The MacMahon Master Theorem for right quantum superalgebras and higher Sugawara operators for $\widehat{\mathfrak{gl}}_{m|n}$

A. I. Molev and E. Ragoucy

Abstract

We prove an analogue of the MacMahon Master Theorem for the right quantum superalgebras. In particular, we obtain a new and simple proof of this theorem for the right quantum algebras. In the super case the theorem is then used to construct higher order Sugawara operators for the affine Lie superalgebra $\widehat{\mathfrak{gl}}_{m|n}$ in an explicit form. The operators are elements of a completed universal enveloping algebra of $\widehat{\mathfrak{gl}}_{m|n}$ at the critical level. They occur as the coefficients in the expansion of a noncommutative Berezinian and as the traces of powers of generator matrices. The same construction yields higher Hamiltonians for the Gaudin model associated with the Lie superalgebra $\mathfrak{gl}_{m|n}$.

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School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia
alexm@maths.usyd.edu.au

LAPTH, Chemin de Bellevue, BP 110
F-74941 Annecy-le-Vieux cedex, France
ragoucy@lapp.in2p3.fr

1 Introduction

1.1. MacMahon Master Theorem. A natural quantum analogue of the celebrated MacMahon Master Theorem was proved by Garoufalidis, Lê and Zeilberger in [13]. A few different proofs and generalizations of this analogue have since been found; see Foata and Han [9, 10], Hai and Lorenz [19] and Konvalinka and Pak [24].

In the quantum MacMahon Master Theorem of [13] the numerical matrices are replaced with the *right quantum matrices* $Z = [z_{ij}]$ whose matrix elements satisfy some quadratic relations involving a parameter q. As explained in [10], this parameter may be taken to be equal to 1 without loss of generality. Then the relations for the matrix elements z_{ij} take the form

$$[z_{ij}, z_{kl}] = [z_{kj}, z_{il}] \qquad \text{for all} \quad i, j, k, l \in \{1, \dots, N\}, \tag{1.1}$$

where $[x, y] = xy - yx$.

The proof of the quantum MacMahon Master Theorem given in [19] is based on the theory of Koszul algebras. In that approach, (1.1) are the defining relations of the bialgebra $\underline{\mathrm{end}}\, \mathcal{A}$ associated with the symmetric algebra $\mathcal{A}$ of a vector space. The construction of the bialgebra $\underline{\mathrm{end}}\, \mathcal{A}$ associated with an arbitrary quadratic algebra or superalgebra $\mathcal{A}$ is due to Manin [26, 27], and the matrices Z satisfying (1.1) are also known as *Manin matrices*; see [2], [3], [4]. In the super case the construction applied to the symmetric algebra of a $\mathbb{Z}_2$-graded vector space (i.e., superspace) of dimension $m+n$ leads to the defining relations of the *right quantum superalgebra* $\mathcal{M}_{m|n}$. This superalgebra is generated by elements z_{ij} with $\mathbb{Z}_2$-degree (or parity) $\bar{\imath}+\bar{\jmath}$, where $\bar{\imath} = 0$ for $1 \leqslant i \leqslant m$ and $\bar{\imath} = 1$ for $m+1 \leqslant i \leqslant m+n$. The defining relations have the form

$$[z_{ij}, z_{kl}] = [z_{kj}, z_{il}](-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}} \qquad \text{for all} \quad i, j, k, l \in \{1, \dots, m+n\}, \tag{1.2}$$

where $[x, y] = xy - yx(-1)^{\deg x \deg y}$ is the super-commutator of homogeneous elements x and y, as presented e.g. in [18, Example 3.14][1]. We will call any matrix $Z = [z_{ij}]$ satisfying (1.2) a Manin matrix.

