GEOMETRIC DECOMPOSITIONS OF 4-DIMENSIONAL BUNDLE SPACES

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Abstract. We consider geometric decompositions of aspherical 4-manifolds which fibre over 2-orbifolds. We show first that no such manifold admits infinitely many fibrations over hyperbolic base orbifolds. If E is Seifert fibred over a hyperbolic surface B and either B has at most one cone point of order 2 or the monodromy has image in $SL(2,\mathbb{Z})$ then E it has a decomposition induced from a decomposition of B.

An n-manifold M admits a *geometric decomposition* if it has a finite collection $\mathcal{S}$ of disjoint connected 2-sided hypersurfaces such that each component of $M - \cup_{S\in\mathcal{S}} S$ is geometric of finite volume, i.e., is homeomorphic to $\Gamma\backslash X$, for some geometry $\mathbb{X}$ and lattice Γ. We shall call the hypersurfaces S *cusps* and the components of $M - \cup_{S\in\mathcal{S}} S$ *pieces* of M. The decomposition is *proper* if the set of cusps is nonempty.

We shall consider the possible geometric decompositions of aspherical orbifold bundles in dimension 4. An *orbifold bundle* E is the total space of an orbifold fibration $p : E \to B$ over a 2-dimensional base orbifold, with regular fibre F an aspherical surface. (Here "surface" shall mean closed 2-orbifold without exceptional points.) Let $\pi = \pi_1(E)$, $\phi = \pi_1(F)$ and $\beta = \pi_1^{orb}(B)$, and let $\theta : \beta \to Out(\phi)$ be the characteristic homomorphism (or monodromy).

We show first that if $\chi(E) > 0$ then E admits only finitely many orbifold fibrations. In §2 we extend and correct some results on geometries on bundle spaces from Chapter 13 of [4] to the case of orbifold bundles. In §3 we constrain the possible geometries of pieces of a given orbifold bundle. In §4 and §5 we introduce the notions of (algebraically) horizontal and vertical decompositions. In particular, we show that no decomposition can have both algebraically horizontal and algebraically vertical cusps, and that if E is Seifert fibred and $\mathrm{Im}(\theta) \leq SL(2,\mathbb{Z})$ or

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B has at most one cone point of order 2 then E has a vertical decomposition. The final section considers uniqueness of algebraically vertical decompositions inducing a given splitting of $\pi_1(E)$.

If G is a group G', ζG and $\sqrt{G}$ shall denote the commutator subgroup, centre and Hirsch-Plotkin radical of G, respectively. (In all the cases considered here $\sqrt{G}$ is the maximal normal nilpotent subgroup of G, and in many cases $\sqrt{G}$ is abelian.)

1. Bounding the orbifold fibrations of a given 4-manifold

The orbifold bundles with flat fibre ($\chi(F) = 0$) are precisely the Seifert bundles in 4 dimensions. Every torsion-free group which is virtually an extension of a surface group by Z^2 arises in this way, and two such Seifert bundles are isomorphic if and only if their group extensions are equivalent [12]. The extension is in turn determined by the group π, since $\chi(B) < 0$ implies that ϕ is the unique maximal solvable normal subgroup of π. (Note that in [12] "Seifert bundle" is used to mean a codimension 2 foliation with all leaves compact, in other words, what we call an orbifold bundle here. He gives also a corresponding result for orbifold bundles with $\chi(F) < 0$, subject to an additional arithmetic hypothesis which implies that ϕ is a characteristic subgroup.) Thus if E is Seifert fibred over a hyperbolic base the Seifert fibration is essentially unique. If however E is an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold it may fibre over the torus T in infinitely many ways, with fibre of arbitrarily high genus [2]!

If π is a torsion-free extension of an aspherical 2-orbifold group β by a PD_2-group ϕ with $\chi(\phi) < 0$ then the extension is realized by an orbifold bundle p, and the bundle is determined up to bundle isomorphism by the group extension [12]. If moreover the action $\theta : \beta \to Out(\phi)$ has infinite image and nontrivial kernel then ϕ is unique and so p is determined by π. (See Theorems 5.5 and 7.3 of [4] and Theorem 5.3 of [12].) If θ has finite image there is at most one other such subgroup, and π is the group of an $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. (See Theorem 7.3 of [4].) We shall show that any orbifold bundle space E with $\chi(E)$ nonzero has only finitely many orbifold fibrations.

