2-KNOTS WITH SOLVABLE GROUP

JONATHAN A. HILLMAN

Abstract. The 2-knots with torsion-free, elementary amenable knot group and which have not yet been fully classified are fibred, with closed fibre the Hantzsche-Wendt flat 3-manifold HW or a Nil-manifold with base orbifold $S(3,3,3)$. We give explicit normal forms for the strict weight orbits of normal generators for the groups of such knots, and determine when the knots are amphicheiral or invertible.

The largest class of groups π over which TOP surgery techniques in dimension 4 are known to hold is the class SA obtained from groups of subexponential growth by extensions and increasing unions. No such group has a noncyclic free subgroup. The known 2-knot groups in this class are either torsion-free and solvable or have finite commutator subgroup. (It seems plausible that there may be no others. See Theorem 15.13 of [5] and §4 below for evidence in this direction.)

If the group of a nontrivial 2-knot K is torsion-free and elementary amenable then K is either the Fox knot (Example 10 of [2]) or is fibred, with closed fibre $\mathbb{R}^3/\mathbb{Z}^3$, the Hantzsche-Wendt flat 3-manifold $HW = \mathbb{R}^3/G_6$ or a Nil^3-manifold. (See Lemma 1.) Each such knot is determined up to Gluck reconstruction, TOP isotopy and change of orientations by its group π and weight orbit (the orbit of a weight element under the action of $Aut(\pi)$). This orbit is unique for the Fox knot and for the fibred knots with closed fibre $\mathbb{R}^3/\mathbb{Z}^3$ (the Cappell-Shaneson knots) or a coset space of the Lie group Nil. In each of these cases the questions of amphicheirality, invertibility and reflexivity have been decided, and so such knots may be considered completely classified. (See [5, 6, 7].)

We shall give explicit normal forms for the strict weight orbits. Using these, we shall show that (with at most six exceptions) no 2-knot with closed fibre HW is amphicheiral or invertible. The remaining knots have closed fibre the 2-fold branched cover of S^3, branched over a Montesinos knot $k(e,\eta) = K(0|e;(3,\eta),(3,1),(3,1))$, with e even and $\eta = \pm 1$. This include the 2-twist spins of these Montesinos knots, which are strongly +amphicheiral but not invertible, and are reflexive. None of the other knots are amphicheiral or invertible.

When the commutator subgroup of a 2-knot group is finite the list of possible groups and weight orbits is known, but the surgery obstruction groups are large, and there are in general infinitely many TOP locally flat knots with a given such group. Thus it is reasonable to restrict attention to those which are fibred. The closed fibre is then a spherical manifold S^3/π'. In this case the question of reflexivity has been settled for 10 of the 17 possible families of such knots [13].

It is likely that none of the remaining knots are reflexive, but this has not yet been confirmed.

1991 *Mathematics Subject Classification.* 57Q45.

Key words and phrases. amphicheiral. flat manifold. Hantzsche-Wendt. invertible. 2-knot. Nil-manifold. reflexive. strict weight orbit.

1. KNOT GROUPS

An automorphism ϕ of a group G is *meridianal* if $\langle\langle g^{-1}\phi(g) \mid g \in G\rangle\rangle_G = G$. When G is finitely generated and solvable this holds if and only if $H_1(\phi) - 1$ is an automorphism of the abelianization $H_1(G)$. If ϕ and ψ are meridianal automorphisms of G then the semidirect products $G\rtimes_\phi Z$ and $G\rtimes_\psi Z$ are isomorphic if and only if the outer automorphism class $[\phi]$ is conjugate to $[\psi]$ or $[\psi]^{-1}$ in $Out(G)$. There is an isomorphism preserving the stable letters of the HNN extensions if and only if ϕ and ψ are conjugate in $Aut(G)$. (See Lemma 1.1 of [5].)

Let $t \in \pi = G\rtimes_\phi Z$ be an element whose normal closure $\langle\langle t\rangle\rangle_\pi$ is the whole group. Every such "weight element" w is of the form $w = gt$ or $w = (gt)^{-1}$, for some $g \in G$. The strict weight orbit of w is the set $\{\alpha(w) \mid \alpha \in Aut(\pi),\ \alpha(w) \equiv w \ mod\ G\}$.

Let $c_t \in Aut(G)$ be the automorphism induced by conjugation by t in π. By Theorem 14.1 of [5], two weight elements t, u such that $[c_t] = [c_u]$ in $Out(G)$ are in the same strict weight orbit if and only if there is an automorphism ψ of G such that $c_u = \psi.c_t.\psi^{-1}$. In particular, c_t and c_u have the same order.

Although it is possible to study the automorphism groups considered below by purely algebraic means, we shall use embeddings in the appropriate affine groups to guide the construction of homeomorphisms and isotopies.

2. SELF-HOMEOMORPHISMS OF KNOT EXTERIORS

We assume that the spheres S^n are oriented. Let $K : S^2 \to S^4$ be a 2-knot with exterior X, and fix a homeomorphism $\partial X \cong S^2 \times S^1$ which is compatible with the orientations of the spheres S^1, S^2 and $X \subset S^4$. Let $\tau(x, y) = (\rho(y)(x), y)$ for all (x, y) in $S^2 \times S^1$, where $\rho : S^1 \to SO(3)$ is an essential map. The *Gluck reconstruction* of K is the knot K^* given by the composite inclusion

$$S^2 \subset S^2 \times D^2 \subset X \cup_\tau S^2 \times D^2 \cong S^4.$$

The knot K is *reflexive* if K^* is isotopic to one of the four knots K, r_4K, Kr_2 or $-K = r_4Kr_2$ obtained by composition with reflections r_n of S^n.

A self-homeomorphism of X extends "radially" to a self-homeomorphism of the knot manifold $M(K) = X \cup D^3 \times S^1$ which maps the cocore $C = \{0\} \times S^1$ to itself. If h preserves both orientations or reverses both orientations then it fixes the meridian, and we may assume that $h|_C = id_C$. If h reverses the meridian t, we may still assume that it fixes a point on C. We take such a fixed point as the basepoint for $M(K)$. Let h'_* be the induced automorphism of π'.

If K is invertible or $\pm$amphicheiral there is a self-homeomorphism h of (S^4, K) which changes the orientations appropriately, but does not twist the normal bundle of $K(S^2) \subset S^4$. If it is reflexive there is such a self-homeomorphism which changes the framing of the normal bundle. Thus if K is $-$amphicheiral there is such an h which reverses the orientation of $M(K)$ and h'_* commutes with the meridianal automorphism c_t. If K is invertible or $+$ampicheiral there is a homeomorphism h such that $h'_*c_th'_* = c_t^{-1}$ and which preserves or reverses the orientation.