Our first main result (Theorem 2.2) is an analogue of the MacMahon Master Theorem for the right quantum superalgebra $\mathcal{M}_{m|n}$. We will identify the matrix Z with an element of the tensor product superalgebra $\mathrm{End}\, \mathbb{C}^{m|n} \otimes \mathcal{M}_{m|n}$ by

$$Z = \sum_{i,j=1}^{m+n} e_{ij} \otimes z_{ij} (-1)^{(\bar{\imath}+1)\bar{\jmath}},$$

where the e_{ij} denote the standard matrix units. Taking multiple tensor products

$$\mathrm{End}\, \mathbb{C}^{m|n} \otimes \dots \otimes \mathrm{End}\, \mathbb{C}^{m|n} \otimes \mathcal{M}_{m|n} \tag{1.3}$$

[1] The relations given in [18] correspond to our *left* quantum superalgebra as defined in Sec. 2.2 below.

with k copies of $\operatorname{End}\mathbb{C}^{m|n}$, for any $a = 1, \dots, k$ we will write Z_a for the matrix Z corresponding to the a-th copy of the endomorphism superalgebra so that the components in all remaining copies are the identity matrices. The symmetric group $\mathfrak{S}_k$ acts naturally on the tensor product space $(\mathbb{C}^{m|n})^{\otimes k}$. We let H_k and A_k denote the images of the normalized symmetrizer

$$h_k = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma \in \mathbb{C}[\mathfrak{S}_k] \tag{1.4}$$

and antisymmetrizer

$$a_k = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot \sigma \in \mathbb{C}[\mathfrak{S}_k] \tag{1.5}$$

in (1.3), respectively. Recall that the *supertrace* of an even matrix $X = [x_{ij}]$ is defined by

$$\operatorname{str} X = \sum_{i=1}^{m+n} x_{ii} (-1)^{\bar{\imath}}. \tag{1.6}$$

Now set

$$\text{Bos} = \sum_{k=0}^{\infty} \operatorname{str} H_k Z_1 \dots Z_k, \qquad \text{Ferm} = \sum_{k=0}^{\infty} (-1)^k \operatorname{str} A_k Z_1 \dots Z_k, \tag{1.7}$$

where str denotes the supertrace taken with respect to all k copies of $\operatorname{End}\mathbb{C}^{m|n}$. Our analogue of the MacMahon Master Theorem for the right quantum superalgebra $\mathcal{M}_{m|n}$ now reads

$$\text{Bos} \times \text{Ferm} = 1. \tag{1.8}$$

In each of the particular cases $n = 0$ and $m = 0$ this identity turns into the quantum MacMahon Master Theorem of [13].

Our proof of the identity (1.8) is based on the use of the matrix form of the defining relations (1.2). These relations can be written as

$$(1 - P_{12}) [Z_1, Z_2] = 0 \tag{1.9}$$

which is considered as an identity in the superalgebra (1.3) with $k = 2$ and P_{12} is the image of the transposition $(12) \in \mathfrak{S}_2$. The proof of (1.8) is derived from (1.9) by using some elementary properties of the symmetrizers and antisymmetrizers.

1.2. Noncommutative Berezinian. It is well-known that in the super-commutative specialization the expressions (1.7) coincide with the expansions of the *Berezinian* so that

$$\text{Ferm} = \operatorname{Ber}(1 - Z) \qquad \text{and} \qquad \text{Bos} = \big[\operatorname{Ber}(1 - Z)\big]^{-1};$$

see [1], [23], [32]. In order to prove their noncommutative analogues, we introduce the *affine right quantum superalgebra* $\widehat{\mathcal{M}}_{m|n}$. It is generated by a countable set of elements

