the cases lead to $\lim_{r\to 0} u(r) = 0$, which is a contradiction. Hence u is of Category **A** for every $q \in (1, q^\#)$.

Step 5. *Radial solutions of Type* **II** *are of Category* **C**: Let u be a positive radial solution of $L_\mu v + v^q = 0$ in $B_1^*(0)$ such that $\limsup_{r\to 0} f^+(r) = \infty$. By Step 3 we have $\lim_{r\to 0} u(r)/\Phi_\mu^+(r) = \infty$. Proving that u is of Category C means that $\lim_{r\to 0} r^{2/(q-1)} u(r) = \widetilde{M}$ with $\widetilde{M}$ given by (7). By Step 1, it remains to show that $\liminf_{r\to 0} r^{2/(q-1)}\, u(r) \ge \widetilde{M}$. This will be achieved by establishing the following inequality

$$\widetilde{M} r^{-2/(q-1)} \le u(r) + \widetilde{M}\Phi_\mu^+(r) \quad \text{for every } r \in (0,1). \tag{15}$$

Since $\lim_{r\to 0} r^{-2/(q-1)}/\Phi_\mu^+(r) = \infty$, we cannot directly conclude (15) for r close to zero. The idea is to fix $\varepsilon > 0$ small and devise a suitable family of sub-solutions φ_ε of $L_\mu v + v^q = 0$ in $B_1^*(0)$ such that:

(P_1) $\varphi_\varepsilon(r)$ increases to $\widetilde{M} r^{-2/(q-1)}$ as ε decreases to 0;

(P_2) $\varphi_\varepsilon(r) \le u(r) + \widetilde{M}\Phi_\mu^+(r)$ for every $r \in (0,1)$.

The construction of φ_ε completes Step 5. Indeed, letting $\varepsilon \to 0$ in (P_2) and using (P_1) yields (15).

Let $\alpha > 0$ to be specified in (17) and define φ_ε by

$$\varphi_\varepsilon(r) := \left(\widetilde{M}^{-\frac{(q-1)}{2}} r + (\varepsilon r^\alpha)^{\frac{q-1}{2}} \right)^{-\frac{2}{q-1}} \quad \text{for every } r \in (0,1). \tag{16}$$

For sufficiently small $\tau = \tau(N,\mu) > 0$, we can choose a smaller positive number ν that is independent of q such that $L_\mu \varphi_\varepsilon + (\varphi_\varepsilon)^q \le 0$ in $B_1^*(0)$ for the particular choice of α given by

$$\alpha := \begin{cases} (N-2)/2 + \sqrt{\mu^* - \mu} & \text{if } q^* - \tau < q < q^*, \\ 2/(q - 1 + \nu) & \text{if } 1 < q \le q^* - \tau. \end{cases} \tag{17}$$

We see that (P_1) holds for φ_ε in (16). We only need to prove (P_2). The key ingredient is to establish that

$$\lim_{r\to 0} r^\alpha w(r) = \infty \text{ for any positive radial solution } w \text{ of } L_\mu v + v^q = 0 \text{ in } B_1^*(0), \text{ subject to } \lim_{r\to 0} v(r)/\Phi_\mu^+(r) = \infty. \tag{18}$$

Assuming the validity of (18), we verify (P_2) and complete Step 5. Indeed, (18) implies in particular that $r^\alpha u(r) \to \infty$ as $r \to 0$. Thus for some $r_\varepsilon > 0$ we have $r^{-\alpha}/\varepsilon \le u(r)$ for every $r \in (0, r_\varepsilon]$. From (16) we have $\varphi_\varepsilon(r) \le r^{-\alpha}/\varepsilon$ for $r \in (0,1)$. Hence the inequality in (P_2) holds for every $r \in (0, r_\varepsilon]$. Since $\varphi_\varepsilon(1) \le \widetilde{M}$ and $u + \widetilde{M}\Phi_\mu^+$ is a super-solution of $L_\mu v + v^q = 0$ in $B_1^*(0)$, by the comparison principle, (P_2) holds in $[r_\varepsilon, 1)$. This proves the validity of (P_2).

Proof of (18). Since $\nu > 0$, there exists a large integer $m > 0$ such that $\nu = (q^* - \tau - 1)/m$. Set

$$J_0 := (q^* - \tau, q^*) \quad \text{and} \quad J_i := (q^* - \tau - i\nu, q^* - \tau - (i-1)\nu] \ \text{ for } i = 1, 2, \ldots, m.$$

Hence $(1, q^*) = \cup_{i=0}^{m} J_i$. To achieve (18) for any $q \in (1, q^*)$, we proceed by induction.