Theorem 1. *Let π be a torsion-free group which has a normal PD_2-subgroup ϕ with quotient an hyperbolic 2-orbifold group. If $\chi(\pi) > 0$ the set of such subgroups is finite.*

Proof. Let $\mathcal{B}$ be the set of normal PD_2-subgroups of π such that $\beta = \pi/\phi$ is an hyperbolic orbifold group. Then $|\chi^{orb}(\beta)| \leq \chi(\pi)$, since $|\chi(\phi)| \geq 1$. Hence there are only finitely many possible isomorphism classes of quotients. For each $\phi \in \mathcal{B}$ let d_ϕ be the least index of a torsion-free normal subgroup in π/ϕ. Then $d = lcm\{d_\phi \mid \phi \in \mathcal{B}\}$ is finite.

Let $\hat{\pi}$ be the intersection of all subgroups of π of index dividing d. This is a characteristic subgroup of finite index. If $\phi \in \mathcal{B}$ then $\hat{\pi}/\phi \cap \hat{\pi}$ is a PD_2-group. There are finitely many such normal subgroups of $\hat{\pi}$ [8]. (See also Corollary 5.6.1 of [4].) If ψ is another such group and $\phi \cap \hat{\pi} = \psi \cap \hat{\pi}$ the image of ψ in π/ϕ is a finite normal subgroup, and so is trivial. Thus $\psi \leq \phi$, and hence $\psi = \phi$. Therefore $\mathcal{B}$ is finite. □

Note that if $\phi \in \mathcal{B}$ then $|\chi(\phi)| \leq 42\chi(\pi)$, since $\frac{1}{42} = |\chi^{orb}(S^2(2,3,7))|$ is the minimal area of any hyperbolic 2-orbifold.

If $\beta = Z^2$ and π/π' has rank 2 then ϕ is the unique normal PD_2-subgroup with quotient Z^2. If however $\beta = Z^2$ and π/π' has rank at least 3 there may be infinitely many such subgroups ϕ. (See Theorem 4.2 of [2].) In particular, an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold has an unique Seifert fibration but may fibre over T in infinitely many ways.

A closed 4-manifold is a *virtual bundle space* if it has a finite regular covering space which fibres over a surface. If a torsion-free group is virtually an extension of one surface group by another is it the group of an aspherical 4-manifold? We may assume that π has a normal subgroup G which is an extension of a PD_2-group G/K by a normal PD_2-subgroup K. If K is characteristic in G (and hence normal in π) then π is the group of an orbifold bundle, by Theorem 7.3 of [4]. If $\chi(\pi) > 0$ and π is virtually a product then it has an index-2 subgroup which is an orbifold bundle group, but need not itself be such a group. (See Corollary 9.8.1 of [4].) Is it nevertheless realizable (by an $\mathbb{H}^2 \times \mathbb{H}^2$-manifold)? Suppose that $\chi(\phi) < 0$ and $\theta : \beta \to Out(\phi)$ is injective (type III of [7]). Does π have a characteristic PD_2-subgroup?

2. Geometries on orbifold bundles with hyperbolic fibre

Suppose that F is hyperbolic ($\chi(F) < 0$). The next result extends Theorem 13.5 of [4], and corrects the assertion regarding decompositions. (See the subsequent counter-example.)

Theorem 2. *Let E be the total space of an aspherical F-bundle over a 2-orbifold B with $\chi(B) = 0$ and $\chi(F) < 0$. Then*

(1) E admits the geometry $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if θ has finite image;
(2) E admits the geometry $\mathbb{H}^3 \times \mathbb{E}^1$ if and only if $\mathrm{Ker}(\theta)$ *has two ends and* $\mathrm{Im}(\theta)$ *contains the class of a pseudo-Anasov homeomorphism of F;*
(3) otherwise E is not geometric.