3. SECTIONS OF THE MAPPING TORUS

Let θ be a self-homeomorphism of a 3-manifold F, with mapping torus $M(\theta) = F \times [0, 1]/\sim$, where $(f, 0) \sim (\theta(f), 1)$ for all $f \in F$, and canonical projection $p_\theta : M(\theta) \to S^1$, given by $p_\theta([f, s]) = e^{2\pi is}$ for all $[f, s] \in M(\theta)$. The mapping torus $M(\theta)$ is orientable if and only if θ is orientation-preserving. If $\theta' = h\theta h^{-1}$ for some self-homeomorphism h of F then $[f,s] \mapsto [h(f),s]$ defines a homeomorphism $m(h) : M(\theta) \to M(\theta')$ such that $p_{\theta'}m(h) = p_\theta$. Similarly, if θ' is isotopic to θ then $M(\theta') \cong M(\theta)$.

If $P \in F$ is fixed by θ then the image of $P \times [0,1]$ in $M(\theta)$ is a section of p_θ. In particular, if the fixed point set of θ is connected there is a canonical isotopy class of sections. If moreover $\theta_* = \pi_1(\theta)$ is meridianal these determine a preferred conjugacy class of weight elements in the group $\pi_1(M(\theta))$. (Two sections are isotopic if and only if they represent conjugate elements of π.)

In general, we may isotope θ to have a fixed point P. Let $t \in \pi_1(M(\theta))$ correspond to the constant section of $M(\theta)$, and let $u = gt$ with $g \in \pi_1(F)$. Let $\gamma : [0,1] \to F$ be a loop representing g. There is an isotopy h_s from $h_0 = id_F$ to $h = h_1$ which drags P around γ, so that $h_s(P) = \gamma(s)$ for all $0 \leq s \leq 1$. Then $H([f,s]) = [(h_s)^{-1}(f), s]$ defines a homeomorphism $M(\theta) \cong M(h^{-1}\theta)$. Under this homeomorphism the constant section of $p_{h^{-1}\theta}$ corresponds to the section of p_θ given by $m_u(t) = [\gamma(t), t]$, which represents u. If F is a geometric 3-manifold we may assume that γ is a geodesic path.

Suppose henceforth that θ is orientation-preserving and θ_* is meridianal. Then surgery on a section gives a 2-knot. There are two possible framings for the surgery, but the exteriors of the two knots are homeomorphic.

This is the situation for twist-spins, where F is a cyclic branched cover of S^3, branched over a classical knot, and θ generates the covering group. The subset fixed by θ is connected and nonempty, since it is the branch locus. The knot exterior is the complement of an open regular neighbourhood of the canonical section of the mapping torus of θ.

If F has universal cover $\widetilde{F} \cong \mathbb{R}^3$ and h is a self-homeomorphism of $M(\theta)$ which fixes a section setwise the behaviour of h with respect to the orientations is detected by the effect of h'_* on $H_3(F;\mathbb{Z})$ and whether $h'_* c_t h'_* = c_t$ or c_t^{-1}. As in [1, 4, 7] (and Chapter 18 of [5]), in order to determine whether h changes the framing it shall suffice to pass to the irregular covering space $M(\widetilde{\theta}) = \widetilde{F} \times_{\widetilde{\theta}} S^1$. We seek a coordinate homeomorphism $\widetilde{F} \cong \mathbb{R}^3$ which gives convenient representations of the maps in question, and then use an isotopy from the identity to $\widetilde{\theta}$ to identify $M(\widetilde{\theta})$ with $\mathbb{R}^3 \times S^1$.

4. TORSION-FREE ELEMENTARY AMENABLE IMPLIES SOLVABLE

We shall let Φ denote the group of the Fox knot. This is an ascending HNN extension $\Phi \cong Z*_2$, with presentation $\langle a, t \mid tat^{-1} = a^2 \rangle$.

Theorem 1. *Let K be a 2-knot whose group $\pi = \pi K$ is torsion-free and elementary amenable. Then K is trivial, the Fox knot, or is fibred with closed fibre a flat 3-manifold or a Nil-manifold.*

Proof. If π is torsion-free and has more than one end then $\pi \cong Z$, and so K is trivial [3]. If π has one end and $H^2(\pi;\mathbb{Z}[\pi) = 0$ then $M(K)$ is aspherical, by Theorem 3.5 of [5], and so $H^4(\pi;\mathbb{Z}[\pi) \neq 0$. Otherwise $H^2(\pi;\mathbb{Z}[\pi) \neq 0$. In all cases $H^s(\pi;\mathbb{Z}[\pi]) \neq 0$ for some $s \leq 4$, and so π is virtually solvable, by Proposition 3 of [8]. It then follows that either $\pi \cong Z$ or $\pi \cong \Phi = Z*_2$, or that π is virtually poly-Z of Hirsch length 4. (See Theorem 15.13 of [5].) If $\pi \cong \Phi$ then K is the Fox knot or its reflection [6], while the remaining cases are covered in Chapter 16 of [5]. □

Can we relax the condition on torsion? Let π be an elementary amenable knot group. Since π is finitely presentable and has an infinite cyclic quotient it is an HNN extension with finitely generated base and associated subgroups. Since it has no noncyclic free subgroups the HNN extension is ascending: $\pi \cong H*_\phi$, where H is finitely generated and $\phi : H \to H$ is injective. If moreover H is FP_3 and virtually indicable then either π' is finite or π is torsion-free, by Theorem 15.13 of [5].

The additional hypotheses on H could be removed if we had a better understanding of when $H^2(\pi;\mathbb{Z}[\pi]) = 0$. Suppose that whenever G is a finitely presentable group such that either

1. (1) G has an elementary amenable normal subgroup E such that
   - (a) $h(E) > 2$; or
   - (b) $h(E) = 2$ and G/E is infinite; or
   - (c) $h(E) = 1$ and G/E has one end; or
2. (2) $G \cong B*_\phi$ is an ascending HNN extension with finitely generated, 1-ended base B;

then $H^2(G;\mathbb{Z}[G]) = 0$.

We may then argue as follows. Since π is finitely presentable and has an infinite cyclic quotient it is an HNN extension with finitely generated base and associated subgroups. Since it has no noncyclic free subgroups the HNN extension is ascending: $\pi \cong H*_\phi$, where H is finitely generated and $\phi : H \to H$ is injective. Since π is elementary amenable and infinite $\beta_1^{(2)}(\pi) = 0$. If $h(\pi) = 1$ then π' is finite. Suppose that π' is infinite. Then π has one end. If $h(\pi) > 2$ or $h(\pi) = 2$ and the HNN base H has one end then $H^2(\pi;\mathbb{Z}[\pi]) = 0$ and so the knot manifold $M(K)$ is aspherical, by Theorem 3.5 of [5]. Hence π is torsion-free and virtually solvable, by Theorem 1.11 of [5]. (Closer examination shows that it must be polycyclic. See Chapter 16 of [5].) Otherwise H must have two ends. Let T be the maximal finite normal subgroup of H. Then $\phi(T) = T$, since ϕ is injective, and so T is normal in π. Hence $T = 1$ and $\pi \cong \Phi$, by Theorem 15.2 of [5].