$z_{ij}^{(r)}$ of parity $\bar{\imath}+\bar{\jmath}$, where r runs over the set of positive integers. The defining relations of $\widehat{\mathcal{M}}_{m|n}$ take the form

$$(1-P_{12})\,[Z_1(u),Z_2(u)]=0, \tag{1.10}$$

where u is a formal variable, and the matrix elements of the matrix $Z(u)=[z_{ij}(u)]$ are the formal power series

$$z_{ij}(u)=\delta_{ij}+z_{ij}^{(1)}\,u+z_{ij}^{(2)}\,u^2+\cdots. \tag{1.11}$$

The relation (1.10) is understood in the sense that all coefficients of the powers of u on the left hand side vanish. The formal power series $Z(u)$ in u with matrix coefficients is invertible and we denote by $z'_{ij}(u)$ the matrix elements of its inverse so that $Z^{-1}(u)=[z'_{ij}(u)]$. Now we set

$$\begin{aligned}\mathrm{Ber}\,Z(u)=&\sum_{\sigma\in\mathfrak{S}_m}\mathrm{sgn}\,\sigma\cdot z_{\sigma(1)1}(u)\dots z_{\sigma(m)m}(u)\\ &\times\sum_{\tau\in\mathfrak{S}_n}\mathrm{sgn}\,\tau\cdot z'_{m+1,m+\tau(1)}(u)\dots z'_{m+n,m+\tau(n)}(u).\end{aligned} \tag{1.12}$$

In the super-commutative specialization $\mathrm{Ber}\,Z(u)$ coincides with the ordinary Berezinian of the matrix $Z(u)$; see [1].

We prove the following expansions of the noncommutative Berezinian, where Z is a Manin matrix:

$$\mathrm{Ber}\,(1+uZ)=\sum_{k=0}^{\infty}u^k\,\mathrm{str}\,A_kZ_1\dots Z_k \tag{1.13}$$

and hence

$$\big[\mathrm{Ber}\,(1-uZ)\big]^{-1}=\sum_{k=0}^{\infty}u^k\,\mathrm{str}\,H_kZ_1\dots Z_k. \tag{1.14}$$

Furthermore, we derive the following super-analogues of the Newton identities:

$$\big[\mathrm{Ber}\,(1+uZ)\big]^{-1}\partial_u\mathrm{Ber}\,(1+uZ)=\sum_{k=0}^{\infty}(-u)^k\,\mathrm{str}\,Z^{k+1} \tag{1.15}$$

which allow one to express the elements $\mathrm{str}\,Z^k$ in terms of the coefficients of either series (1.13) or (1.14). In the even case of $n=0$ the Newton identities were proved in the papers [3, 4] which also contain various generalizations of the matrix algebra properties to the class of Manin matrices. However, the proof of the identities given in these papers relies on the existence of the adjoint matrix and does not immediately extend to the super case. To prove (1.15) we employ instead an appropriate super-extension of the R-matrix arguments used in [21].

1.3. Segal–Sugawara vectors. The above properties of Manin matrices will be used in our construction of the *higher order Sugawara operators* for the affine Lie superalgebra $\widehat{\mathfrak{gl}}_{m|n}$

with $m \neq n$. The commutation relations of the Lie superalgebra $\widehat{\mathfrak{gl}}_{m|n} = \mathfrak{gl}_{m|n}[t, t^{-1}] \oplus \mathbb{C}K$ have the form

$$\big[e_{ij}[r], e_{kl}[s]\big] = \delta_{kj}\, e_{il}[r+s] - \delta_{il}\, e_{kj}[r+s](-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})} \\ + K\Big(\delta_{kj}\,\delta_{il}(-1)^{\bar{\imath}} - \frac{\delta_{ij}\,\delta_{kl}}{m-n}(-1)^{\bar{\imath}+\bar{k}}\Big)\, r\,\delta_{r,-s}, \tag{1.16}$$

the element K is central, and we set $e_{ij}[r] = e_{ij}t^r$. We will also consider the extended Lie superalgebra $\widehat{\mathfrak{gl}}_{m|n} \oplus \mathbb{C}\tau$, where the element τ is even and

$$\big[\tau, e_{ij}[r]\big] = -r\, e_{ij}[r-1], \qquad \big[\tau, K\big] = 0. \tag{1.17}$$