If $\mathrm{Ker}(\theta) \neq 1$ *then* E *has a finite covering space with a geometric decomposition.*

Proof. The arguments of Theorem 13.5 of [4] extend to this situtation. The only point that needs explanation is in showing that the algebraic conditions of part (2) suffice. Suppose (2) holds. Then $\mathrm{Im}(\theta)$ has a normal subgroup generated by the image of a pseudo-Anasov homeomorphism ψ. Let N be the mapping torus of ψ and $\nu = \pi_1(N)$. Then N is an $\mathbb{H}^3$-manifold and π has a normal subgroup of finite index of the form $\nu \times Z$. Hence $\sqrt{\pi} \cong Z$, since $\sqrt{\nu} = 1$ and π is torsion-free. Since $\sqrt{\phi} = 1$ the image of $\sqrt{\pi}$ in β is an infinite cyclic normal subgroup. Flat 2-orbifold groups are 2-dimensional crystallographic groups. Since β has an infinite cyclic normal subgroup its holonomy group has exponent 2. Therefore it has at least one other independent infinite cyclic normal subgroup. Thus there is a homomorphism $\lambda : \pi \to Isom(\mathbb{E}^1)$ with $\lambda(\sqrt{\pi}) \cong Z$, and the construction of the cited theorem may be carried through. □

Let $N = X(3_1) \cup X(4_1)$ be the union of the exteriors of the trefoil and figure-eight knots, with boundaries identified so that the meridians and longitudes match. Then $E = N \times S^1$ fibres over T with fibre of genus 2. This manifold is not geometric, but is the union of an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold $X(3_1) \times S^1$ with an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold $X(4_1) \times S^1$. (Thus the argument involving splitting π/ϕ over $\sigma/\sigma \cap \phi$ at the end of p253 of [Hi] is wrong.)

If B is also hyperbolic then $\chi(E) > 0$ and $\pi_1(E)$ has no solvable subgroups of Hirsch length 3. No such bundle space admits the geometry $\mathbb{H}^2(\mathbb{C})$, by Corollary 13.7.2 of [4]. Hence the only possible geometries on E are $\mathbb{H}^2 \times \mathbb{H}^2$ and $\mathbb{H}^4$. There are no known examples of $\mathbb{H}^4$-manifolds which are also bundle spaces.

Theorem 3. *Let* E *be an aspherical orbifold bundle space with* $\chi(E) > 0$. *Then the following are equivalent:*

(1) E *admits the geometry* $\mathbb{H}^2 \times \mathbb{H}^2$*;*
(2) E *is finitely covered by a cartesian product of surfaces;*
(3) θ *has finite image.*

If E *is a* $\mathbb{H}^4$*-manifold then* θ *is injective.*

Proof. The argument of Theorem 13.6 of [4] applies almost without change. □

See [3] for examples of bundle spaces with B hyperbolic and θ injective. Are there infinitely many such with given base and fibre? In [1] it is shown that for any given surfaces B and F there are at most finitely many bundles for which π has no abelian subgroup of rank > 1. (For such groups θ must be injective.) See also [10].

Are there any examples with B hyperbolic, F of genus 2 and θ injective? The genus 2 mapping class group is commensurable with the pure braid group $P_6(S^2)$, and hence with $(F(4) \rtimes F(3)) \rtimes F(2)$, and so this seems unlikely. On the other hand, it is easy to see that $P_5(S^2)$ and the genus 2 mapping class group each contain Z^2.

Such bundle spaces need not be geometric. Let $B = F = T\#T$. Then B retracts onto $S^1 \vee S^1$. Mapping one generator of $F(2)$ to the involution τ which swaps the summands of F and the other to $c\tau c^{-1}$ where c is a Dehn twist gives rise to a bundle with base and fibre of genus 2 and $\mathrm{Im}(\theta) \cong D$.

3. THE POSSIBLE PIECES

If $\chi(B) = \chi(F) = 0$ then E has geometry $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$, and has no proper geometric decomposition. Thus we may assume henceforth that F or B is hyperbolic.

Theorem 4. *If an aspherical orbifold bundle space E has a proper decomposition then either*

(1) $\chi(B) = 0$, the pieces are $\mathbb{H}^3 \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{E}^2$-manifolds and the cusps are flat; or

(2) $\chi(F) = 0$, the pieces are $\mathbb{H}^2 \times \mathbb{E}^2$-manifolds and the cusps are flat; or

(3) $\chi(F) = 0$, the pieces are $\mathbb{F}^4$-manifolds and the cusps are $\mathbb{N}il^3$-manifolds; or

(4) $\chi(E) > 0$, the pieces are reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds and the cusps are $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds.

Proof. This follows from Theorem 7.1 of [4], with the following observations. Firstly, nonuniform $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are also $\mathbb{H}^2 \times \mathbb{E}^2$-manifolds, and vice versa [6].