5. The Hantzsche-Wendt flat 3-manifold

The group of affine motions of 3-space is $Aff(3) = \mathbb{R}^3 \rtimes GL(3,\mathbb{R})$. The action is given by $(v, A)(x) = Ax+v$, for all $x \in \mathbb{R}^3$. Therefore $(v, A)(w, B) = (v+Aw, AB)$.

Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$, and let $X, Y, Z \in GL(3,\mathbb{Z})$ be the diagonal matrices $X = diag[1,-1,-1]$, $Y = diag[-1, 1,-1]$ and $Z = diag[-1,-1, 1]$. Let $x = (\frac{1}{2}e_1, X)$, $y = (\frac{1}{2}(e_2 - e_3), Y)$ and $z = (\frac{1}{2}(e_1 - e_2 + e_3), Z)$. The subgroup of $Aff(3)$ generated by x and y is the Hantzsche-Wendt flat 3-manifold group G_6, with presentation

$$\langle x, y, z \mid xy^2x^{-1}y^2 = yx^2y^{-1}x^2 = 1,\ z = xy\rangle.$$

The translation subgroup $T = G_6 \cap \mathbb{R}^3$ is free abelian, with basis $\{x^2, y^2, z^2\}$. (This is the maximal abelian normal subgroup of G_6.) The holonomy group $H = \{I, X, Y, Z\} \cong (Z/2Z)^2$ is the image of G_6 in $GL(3,\mathbb{R})$. (Thus $H \cong G_6/T$.) We may clearly take $\{1, x, y, z\}$ as coset representatives for H in G_6. The commutator subgroup G_6' is free abelian, with basis $\{x^4, y^4, x^2y^2z^2\}$. Thus $2T < G_6' < T$, $T/G_6' \cong (Z/2Z)^2$ and $G_6'/2T \cong Z/2Z$.

The orbit space $HW = G_6\backslash\mathbb{R}^3$ is the *Hantzsche-Wendt* flat 3-manifold. If $\theta : HW \to HW$ is a self-homeomorphism of finite order then $M(\theta)$ is a flat 4-manifold. The group $N = N_{Aff(3)}(G_6)$ acts on HW via isometries, and $Isom(HW) \cong$

$N/G_6 = Out(G_6)$. The orientation-preserving subgroup is represented by pairs (v, A) with $\det(A) = 1$.

6. THE AUTOMORPHISM GROUP OF G_6

Let $C = C_{Aff(3)}(G_6)$ and $N = N_{Aff(3)}(G_6)$ be the centralizer and normalizer of G_6 in $Aff(3)$, respectively. Every automorphism of G_6 is induced by conjugation in $Aff(3)$, by a theorem of Bieberbach, and so $Aut(6) \cong N/C$ and $Out(G_6) \cong N/CG_6$. In Chapter 8.§2 of [5] we showed that $Aut(G_6)$ was generated by the automorphisms a, b, c, d, e, f, i and j which send x to $x^{-1}, x, x, x, y^2x, z^2x, y, z$ and y to $y, y^{-1}, z^2y, x^2y, y, z^2y, x, x$, respectively, and deduced a presentation for $Out(G_6)$. Here we shall identify these generators in terms of conjugation by elements of N.

We shall also need more detailed knowledge of $Aut(G_6)$. The subgroup generated by $\{a, b, c, d, e, f\}$ is normal, and is a semidirect product $Z^3 \rtimes (Z/2Z)^2$ with presentation

$$\langle a, b, c, d, e, f \mid a^2 = b^2 = c^2 = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ de = ed,\ df = fd,$$
$$ef = fe,\ ada = d^{-1},\ ae = ea,\ af = fa,\ bd = db,\ beb = e^{-1},\ bf = fb,$$
$$cd = dc,\ ce = ec,\ cfc = f^{-1}\rangle.$$

This subgroup contains the inner automorphisms $c_x = bcd$, $c_y = acef$ and $c_z = c_xc_y$ determined by conjugation by x and y. In particular, $c_x^2 = d^2$, $c_y^2 = e^2$ and $c_z^2 = f^2$. Adjoining the generator j gives another normal subgroup, in which $j^3 = abce$, so $j^6 = 1$, and j acts on $\langle a, b, c, d, e, f\rangle$ as follows:

$$jaj^{-1} = c,\ jbj^{-1} = ad^{-1},\ jcj^{-1} = be,\ jdj^{-1} = f,\ jej^{-1} = d,\ jfj^{-1} = e^{-1}.$$

This subgroup has index 2 in $Aut(G_6)$. The remaining generator i is an involution ($i^2 = 1$), and there are further relations

$$idi = e,\ iei = d,\ ifi = f^{-1},\ iai = b,\ ibi = a,\ ici = cf,\ jiji = d.$$

(In particular, $Aut(G_6)$ is generated by $\{a, i, j\}$, but this does not seem useful.)

If $(v, A) \in Aff(3)$ commutes with all elements of G_6 then $AB = BA$ for all $B \in H$, so A is diagonal, and $v + Aw = w + Bv$ for all $(w, B) \in G_6$. Taking $B = I$, we see that $Aw = w$ for all $w \in \mathbb{Z}^3$, so $A = I$, and then $v = Bv$ for all $B \in H$, so $v = 0$. Thus $C = 1$, and so $Aut(G_6) \cong N$.

If $(v, A) \in N$ then $A \in N_{GL(3,\mathbb{R})}(H)$ and A preserves $T = \mathbb{Z}^3$, so $A \in N_{GL(3,\mathbb{Z})}(H)$. Therefore $W = AXA^{-1}$ is in H. Hence $WA = AX$ and so $WAe_1 = Ae_1$ is up to sign the unique basis vector fixed by W. Applying the same argument to AYA^{-1} and AZA^{-1}, we see that $N_{GL(3,\mathbb{R})}(H)$ is the group of "signed permutation matrices", generated by the diagonal matrices and permutation matrices. Let

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

If A is a diagonal matrix in $GL(3, \mathbb{Z})$ then $(0, A) \in N$. Thus $\tilde{a} = (0, -X)$, $\tilde{b} = (0, -Y)$ and $\tilde{c} = (0, -Z)$ are in N. It is easily seen that $N \cap \mathbb{R}^3 = \frac{1}{2}\mathbb{Z}^3$, with basis $\tilde{d} = (\frac{1}{2}e_1, I)$, $\tilde{e} = (\frac{1}{2}e_2, I)$ and $\tilde{f} = (\frac{1}{2}e_3, I)$. It is also easily verified that $\tilde{i} = (-\frac{1}{4}e_3, P)$ and $\tilde{j} = (\frac{1}{4}(e_1 - e_2), J)$ are in N, and that N is generated by $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{i}, \tilde{j}\}$.