For any $\kappa \in \mathbb{C}$ the *affine vertex algebra* $V_\kappa(\mathfrak{gl}_{m|n})$ can be defined as the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{gl}}_{m|n})$ by the left ideal generated by $\mathfrak{gl}_{m|n}[t]$ and $K - \kappa$; see e.g. [12], [22]. The *center* of the vertex algebra $V_\kappa(\mathfrak{gl}_{m|n})$ is its subspace spanned by all elements $b \in V_\kappa(\mathfrak{gl}_{m|n})$ such that $\mathfrak{gl}_{m|n}[t]\, b = 0$. The axioms of the vertex algebra imply that the center is a commutative associative superalgebra. The center of the vertex algebra $V_\kappa(\mathfrak{gl}_{m|n})$ is trivial for all values of κ except for the *critical value* $\kappa = n - m$. In the latter case the center is big and we denote it by $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$. Any element of $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ is called a *Segal–Sugawara vector*. As a vector superspace, the vertex algebra $V_{n-m}(\mathfrak{gl}_{m|n})$ can be identified with the universal enveloping algebra $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$. Moreover, the multiplication in the commutative algebra $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ coincides with the multiplication in $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$. Therefore, the center $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ is naturally identified with a commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$.

Another principal result of this paper (Theorem 3.2 and Corollary 3.3) is an explicit construction of several families of Segal–Sugawara vectors. The construction is based on the observation that the matrix elements of the matrix

$$\tau + \widehat{E}[-1] = \big[\delta_{ij}\tau + e_{ij}[-1](-1)^{\bar{\imath}}\big]$$

satisfy the defining relations (1.2) of the right quantum superalgebra. That is, $\tau + \widehat{E}[-1]$ is a Manin matrix. We use this fact to show that all coefficients s_{kl} in the expansion

$$\mathrm{str}(\tau + \widehat{E}[-1])^k = s_{k0}\,\tau^k + s_{k1}\,\tau^{k-1} + \cdots + s_{kk}$$

are Segal–Sugawara vectors. The relations (1.13), (1.14) and (1.15) then lead to explicit formulas for a few other families of Segal–Sugawara vectors. In particular, such vectors also occur as the coefficients b_{kl} in the expansion of the Berezinian

$$\mathrm{Ber}\,\big(1 + u(\tau + \widehat{E}[-1])\big) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} b_{kl}\, u^k \tau^{k-l}. \tag{1.18}$$

1.4. Commutative subalgebras and higher Gaudin Hamiltonians. Applying the state-field correspondence map to the Segal–Sugawara vectors of any of the families above, we get explicit formulas for elements of the center of the local completion of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{gl}}_{m|n})$ at the critical level. Such elements are called the (higher order) *Sugawara operators*. In particular, such operators can be calculated as the Fourier coefficients of the fields $b_{kl}(z)$ defined by the expansion of the normally ordered Berezinian

$$: \mathrm{Ber}\,\big(1+u(\partial_z+\widehat{E}(z))\big) : \; = \sum_{k=0}^{\infty}\sum_{l=0}^{k} b_{kl}(z)\, u^k \partial_z^{k-l}, \tag{1.19}$$

where $\widehat{E}(z)=[e_{ij}(z)(-1)^{\bar{i}}]$ and

$$e_{ij}(z)=\sum_{r\in\mathbb{Z}} e_{ij}[r]\, z^{-r-1}.$$

Similarly, all Fourier coefficients of the fields $s_{kl}(z)$ defined by the expansion of the normally ordered supertrace

$$: \mathrm{str}\big(\partial_z+\widehat{E}(z)\big)^k : \; = s_{k0}(z)\,\partial_z^k + s_{k1}(z)\,\partial_z^{k-1}+\dots+s_{kk}(z) \tag{1.20}$$

are Sugawara operators.