Secondly, if $\chi(F) = 0$ then $\sqrt{\phi} \cong Z^2$ is an abelian normal subgroup. Hence $\sqrt{\phi}$ is contained in the group of every cusp, and hence of every piece. Thus there can be no pieces of type $\mathbb{H}^3 \times \mathbb{E}^1$.

Thirdly, if $\Gamma\backslash X$ is a piece of a geometric decomposition then $c.d.\Gamma = 3$ and so $\phi \cap \Gamma \neq 1$. Hence if $\chi(F) < 0$ we must have $\phi \cap \sqrt{\Gamma} = 1$, and so $\phi \cap \Gamma$ centralizes $\sqrt{\Gamma}$. It follows that $\Gamma\backslash X$ cannot have type $\mathbb{F}^4$ if $\chi(B) = 0$.

Finally, if $\chi(E) \neq 0$ then π has no poly-Z subgroups of Hirsch length 3, and so we may eliminate pieces with geometry $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$ or irreducible $\mathbb{H}^2 \times \mathbb{H}^2$. Moreover, no reducible $\mathbb{H}^2 \times \mathbb{H}^2$ piece can be finitely covered by the product of two punctured surfaces, since the inclusions of the cusps must be π_1-injective. Thus the cusps must be $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds. □

Each of the possibilities allowed by this theorem may be realized by some closed 4-manifold which fibres over a surface. The conclusions of this theorem apply equally well to virtual bundle spaces.

4. Horizontal and vertical cusps

A cusp S of a geometric decomposition of E is *horizontal* if it is transverse to all the fibres $F_b = p^{-1}(b)$. If the base is a surface then $p|_S$ is a submersion, and the leaf space of the foliation of S by the components of $S \cap F_b$ (for $b \in B$) is a surface which finitely covers B.

The following example shows that $S \cap F_b$ need not be connected. Let $M(\tau)$ be the mapping torus of the involution τ which swaps the summands of $F = T\#T$. Then $E = M(\tau) \times S^1$ fibres over T with fibre F. Let C be a non-separating essential simple closed curve in T_o, and let $D = c \cup \tau(C)$. Then $M(\tau|_D) \times S^1$ is a cusp in E which meets each fibre in two circles.

A cusp S is *vertical* if it is a union of fibres. Thus $S = p^{-1}(A)$ for some 1-dimensional suborbifold $A \subset B$. Since S is connected A must be either a circle S^1 or a reflector interval $\mathbb{I}$. Let $D = Z/2Z * Z/2Z = \pi_1^{orb}(\mathbb{I})$ be the infinite dihedral group.

Lemma 5. *Let S be a cusp and $\sigma = \pi_1(S)$. Then*

(1) $\phi \leq \sigma \Leftrightarrow p_*\sigma$ has two ends;
(2) if S is horizontal then $\sigma \cap \phi \cong Z$, and $[\beta : p_*\sigma]$ is finite;
(3) if S is vertical then $\phi \leq \sigma$, $p_*\sigma \cong Z$ or D and β splits over $p_*\sigma$.

Proof. If S is flat or is a $\mathbb{N}il^3$-manifold then σ is virtually poly-Z and so $h(\sigma \cap \phi) + h(p_*\sigma) = h(\sigma) = 3$. Hence either $\sigma \cap \phi \cong Z$ and $p_*\sigma$ is virtually Z^2, in which case $\chi(B) = 0$, or $\sigma \cap \phi$ is virtually Z^2 and $p_*\sigma$ has two ends. In the latter case $[\phi : \sigma \cap \phi]$ is finite and $\chi(F) = 0$.

If S is an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold then F and B are hyperbolic, and so centralizers in ϕ are cyclic, while centralizers in β are finite or have two ends. If $\sqrt{\sigma} \cap \phi = 1$ then $\sigma \cap \phi$ is a PD_2-group. Hence $[\phi : \sigma \cap \phi]$ is finite. If $\sqrt{\sigma} \cap \phi \neq 1$ then $\sqrt{\sigma} \cap \phi \cong Z$ and so $p_*\sigma$ is virtually a PD_2-group. Since $p_*\sigma$ can have no non-trivial finite normal subgroup we must have $\sigma \cap \phi = \sqrt{\sigma} \cap \phi \cong Z$.