Conjugation by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}$, $\tilde{f}$, $\tilde{i}$ and $\tilde{j}$ induces the automorphisms a, b, c, d, e, f, i, j as defined above. (We shall henceforth drop the tildes.) Then $Out(G_6)$ has a presentation

$$\langle a, b, c, e, i, j \mid a^2 = b^2 = c^2 = e^2 = i^2 = j^6 = 1,\ a, b, c, e \text{ commute},\ iai = b,\ ici = ae,$$
$$jaj^{-1} = c,\ jbj^{-1} = abc,\ jcj^{-1} = be, jej^{-1} = bc, j^3 = abce,\ (ji)^2 = bc\rangle.$$

The natural homomorphism from $Out(G_6)$ to $Aut(G_6/T) \cong GL(2, \mathbb{F}_2)$ is onto, as the images of i and j generate $GL(2, \mathbb{F}_2)$, and its kernel is the normal subgroup $\langle a, b, c, e\rangle \cong (Z/2Z)^4$. Thus $Out(G_6)$ has order 96. The generators a, b, c, and j represent orientation reversing isometries.

Since G_6 is solvable and $H_1(G_6) \cong (Z/4Z)^2$, an automorphism (v, A) of G_6 is meridianal if and only if its image in $Aut(G_6/T) \cong GL(2, 2)$ has order 3. Thus its image in $Out(G_6)$ is conjugate to $[j]$, $[j]^{-1}$, $[ja]$ or $[jb]$. The latter pair are orientation-preserving and each is conjugate to its inverse (via $[i]$). However $[ja]$ is not conjugate to $[jb]^{\pm}$. Let $G(+) = G_6 \rtimes_{[ja]} Z$ and $G(-) = G_6 \rtimes_{[jb]} Z$.

Comparison with [14]. As the group element labeled z and the automorphisms labeled a, b, c by Zimmermann differ from ours, we shall add the subscript "Z" for clarity. The presentation for G_6 used in [14] reduces to

$$\langle x, y, z_Z \mid xy^2x^{-1}y^2 = yx^2y^{-1}x^2 = 1,\ z_Zyx = x^2z_Z^2 = z_Z^2x^2\rangle.$$

Thus $z_Z = yx^{-1}$, so $z_Z = y^2z^{-1}$, and $z_Z^2 = z^{-2}$. His choice of representatives for a generating set for $Out(G_6)$ is $\{a_Z, b_Z, c_Z, I, S, T\}$, where $a_Z = d^{-1}$, $b_Z = e^{-1}$, $c_Z = f^{-1}$, $I = xade$, $S = ideab$ and $T = j^{-1}i^{-1}xadf$. He observes also that $Out(G_6)$ is an extension of $S_3 \times Z/2Z$ by the normal subgroup $\langle\langle d, e, f\rangle\rangle \cong (Z/2Z)^3$, but the extension does not split, since the centre of $Out(G_6)$ is $\langle ab\rangle = Z/2Z$, which is too small.

7. 2-knots with group $G(+)$

The orthogonal matrix $-JX$ is a rotation though $\frac{2\pi}{3}$ about the axis in the direction $e_1 + e_2 - e_3$. The fixed point set of the isometry $[ja]$ of HW is the image of the line $\lambda(s) = s(e_1 + e_2 - e_3) - \frac{1}{4}e_2$. The knots corresponding to the canonical section are the 3-twist spin of the figure eight knot $\tau_3 4_1$ and its Gluck reconstruction $\tau_3 4_1^*$. The knot $\tau_3 4_1$ is $\pm$amphicheiral and invertible [10], but is not reflexive [4]. We shall show that $\tau_3 4_1$ is *strongly* $\pm$amphicheiral, but not strongly invertible. We shall also show that none of the other 2-knots with group $G(+)$ are amphicheiral or invertible.

Theorem 2. *Let $\pi = G(+)$. Then every strict weight orbit representing a given generator t for π/π' contains an unique element of the form $x^{2n}t$.*

Proof. If $t \in \pi$ represents a generator of $\pi/\pi' \cong Z$ it is a weight element, since π is solvable. Suppose that $c_t = ja = (\frac{1}{4}(e_1 - e_2), -JX)$. If u is another weight element with $[c_u] = [ja]$ then c_u is conjugate in $Aut(G_6)$ to $c_{g''t}$, for some $g'' \in \pi'' = G_6'$, by Theorem 14.1 of [5]. Thus we may assume that $c_u = (\hat{u} + \frac{1}{4}(e_1 - e_2), -JX)$, for some $(\hat{u}, I) \in G_6'$. Every element of G_6' arises in this way.

Let $\lambda(\Sigma x_ie_i) = (e_1 + e_2 - e_3) \bullet \Sigma x_ie_i = x_1 + x_2 - x_3$. Then $\mathrm{Ker}(\lambda) = \mathrm{Im}(I + JX)$. If $\lambda(\hat{v} - \hat{u}) = 0$ and $(\hat{v} - \hat{u}, I) \in G_6' < T$ then $(\hat{v} - \hat{u}) \in \mathrm{Ker}(\lambda|_{\mathbb{Z}^3}) = (I + JX)(\mathbb{Z}^3)$. Therefore $\hat{v} - \hat{u} = (I + JX)(w)$ for some $w \in \mathbb{Z}^3$. Let $\psi = (w, I)$. Then $\psi \in N$ and $c_v = \psi.c_u.\psi^{-1}$.

Conversely, if $c_v = \psi.c_u.\psi^{-1}$ for some $\psi = (w, A) \in N$ then $\hat{v} = \hat{u} + (I + JX)w$ and $A(-JX)A^{-1} = -JX$. Hence $\lambda(\hat{v}) = \lambda(\hat{u})$, since $\lambda(I + JX) = 0$.