(i) If $q \in J_0$, then the assertion of (18) follows from the definition of α in (17) and $\lim_{r\to 0} w(r)/\Phi_\mu^+(r) = \infty$.

(ii) Let $i \in \{0, 1, \ldots, m-1\}$ and assume that (18) is true for any $q \in J_i$. We prove that (18) is true for any $q \in J_{i+1}$.

To this aim, let $q \in J_{i+1}$ and w be an arbitrary positive radial solution of the problem in (18). From the definition of α in (17), we have $\alpha = 2/(q - 1 + \nu)$. We choose $q_1 \in J_i$ such that $q_1 < q + \nu$. Since $w(r) \to \infty$ as $r \to 0$, there exists $0 < r_1 < 1$ such that $(w(r))^q \le (w(r))^{q_1}$ for every $r \in (0, r_1)$. By [9, Remark 3.1], for each $k \in \mathbb{N}$ there exists a unique positive solution v_k of the following equation

$$v''(r) + \frac{N-1}{r} v'(r) + \frac{\mu}{r^2} v(r) = v^{q_1}(r), \quad 0 < r < r_1$$

subject to $\lim_{r\to 0} v(r)/\Phi_\mu^+(r) = k$ and $v(r_1) = 0$. By the comparison principle, v_k is non-decreasing in k and $v_k \le w$ in $(0, r_1)$. Let $v_\infty(r) := \lim_{k\to\infty} v_k(r)$ for $r \in (0, r_1)$, so that $v_\infty \le w$ in $(0, r_1)$. Standard regularity arguments show that, up to a subsequence, $v_k \to v_\infty$ in $C^2_{\text{loc}}(0, r_1)$ as $k \to \infty$ and v_∞ is a positive radial solution of $L_\mu v + v^{q_1} = 0$ in $B_1^*(0)$ with $\lim_{r\to 0} v_\infty(r)/\Phi_\mu^+(r) = \infty$. Since $q_1 \in J_i$, by the induction hypothesis applied to v_∞ and the argument after (18), we have $\lim_{r\to 0} r^{2/(q_1-1)} v_\infty(r) = \widetilde{M}(q_1) > 0$. Using $q_1 < q + \nu$, we find $\lim_{r\to 0} r^{2/(q-1+\nu)} w(r) = \infty$. This concludes Step 5.

Step 6. *Reduction to radial symmetry for Type* **I** *solutions:* We show that any positive solution of $L_\mu u + u^q = 0$ in Ω^* with $\lim_{|x|\to 0} f^+(x) = 0$ must be of Category **A**. Let $\mathbb{S}^{N-1}$ be the unit sphere in $\mathbb{R}^N$ and $(r, \sigma) \in (0, \infty) \times \mathbb{S}^{N-1}$ denote the polar coordinates in $\mathbb{R}^N \setminus \{0\}$. For any function $v(r, \sigma)$, its spherical mean $\overline{v}(r)$ is defined by

$$\overline{v}(r) := \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} v(r, \sigma)\, d\sigma.$$

By averaging the equation $L_\mu u + u^q = 0$ in $B_1^*(0)$ and using Jensen's inequality, we find $L_\mu \overline{u} = -\overline{(u^q)} \le -(\overline{u})^q$ in $(0, 1)$. By Step 3 applied to the sub-solution $\overline{u}$, we know that $\lim_{r\to 0} \overline{u}(r)/\Phi_\mu^-(r)$ exists in $[0, \infty]$. By Lemmas 2.1 and 2.3 in [9], the ratio $(u(r, \sigma) - \overline{u}(r))/\Phi_\mu^-(r)$ converges to 0 as $r \to 0$, uniformly in $\sigma \in \mathbb{S}^{N-1}$. Hence $u(r, \sigma)/\Phi_\mu^-(r)$ admits a limit in $[0, \infty]$ as $r \to 0$, uniformly in $\sigma \in \mathbb{S}^{N-1}$. Consequently, $\lim_{|x|\to 0} f^-(x)$ exists in $[0, \infty]$. To conclude Step 6, we apply Step 4 to w_∞ constructed in Step 2.

Step 7. *Reduction to radial symmetry for Type* **II** *solutions:* Let u be a positive solution of $L_\mu v + v^q = 0$ in Ω^* such that $\limsup_{|x|\to 0} f^+(x) = \infty$. We construct w_∞ as in Step 2. By (10) and Step 3, we find $\lim_{r\to 0} w_\infty(r)/\Phi_\mu^+(r) = \infty$. Applying Step 5 to w_∞, together with (10) and Step 1, it follows that u must be of category **C**. □

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