If $[\phi : \sigma \cap \phi]$ is finite then $\phi \leq \sigma$, since ϕ is a normal subgroup of a free product (or HNN extension) with amalgamation over σ.

The other two implications are clear. □

If $\sigma \cap \phi \cong Z$ and $[\beta : p_*\sigma]$ is finite we shall say that S is *algebraically horizontal.* If $\phi \leq \sigma$, $p_*\sigma \cong Z$ or D and β splits over $p_*\sigma$ we shall say that S is *algebraically vertical.* It is clear from the lemma that these possibilities are disjoint and exhaustive. Are there cusps which are neither horizontal nor vertical (up to isotopy)?

Lemma 6. *Let S be an algebraically horizontal cusp. Then the bundle projection induced over some finite covering of the base admits a section.*

Proof. Since S is an algebraically horizontal cusp it is flat (if $\chi(E) = 0$) or is an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold (if $\chi(E) \neq 0$). After passing to a finite covering, if necessary, we may assume that B is a surface, $S \cong B \times S^1$ and $p|_S$ is homotopic to the projection $pr_1 : S \to B$. Since pr_1 has an obvious section, there is a section $s : B \to E$ with image contained in S, by the HLP. $\square$

In particular, π is virtually a semidirect product. (A more transparent necessary condition is that π/ϕ' must be virtually a semidirect product.) If $\chi(\phi) < 0$ then $\zeta\phi = 1$ and so π is a semidirect product if and only if θ factors through $Aut(\phi)$.

Lemma 7. *Let S be an algebraically vertical cusp. If S is not a $\mathbb{N}il^3$-manifold then* $\mathrm{Ker}(\theta) \neq 1$.

Proof. After passing to the covering induced by a finite covering of B we may assume that $S \cong F \times S^1$, and hence that $p_*\sigma \leq \mathrm{Ker}(\theta)$. $\square$

Let π be the group with presentation

$$\langle a, b, c, d, e, f, x, y \mid [a, b][c, d][e, f] = 1, xax^{-1} = ab, ycy^{-1} = cd,$$
$$[x, y] = e, xc = cx, ya = ay,\ x, y \leftrightharpoons b, d, e, f\rangle.$$

Then π is the group of an orientable 4-manifold E which fibres over T with fibre of genus 3. Since θ is injective E is not geometric. The cusps in any geometric decomposition of a bundle space with flat base are infrasolvmanifolds, and so cannot be algebraically vertical. Since no subgroup of finite index in $\beta = Z^2$ admits a section E has no algebraically horizontal cusps, and hence E has no geometric decomposition.

5. HORIZONTAL AND VERTICAL DECOMPOSITIONS

A geometric decomposition of an orbifold bundle space E is horizontal or vertical if all the cusps are horizontal or vertical, respectively. Some bundle spaces (such as direct products $B \times F$) may admit both horizontal and vertical decompositions. However no decomposition can involve both types.

Lemma 8. *No geometric decomposition of an aspherical bundle space E has both algebraically horizontal and algebraically vertical cusps.*

Proof. If $\chi(E) = 0$ then either $\chi(B) = 0$ and every cusp is algebraically horizontal or $\chi(F) = 0$ and every cusp is algebraically vertical. Suppose that $\chi(E) > 0$ and S is an algebraically horizontal cusp. Then S is an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold. After passing to a finite covering, if necessary, we may assume that there is a section $s : B \to E$ with image contained in S, by Lemma 4. Clearly $s(B) \cap F_b = s(b)$, for all $b \in B$ and so $s_*[B] \bullet [F] \neq 0$. (Here it suffices to use $\mathbb{F}_2$ coefficients.) Thus it is not possible to homotope F off $s(B)$. In particular S must meet every algebraically vertical cusp. □

I am grateful to Peter Scott for the argument for the following lemma.