In particular, $x^{2n}t$ is a weight element representing $[ja]$, for all $n \in \mathbb{Z}$, and $x^{2m}t$ and $x^{2n}t$ are in the same strict weight orbit if and only if $m = n$. □

Lemma 3. *If $n = 0$ then $C_{Aut(G_6)}(ja) = \langle ja, def^{-1}, abce\rangle$, and $N_{Aut(G_6)}(\langle ja\rangle) = \langle ja, ice, abce\rangle$. The subgroup which preserves the orientation of $\mathbb{R}^3$ is $\langle ja, ice\rangle$.*

If $n \neq 0$ then $N_{Aut(G_6)}(\langle d^{2n}ja\rangle) = C_{Aut(G_6)}(d^{2n}ja) = \langle d^{2n}ja, def^{-1}\rangle$. This subgroup acts orientably on $\mathbb{R}^3$.

Proof. This is straightforward. (Note that $abce = j^3$ and $def^{-1} = (ice)^2$.) □

Lemma 4. *The mapping torus $M([ja])$ has an orientation reversing involution which fixes a canonical section pointwise, and an orientation reversing involution which fixes a canonical section setwise but reverses its orientation. There is no orientation preserving involution of M which reverses the orientation of any section.*

Proof. Let $\omega = abcd^{-1}f = abce(ice)^{-2}$, and let $p = \lambda(\frac{1}{4}) = \frac{1}{4}(e_1 - e_3)$. Then $\omega = (2p, -I_3)$, $\omega^2 = 1$, $\omega ja = ja\omega$ and $\omega(p) = ja(p) = p$. Hence $\Omega = m([\omega])$ is an orientation reversing involution of $M([ja])$ which fixes the canonical section determined by the image of p in HW.

Let $\Psi([f, s]) = [[iab](f), 1 - s]$ for all $[f, s] \in M([ja])$. Since $(iab)ja(iab)^{-1} = (ja)^{-1}$ this is well-defined, and since $(iab)^2 = 1$ it is an involution. It is clearly orientation reversing, and since $iab(\lambda(\frac{1}{8})) = \lambda(\frac{1}{8})$ it reverses the section determined by the image of $\lambda(\frac{1}{8})$ in HW.

On the other hand, $\langle ja, ice\rangle \cong Z/3Z \rtimes_{-1} Z$, and the elements of finite order in this group do not invert ja. □

Theorem 5. *Let K be a 2-knot with group $G(+)$ and weight element $u = x^{2n}t$, where t is the canonical section. If $n = 0$ then K is strongly $\pm$amphicheiral, but is not strongly invertible. If $n \neq 0$ then K is neither amphicheiral nor invertible.*

Proof. Suppose first that $n = 0$. Since $-JX$ has order 3 it is conjugate in $GL(3, \mathbb{R})$ to a block diagonal matrix $\Lambda(-JX)\Lambda^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & R(\frac{2\pi}{3}) \end{pmatrix}$, where $R(\theta) \in GL(2, \mathbb{R})$ is rotation through θ. Let $R_s = R(\frac{2\pi}{3}s)$ and $\xi(s) = ((I_3 - A_s)p, A_s)$, where $A_s = \Lambda^{-1}\begin{pmatrix} 1 & 0 \\ 0 & R_s \end{pmatrix}\Lambda$, for $s \in \mathbb{R}$. Then ξ is a 1-parameter subgroup of $Aff(3)$, such that $\xi(s)(p) = p$ and $\xi(s)\omega = \omega\xi(s)$ for all s. In particular, $\xi|_{[0,1]}$ is a path from $\xi(0) = 1$ to $\xi(1) = ja$ in $Aff(3)$. Let $\Xi : \mathbb{R}^3 \times S^1 \to M(ja)$ be the homeomorphism given by $\Xi(v, e^{2\pi is}) = [\xi(v), s]$ for all $(v, s) \in \mathbb{R}^3 \times [0, 1]$. Then $\Xi^{-1}\Omega\Xi = \omega \times id_{S^1}$ and so Ω does not change the framing. Therefore K is strongly $-$amphicheiral.

Similarly, if we let $\zeta(s) = ((I_3 - A_s)\lambda(\frac{1}{8}), A_s)$ then $\zeta(s)(\lambda(\frac{1}{8})) = \lambda(\frac{1}{8})$ and $iab\zeta(s)iab = \zeta(s)^{-1}$, for all $s \in \mathbb{R}$, and $\zeta|_{[0,1]}$ is a path from 1 to ja in $Aff(3)$. Let $Z : \mathbb{R}^3 \times S^1 \to M(ja)$ be the homeomorphism given by $Z(v, e^{2\pi is}) = [\zeta(s)(v), s]$ for all $(v, s) \in \mathbb{R}^3 \times S^1$. Then $Z^{-1}\Psi Z(v, z) = (\zeta(1 - s)^{-1}iab\zeta(s)(v), z^{-1}) = (\zeta(1 - s)^{-1}\zeta(s)^{-1}iab(v), z^{-1}) = ((ja)^{-1}iab(v), z^{-1})$ for all $(v, z) \in \mathbb{R}^3 \times S^1$. Hence Ψ does not change the framing, and so K is strongly $+$amphicheiral. However it is not strongly invertible, by Lemma 4.

If $n \neq 0$ every such self-homeomorphism h preserves the orientation and fixes the meridian, by Lemma 3, and so K is neither amphicheiral nor invertible. □

8. 2-knots with group $G(-)$

A similar analysis applies when the knot group is $G(-)$, i.e., when the meridianal automorphism is $jb = (\frac{1}{4}(e_1 - e_2), -JY)$. All 2-knots with group $G(-)$ are fibred, and the characteristic map $[jb]$ has finite order, but none of these knots are twist-spins, as we shall show below.

Theorem 6. *Let $\pi = G(-)$. Then every strict weight orbit representing a given generator t for π/π' contains an unique element of the form $x^{2n}t$.*

Proof. The proof is very similar to that of Theorem 2. The main change is that the orthogonal matrix $-JY$ is now a rotation though $\frac{2\pi}{3}$ about the axis in the direction $e_1 - e_2 + e_3$. Thus we should define the homomorphism λ by dot product with $e_1 - e_2 + e_3$. □

Corollary. *No 2-knot with group $G(-)$ is a twist-spin.*

Proof. Suppose that $G(-)$ is the group of the r-twist-spin of a classical knot. Then the rth power of a meridian is central. The power $(x^{2n}t)^r$ is central in $G(-)$ if and only if $(d^{2n}jb)^r = 1$ in $Aut(G_6)$. But $(d^{2n}jb)^3 = d^{2n}f^{2n}e^{-2n}(jb)^3 = (de^{-1}f)^{2n+1}$. Therefore $d^{2n}jb$ has infinite order, and so $G(-)$ is not the group of a twist-spin. □

Lemma 7. *If $n = 0$ then $C_{Aut(G_6)}(jb) = \langle jb\rangle$, and $N_{Aut(G_6)}(\langle jb\rangle) = \langle jb, i\rangle$. If $n \neq 0$ then $N_{Aut(G_6)}(\langle d^{2n}jb\rangle) = C_{Aut(G_6)}(d^{2n}jb) = \langle d^{2n}jb, de^{-1}f\rangle$. These subgroups act orientably on $\mathbb{R}^3$.* □

The isometry $[jb]$ has no fixed points in $G_6\backslash\mathbb{R}^3$. We shall defined a preferred section as follows. Let $\gamma(s) = \frac{2s-1}{8}(e_1 - e_2) - \frac{1}{8}e_3$, for $s \in \mathbb{R}$. Then $\gamma(1) = jb(\gamma(0))$, and so $\gamma|_{[0,1]}$ defines a section of $p_{[jb]}$. We shall let the image of $(\gamma(0), 0)$ be the basepoint for $M([jb])$.