Lemma 9. *Let $\mathcal{G}$ be a finite graph of groups and $\theta : \beta \to \pi\mathcal{G}$ a homomorphism. Then B has a corresponding decomposition along a codimension-1 suborbifold.*

Proof. Let M be a finite regular covering of B which is a closed surface, and let $H = Aut(M/B)$. Then there is a β-equivariant map from $\widetilde{M} = \widetilde{B}$ to a β-tree $\mathcal{T}$ corresponding to the splitting. This induces a H-equivariant map from M to $\pi_1(M)\backslash\mathcal{T}$. Using Stallings' method of "binding ties", we may construct a H-equivariant homotopy of this map to one for which the preimage of each edge of $\pi_1(M)\backslash\mathcal{T}$ is a single closed curve in M. This projects to a 1-orbifold in B which induces the given splitting. □

If E has a horizontal decomposition with set of cusps $\mathcal{S}$ then $\{S \cap F \mid S \in \mathcal{S}\}$ is a compact 1-manifold, and is invariant (up to isotopy) under the action θ. Does E have a horizontal decomposition iff ϕ has a β-equivariant splitting? If S has an algebraically horizontal decomposition and $\mathrm{Im}(\theta) \cong Z$ is $\mathrm{Im}(\theta)$ generated by the image of a reducible self-homeomorphism of F?

6. Seifert fibred 4-manifolds

All Seifert fibred 4-manifolds over flat bases are geometric, of type $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ [9, 11]. Torus bundles over hyperbolic surfaces need not be geometric. (See the examples in §3 of Chapter 13 of [4].) However they all have vertical decompositions, as do "most" Seifert fibred 4-manifolds over hyperbolic bases. Let Kb be the Klein bottle.

Theorem 10. *Let E be Seifert fibred over an hyperbolic base. Then E is an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold if $F = Kb$, and otherwise has a vertical* *decomposition with pieces of type* $\mathbb{H}^2 \times \mathbb{E}^2$ *unless* B *has at least two cone points of order* 2*, at which the action has one eigenvalue* -1*.*

Proof. If $F = Kb$ then $\phi \cong Z \times_{-1} Z$ and so $Out(\phi)$ is finite. Hence E is an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold. (See [9, 11] or Chapter 9 of the 2007 revision of [4].) Moreover all pieces in any proper geometric decomposition are of type $\mathbb{H}^2 \times \mathbb{E}^2$.

If $F = T$ then $Out(\phi) = GL(2,\mathbb{Z})$ is virtually free. Conversely, if $\mathrm{Im}(\theta)$ is virtually free then $\mathrm{Im}(\theta) = \pi\mathcal{G}$ where G is a finite graph of finite groups. If $\mathrm{Im}(\theta)$ is finite then E has the product geometry. Otherwise B has a proper decomposition along a 1-dimensional suborbifold into pieces on which θ has finite image, by Lemma 9. We may clearly assume this decomposition is minimal, and that B has at most one cone point of order 2, at which the action has one eigenvalue -1. Since adjoining $D(p)$ to a contiguous suborbifold of B merely adds a relation to the orbifold fundamental group, there are no such pieces in the decomposition of B. Since $\chi(B) < 0$ and there are no pieces of type $D(p,p)$ we may thus assume that every piece is hyperbolic. The corresponding pieces of E have the product geometry.

The argument simplifies if $\mathrm{Im}(\theta) \leq SL(2,\mathbb{Z})$. For if $p_*g \in \beta$ has finite order n then g^n is a nontrivial element of ϕ which is fixed by $\theta(p_*g)$. Since $\theta(p_*g)$ has determinant 1 both eigenvalues are 1, and since it has finite order $\theta(p_*g) = I$. Hence θ factors through the fundamental group of the (possibly bounded) surface underlying the base orbifold B. □

The following example shows that the condition on cone points is needed. Let

$$\pi = \langle a,b,u,v,x,y \mid ab = ba, u^2 = v^2 = a, ubu^{-1} = b^{-1}, vbv^{-1} = b^{-1},$$
$$xax^{-1} = ab, xbx^{-1} = b^{-1}, ya = ay, yb = by, [x,y]uv = 1\rangle.$$

Then $\pi = \pi_1(E)$, where E is a Seifert manifold with base $B = T(2,2)$ and regular fibre T. The subgroup ϕ is generated by a,b and $\mathrm{Im}(\theta)$ is generated by $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 1 & -1\end{smallmatrix}\right)$. Hence $\mathrm{Im}(\theta) \cong D$ is infinite and has infinite index in $GL(2,\mathbb{Z})$. Thus E has no pieces of type $\mathbb{F}^4$. On the other hand in any decomposition of B into hyperbolic pieces the action corresponding to at least one piece has infinite image, and so E has no geometric decomposition at all.

If $\theta(\pi_1^{orb}(S^2(p,q,r)) = \pi\mathcal{G}$ is a finite graph of finite groups then consideration of normal forms shows that if x, y and xy all have finite order then they must all belong to the same vertex subgroup, and so this image is finite. If E is an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold and B is a closed hyperbolic orbifold with orientation cover not of the form $S^2(p,q,r)$ then E has a proper vertical decomposition.

If a Seifert manifold has a decomposition with all pieces of type $\mathbb{F}^4$ then $\mathrm{Im}(\theta)$ has finite index in $GL(2,\mathbb{Z})$. It seems unlikely that this condition characterizes such decompositions. (Let $B = F = T\#T$. Then B retracts onto $S^1 \vee S^1$. Mapping one generator of $F(2)$ to the involution $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the other to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ gives rise to a T bundle with base of genus 2 and $\mathrm{Im}(\theta) = SL(2,\mathbb{Z})$. This bundle surely has no decomposition into pieces of type $\mathbb{F}^4$.)

No Seifert fibred 4-manifold has an algebraically horizontal cusp. Must every cusp be isotopic to one which is vertical?

7. uniqueness up to s-cobordism?

Suppose that two algebraically vertical splittings of E induce the same splitting of β. Must they be isotopic? We shall explore some aspects of this question.

Our first reduction is to assume the splittings each have just one cusp, S and S', say. Since they are 2-sided, a necessary condition that they be isotopic is that we may isotope one off the other. We shall assume this, and show S and S' together bound a codimension-0 submanifold of E which is an s-cobordism.

Lemma 11. *Let $(G;\sigma,\tau : H \to G)$ be a PD_n-triple with $\tau = c_g \circ \sigma$ for some $g \in G$. Then $H = G$.*

Proof. The HNN extension $G^* = G*_H = \langle G, t \mid t\sigma(h)t^{-1} = g\sigma(h)g^{-1}\rangle$ is a PD_n-group. The element $g^{-1}t$ centralizes H, and so $\langle H, g^{-1}t\rangle \cong H \times Z$. Since this has cohomological dimension n it has finite index in G^*. Hence $[G : H] < \infty$ and so G is a PD_{n-1}-group. Hence $G = H$. □

We might hope to generalize this as follows. Let M be an n-manifold with two boundary components. Suppose that these are homotopy equivalent, and the inclusions are freely homotopic. What additional assumptions imply that M is an h-cobordism? The following example shows that something else is needed. Let Δ be a contractible n-manifold with $\pi_1(\partial\Delta) \neq 1$. Let $M = \Delta\#\Delta$. The two inclusions of $\partial\Delta$ as boundary components of M are freely homotopic, since $M \simeq S^{n-1}$ and so maps from $(n-1)$-manifolds into M are detected by the degree. However M is clearly not an h-cobordism.

Lemma 12. *Let $j, j' : S \to M$ be freely homotopic π_1-injective 2-sided embeddings of an aspherical 3-manifold into an aspherical 4-manifold. If $j(S) \cap j(S') = \emptyset$ then there is a codimension-0 submanifold N with boundary $j(S) \cup j(S')$ and which is an s-cobordism. Moreover, N is s-cobordant (rel ∂) to $S \times [0,1]$*

Proof. If $j(S)$ separates M then so does $j'(S)$, and $M - j(S) - j'(S)$ has three components, one with boundary $j(S) \cup j'(S)$ and the others with connected boundary. If $j(S)$ and $j'(S)$ do not separate M then $M - j(S) - j'(S)$ has two components, each with boundary $j(S) \cup j'(S)$. In either case a group-theoretic argument using normal forms in generalized free products shows that for one of these complementary components with disconnected boundary, N say, the two inclusions of S into ∂N are freely homotopic. These inclusions are homotopy equivalences, by Lemma 11, and so N is an h-cobordism from S to itself. It is an s-cobordism since $Wh(\pi_1(S)) = 0$ for all aspherical 3-manifolds S.

Since N is an h-cobordism from S to itself, $N \times S^1$ is an s-cobordism from $S \times S^1$ to itself. If S is a geometric 3-manifold then $S \times S^1$ has the corresponding product geometry, and the Strong Novikov Conjecture holds for $\pi_1(S) \times Z$. Hence $N \times S^1 \cong S \times S^1 \times [0,1]$, and so N is s-cobordant (*rel* ∂) to $S \times [0,1]$. □

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