Theorem 8. *Let K be a 2-knot with group $G(-)$ and weight element $u = x^{2n}t$, where t is the canonical section. If $n = 0$ then K is strongly $+$amphicheiral but not invertible. If $n \neq 0$ then K is neither amphicheiral nor invertible.*

Proof. Suppose first that $n = 0$. Since $i(\gamma(s)) = \gamma(1 - s)$ for all $s \in \mathbb{R}$ the section defined by $\gamma|_{[0,1]}$ is fixed setwise and reversed by the orientation reversing involution $[f, s] \mapsto [[i](f), 1 - s]$. Let B_s be a 1-parameter subgroup of $O(3)$ such that $B_1 = jb$, and let $\zeta(s) = ((I_3 - B_s)\gamma(s), B_s)$ for $s \in \mathbb{R}$. Then $\zeta|_{[0,1]}$ is a path from 1 to jb in $Aff(3)$ such that $\zeta(s)(\gamma(s)) = \gamma(s)$ and $i\zeta(s)i = \zeta(s)^{-1}$ for all $0 \leq s \leq 1$. As in Theorem 5 it follows that this involution does not change the framing and so K is strongly $+$amphicheiral.

The other assertions follow from Lemma 7, as in Theorem 5. □

In particular, only $\tau_3 4_1, \tau_3 4_1^*$ and the knots obtained by surgery on the section of $M([jb])$ defined by $\gamma|_{[0,1]}$ admit orientation-changing symmetries.

9. Normal forms for meridianal automorphisms of $\Gamma(e, \eta)$

The other case of interest is when the commutator subgroup of the knot group is the fundamental group $\Gamma(e, \eta)$ of a Seifert fibred 3-manifold $M(e, \eta)$ which is a 2-fold branched covering of S^3, branched over a Montesinos knot $k(e, \eta) = K(0|e; (3, \eta), (3, 1), (3, 1))$, with e even and $\eta = \pm 1$. (See Chapter 16.§4 of [5].)

This 3-manifold is Seifert fibred over the flat 2-orbifold $S(3,3,3)$, and $\Gamma(e,\eta)$ has a presentation

$$\langle h,x,y,z \mid x^3=y^3=z^{3\eta}=h,\ xyz=h^e\rangle,$$

for some $\eta=\pm1$. Let $u=z^{-1}x$, $v=xz^{-1}$ and $q=3e-\eta-2$. Then $\Gamma(e,\eta)$ also has the presentation

$$\langle u,v,z \mid zuz^{-1}=v,\ zvz^{-1}=v^{-1}u^{-1}z^{3\eta-3},\ vuv^{-1}u^{-1}=z^{3\eta q}\rangle.$$

The image of $z^{3\eta}$ in $\Gamma(e,\eta)$ generates the centre $\zeta\Gamma(e,\eta)$, and $P=\Gamma(e,\eta)/\zeta\Gamma(e,\eta)$ is the orbifold fundamental group of $S(3,3,3)$.

An automorphism ϕ of $\Gamma(e,\eta)$ must preserve characteristic subgroups such as the centre $\zeta\Gamma(e,\eta)$ (generated by z^3) and the maximal nilpotent normal subgroup $\sqrt{\Gamma(e,\eta)}$ (generated by u, v and z^3). Let K be the subgroup of $Aut(\Gamma(e,\eta))$ consisting of automorphisms which induce the identity on $\Gamma(e,\eta)/\sqrt{\Gamma(e,\eta)}\cong Z/3Z$ and $\sqrt{\Gamma(e,\eta)}/\zeta\Gamma(e,\eta)\cong Z^2$. Automorphisms in K also fix the centre, and are of the form $k_{m,n}$, where

$$k_{m,n}(u)=uz^{3\eta s},\ k_{m,n}(v)=vz^{3\eta t}\quad\text{and}\quad k_{m,n}(z)=z^{3\eta p+1}u^mv^n,$$

for $(m,n)\in\mathbb{Z}^2$. These formulae define an automorphism if and only if

$$s-t=-nq,\quad s+2t=mq\quad\text{and}\quad 6p=(m+n)((m+n-1)q+2(\eta-1)).$$

In particular, conjugation by u and v give $c_u=k_{-2,-1}$ and $c_v=k_{1,-1}$, respectively. If $\eta=1$ then $q=3e$, so $s=(m-2n)e$, $t=-(m+n)e$ and $p=\binom{m+n}{2}e$ are integers for all $m,n\in\mathbb{Z}$. In this case $K\cong\mathbb{Z}^2$ is generated by $k=k_{1,0}$ and c_u. If $\eta=-1$ then $m+n\equiv0\ mod\,(3)$, and K is generated by c_u and c_v. In this case K has index 3 in $\mathbb{Z}^2$.

We may define automorphisms b and r by the formulae:

$$b(u)=v^{-1}z^{3\eta e-3},\quad b(v)=uvz^{3\eta(e-1)}\quad\text{and}\quad b(z)=z;\quad\text{and}$$

$$r(u)=v^{-1},\quad r(v)=u^{-1}\quad\text{and}\quad r(z)=z^{-1}.$$

It is easily checked that $b^6=r^2=(br)^2=1$ and that conjugation by z gives $c_z=b^4$. Since $\Gamma(e,\eta)/\Gamma(e,\eta)'$ is finite, $Hom(\Gamma(e,\eta),\zeta\Gamma(e,\eta))=0$, and so the natural homomorphism from $Aut(\Gamma(e,\eta))$ to $Aut(P)$ is injective. If $\eta=+1$ this homomorphism is an isomorphism, and $Aut(\Gamma(e,1))=$ has a presentation

$$\langle b,c_u,k,r \mid b^6=r^2=(br)^2=1,\ c_uk=kc_u,\ bc_ub^{-1}=c_u^{-1}k^{-3},\ bkb^{-1}=c_uk^2,$$
$$rc_ur=c_uk^3,\ rkr=k^{-1}\rangle.$$

(Here $c_v=c_uk^3$.) On the other hand, $Aut(\Gamma(e,-1))$ has a presentation

$$\langle b,c_u,c_v,r \mid b^6=r^2=(br)^2=1,\ c_uc_v=c_vc_u,\ bc_ub^{-1}=c_v^{-1},\ bc_vb^{-1}=c_uc_v,$$
$$rc_ur=c_v,\ rc_vr=c_u\rangle.$$

Hence $Out(\Gamma(e,1))\cong S_3\times Z/2Z$, while $Out(\Gamma(e,-1))\cong(Z/2Z)^2$.

In each case an automorphism ϕ is meridianal if and only if $[\phi]$ is conjugate to $[r]$, and so there is an unique corresponding knot group $\pi(e,\eta)=\Gamma(e,\eta)\rtimes_r Z$.

10. EMBEDDINGS IN THE AFFINE GROUP

The group P embeds as a discrete subgroup of $Isom(\mathbb{E}^2)$, via $u \mapsto (e_1, I_2)$, $v \mapsto (e_2, I_2)$ and $z \mapsto (0, -\beta)$, where $\beta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. The images of u and v in P form a basis for the translation subgroup $T(P) \cong Z^2$, and $P = T(P) \rtimes_{-\beta} (Z/3Z)$. It is easily seen that $C_{Aff(2)}(P) = 1$, and so $Aut(P) \cong N_{Aff(2)}(P)$. If $(v, A) \in N_{Aff(2)}(P)$ then $(I_2 + \beta)v \in \mathbb{Z}^2$ and A is in the subgroup D of $GL(2, \mathbb{R})$ generated by the matrices β and $\rho = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$, which has order 12. Thus $Aut(P) = (I_2 + \beta^{-1})T(P) \rtimes D$. Hence $Out(P) \cong D \cong S_3 \times Z/2Z$, where the first factor is generated by the images of u and $(0, -I_2)$ and the second factor is generated by the image of $(0, -\rho)$.

Let $Nil < GL(3, \mathbb{R})$ be the group of 3×3 upper triangular matrices

$$[x, y, w] = \begin{pmatrix} 1 & x & w \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and let $Aut(Nil)$ be the group of Lie automorphisms. As a set, $Aut(Nil)$ is the cartesian product $GL(2, \mathbb{R}) \times \mathbb{R}^2$, with $(A, \mu) = (\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right), (\mu_1, \mu_2))$ acting via $(A, \mu)([x, y, w]) =$

$$[ax + cy, bx + dy, \mu_1 x + \mu_2 y + (ad - bc)w + bcxy + \frac{ab}{2}x(x-1) + \frac{cd}{2}y(y-1)].$$

All such automorphisms are orientation preserving. The product of (A, μ) with $(B, \nu) = (\left(\begin{smallmatrix} g & j \\ h & k \end{smallmatrix}\right), (n_1, n_2))$ is

$$(A, \mu) \circ (B, \nu) = (AB, \mu B + det(A)\nu + \frac{1}{2}\eta(A, B)),$$

where

$$\eta(A, B) = (abg(1-g) + cdh(1-h) - 2bcgh, abj(1-j) + cdk(1-k) - 2bcjk).$$

Let $Aff(Nil) = Nil \rtimes Aut(Nil)$. Then $Aff(Nil)$ acts on the open 3-manifold $Nil \cong \mathbb{R}^3$ by $(n, \sigma)(n') = n\sigma(n')$. The abelianization $Nil \to \mathbb{R}^2 = Nil/\zeta Nil$ extends to an epimorphism $p : Aff(Nil) \to Aff(2)$, given by $p(n, A, \mu) = (\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), A)$ for $n = [x, y, w] \in Nil$, $A \in GL(2, \mathbb{R})$ and $\mu \in \mathbb{R}^2$. We may embed $\Gamma(e, \eta)$ in $Aff(Nil)$ by

$$u \mapsto ([1, 0, 0], \iota), \quad v \mapsto ([0, 1, 0], \iota) \quad \text{and} \quad z \mapsto ([0, 0, \frac{-1}{3q}], \alpha),$$

where $\iota = id_{Nil}$ and $\alpha = (-\beta, (0, \frac{\eta-1}{q}))$. (Note that $vuv^{-1}u^{-1} \mapsto ([0, 0, -1], \iota)$.) Let $N = N_{Aff(Nil)}(\Gamma(e, \eta))$ and $C = C_{Aff(Nil)}(\Gamma(e, \eta))$. As in the flat case, $Aut(\Gamma(e, \eta)) \cong N/C$ and $Out(\Gamma(e, \eta)) \cong N/C\Gamma(e, \eta)$.

It is easily seen that $C = \zeta Nil = \{([0, 0, z], \iota) \mid z \in \mathbb{R}\}$. If $n = [x, y, w]$ and $(n, A, \mu) \in N$ then $(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), A) \in N_{Aff(2)}(P)$, so $A \in D$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in (I_2 + \beta)^{-1}\mathbb{Z}^2$. If $A = I_2$ then (n, I_2, μ) is in N if and only if it normalizes $\sqrt{\Gamma(e, \eta)}$ and $(n, I_2, \mu)z = (n', \iota)z(n, I_2, \mu)$ for some $n' \in \sqrt{\Gamma(e, \eta)}$. The latter condition implies that $(I_2, \mu)\alpha = \alpha(I_2, \mu)$, and so $\mu(\beta + I_2) = 0$. Thus we must have $\mu = 0$ and $(I_2, \mu) = \iota$. The remaining conditions then imply that $x, y \in \frac{1}{q}\mathbb{Z}$. If $\eta = 1$ (so $q = 3e$) this satisfied by all $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in (I_2 + \beta)^{-1}\mathbb{Z}^2 < \frac{1}{3}\mathbb{Z}^2$. If $\eta = -1$ then $x, y \in \mathbb{Z}$. Thus the natural map from $Aut(\Gamma(e, \eta))$ to $Aut(P)$ is an isomorphism if $\eta = 1$, and has image of index 3 if $\eta = -1$.

11. 2-knots with group $\pi(e,\eta)$

Let $R = ([0,0,0],\rho,(0,0))$ in $Aff(Nil)$. Then $R^2 = 1$ and $R([x,y,z]) = [-y,-x,-z]$ for all $[x,y,z] \in Nil$. The fixed point set of the action of R on Nil is the connected curve $\{[s,-s,0] \mid s \in \mathbb{R}\}$. Thus the fixed point set of the involution $[R]$ of $M(e,\eta)$ induced by R is connected and nonempty. The corresponding 2-knot is $\tau_2 k(e,\eta)$. This is reflexive and +amphicheiral, as was first shown by Litherland. (He showed more generally that every 2-twist spin is reflexive. See [4, 11, 12] for proofs.)

Theorem 9. *Let $\pi = \pi(e,\eta)$. Then every strict weight orbit representing a given generator t for π/π' contains an unique element of the form $u^n t$.*

Proof. If $\psi r \psi^{-1} = rk$ for some $k \in K$ then we may assume that $\psi \in K$, and then $k \in (I-\rho)K$. The result follows by the argument of Theorem 3, with minor changes. (Note that $t \mapsto th$ defines an automorphism of π.) □

The next result is based on Lemma 1 of [7]. (See also Lemma 18.1 of [5]).

Theorem 10. *The knot $K = \tau_2 k(e,\eta)$ is strongly +amphicheiral and reflexive but is not invertible.*

Proof. Let $S([m,s]) = [b^3(m),s]$ and $h([m,s]) = [m,1-s]$ for $m \in M(e,\eta)$ and $0 \le s \le 1$. Then S and h define commuting involutions of $M([R])$, which each fix the canonical section setwise.

As remarked in §3, in order to determine how these involutions affect the framing we may pass to the irregular covering space $M(R) = Nil \times_R S^1$. We shall identify the space Nil with $\mathbb{R}^3$, in the obvious way.

Let $R(\theta) \in GL(2,\mathbb{R})$ be rotation through θ, and let $P = \left(\begin{smallmatrix} R(\frac{\pi}{4}) & 0 \\ 0 & 1 \end{smallmatrix}\right) \in GL(3,\mathbb{R})$. Then $PRP^{-1} = diag[1,-1,-1]$. We may isotope PRP^{-1} back to the identity, via $Q_s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R(s\pi) \end{smallmatrix}\right)$, for $0 \le s \le 1$. Let $Q : \mathbb{R}^3 \times S^1 \to M(PRP^{-1})$ be the homeomorphism given by $Q(v, e^{2\pi is}) = [Q_s(v), s]$ for all $(v,s) \in \mathbb{R}^3 \times [0,1]$. Then $Q^{-1}hQ((v,z) = (Q_{2s-1}(v), z^{-1})$ for all $(v,z) \in \mathbb{R}^3 \times S^1$. After reversing the S^1 factor this is just the twist, and so h changes the framing. Thus K is reflexive.

The automorphism b^3 acts linearly, via $b^3([x,y,z]) = [-x,-y,z+(e\eta-1)(x+y)]$, and so $Pb^3P^{-1} = \left(\begin{smallmatrix} -I_2 & 0 \\ \mu & 1 \end{smallmatrix}\right)$, where $\mu = (e\eta-1, e\eta-1)R(-\frac{\pi}{4})$. We may isotope Pb^3P^{-1} to $d = diag[-1,-1,1]$ through invertible matrices which commute with PRP^{-1}. Let $D([v,s]) = [d(v),s]$. Then S and D twist the framing in the same way. Since $Q^{-1}DQ(v,e^{2\pi is}) = (Q_{-s}dQ_s(v), e^{2\pi is}) = (dQ_{2s}(v), e^{2\pi is})$, for all $(v,s) \in \mathbb{R}^3 \times [0,1]$, it follows that S changes the framing. The composite sh is an involution which reverses the orientation and the meridian, but does not twist the framing. Hence K is strongly +amphicheiral.

Since automorphisms of $\Gamma(e,\eta)$ are orientation preserving K is not −amphicheiral or invertible. □

Theorem 11. *Let K be a 2-knot with group $\pi(e,\eta)$ and weight element $u^n t$, where t is the canonical section and $n \ne 0$. Then $N_{Aut(\Gamma(e,\eta))}(\langle u^n r\rangle) = C_{Aut(\Gamma(e,\eta))}(u^n r) = \langle u^n r, uv^{-1}\rangle$, and $u^n r$ is not conjugate to its inverse. Hence K is neither amphicheiral nor invertible.*

Proof. The first assertion is straightforward. Since $u^n t$ is not conjugate to its inverse K is not +amphicheiral, and since automorphisms of $\Gamma(e, \eta)$ are orientation preserving K is not −amphicheiral or invertible. □

References

[1] Cappell, S.E. and Shaneson, J.L. There exist inequivalent knots with the same complement, Ann. Math. 103 (1976), 349–353.

[2] Fox, R.H. A quick trip through knot theory, in *Topology of 3-Manifolds and Related Topics* (edited by M.K.Fort, Jr), Prentice-Hall, Englewood Cliffs, N.J.(1962), 120–167.

[3] Freedman, M.H. The topology of four-dimensional manifolds, J. Diff. Geom. 17 (1982), 357–453.

[4] Gordon, C. McA. Knots in the 4-sphere, Comment. Math. Helvetici 51 (1976), 585–596.

[5] Hillman, J.A. *Four-Manifolds, Geometries and Knots*, Geometry and Topology Monographs, vol. 5, Geometry and Topology Publications (2002).

[6] Hillman, J.A. Strongly minimal PD_4-complexes, Top. Appl. 156 (2009), 1565–1577.

[7] Hillman, J.A. and Plotnick, S. Geometrically fibred two-knots, Math. Ann. 287 (1990), 259–273.

[8] Kropholler, P.H. Soluble groups of type FP_∞ have finite torsion-free rank, Bull. London Math. Soc. 25 (1993), 558–566.

[9] Lee, K.B. and Raymond, F. Rigidity of almost crystallographic groups, in *Combinatorial Methods in Topology and Algebraic Geometry* (editors J.R.Harper and R.Mandelbaum), CONM 44, Amer. Math. Soc., Providence R.I. (1985), 73–78.

[10] Litherland, R.A. Symmetries of twist-spun knots, in *Knot Theory and Manifolds* (edited by D.Rolfsen), Lecture Notes in Mathematics 1144, Springer-Verlag, Berlin - Heidelberg - New York (1985), 97–107.

[11] Montesinos, J.M. On twins in the 4-sphere I, Quarterly J. Math. Oxford (2), 34 (1983), 171–199; II, *ibid* 35 (1984), 73–83.

[12] Plotnick, S. Fibered knots in S^4 – twisting, spinning, rolling, surgery, and branching, in *Four-Manifold Theory* (editors C.McA.Gordon and R.C.Kirby), CONM35, Amer. Math. Soc., Providence R.I. (1984), 437–459.

[13] Plotnick, S. and Suciu, A. Fibered knots and spherical space forms, J. London Math. Soc. 35 (1987), 514–526.

[14] Zimmermann, B. On the Hantzsche-Wendt manifold, Mh. Math. 110 (1990), 321–327.

School of Mathematics and Statistics F07, University of Sydney, Sydney, NSW 2006, Australia

E-mail address: jonh@maths.usyd.edu.au