It was observed in [8] that the center of the affine vertex algebra at the critical level is closely related to Hamiltonians of the Gaudin model describing quantum spin chain. The Hamiltonians are obtained by the application of the Sugawara operators to the vacuum vector of the vertex algebra $V_\kappa(\mathfrak{gl}_{m|n})$ with $\kappa=n-m$. Such an application yields elements of the center $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ of $V_{n-m}(\mathfrak{gl}_{m|n})$. We thus obtain explicit formulas for several families of elements of a commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$.

Using a duality between the superalgebras $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$ and $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$ we also obtain the corresponding families of commuting elements in the superalgebra $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$. More general families of commutative subalgebras of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$ can be brought in by applying some automorphisms of this superalgebra parameterized by diagonal or numerical matrices.

The connection with the Gaudin model is obtained by considering the iterated comultiplication map

$$\mathrm{U}(\mathfrak{gl}_{m|n}[t]) \to \mathrm{U}(\mathfrak{gl}_{m|n}[t])\otimes\ldots\otimes \mathrm{U}(\mathfrak{gl}_{m|n}[t]) \tag{1.21}$$

with k copies of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$ and then taking the images of elements of a commutative subalgebra of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$ in the tensor product of evaluation representations of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$. This yields the higher Hamiltonians of the Gaudin model; see Sec. 3.2 below for more details. Such a scheme was used in [33] to give simple and explicit determinant-type formulas for the higher Gaudin Hamiltonians in the case of $\mathfrak{gl}_n$; see also [5], [6], [29], [30] and references therein. The results of [5] include a calculation of the eigenvalues of the Sugawara operators in the Wakimoto modules which we believe can be extended to the super case as well.

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2 Manin matrices over superalgebras

2.1 MacMahon Master Theorem

As defined in the Introduction, the right quantum superalgebra $\mathcal{M}_{m|n}$ is generated by the elements z_{ij} with $1 \leqslant i, j \leqslant m+n$ such that the parity of z_{ij} is $\bar{\imath}+\bar{\jmath}$. The defining relations are given in (1.2). In what follows we will use the matrix form (1.9) of these relations. In order to explain this notation in more detail, consider the superalgebra

$$\mathrm{End}\,\mathbb{C}^{m|n} \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^{m|n} \otimes \mathcal{M}_{m|n} \tag{2.1}$$

with k copies of $\mathrm{End}\,\mathbb{C}^{m|n}$. For each $a \in \{1, \ldots, k\}$ the element Z_a of the superalgebra (2.1) is defined by the formula

$$Z_a = \sum_{i,j=1}^{m+n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(k-a)} \otimes z_{ij}\,(-1)^{(\bar{\imath}+1)\bar{\jmath}}.$$

Using the natural action of $\mathfrak{S}_k$ on $(\mathbb{C}^{m|n})^{\otimes k}$ we represent any permutation $\sigma \in \mathfrak{S}_k$ as an element P_σ of the superalgebra (2.1) with the identity component in $\mathcal{M}_{m|n}$. In particular, the transposition $(a\,b)$ with $a < b$ corresponds to the element

$$P_{ab} = \sum_{i,j=1}^{m+n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(k-b)} \otimes 1\,(-1)^{\bar{\jmath}}, \tag{2.2}$$

which allows one to determine P_σ by writing an arbitrary $\sigma \in \mathfrak{S}_k$ as a product of transpositions. Recall also that if x, x' are homogeneous elements of a superalgebra $\mathcal{A}$ and y, y' are homogeneous elements of a superalgebra $\mathcal{B}$ then the product in the superalgebra $\mathcal{A}\otimes\mathcal{B}$ is calculated with the use of the sign rule

$$(x \otimes y)(x' \otimes y') = (xx' \otimes yy')\,(-1)^{\deg y\,\deg x'}.$$

Note that all the elements Z_a and P_{ab} are even and P_{ab} commutes with Z_c if $c \neq a, b$. Moreover, $P_{ab}Z_a = Z_b P_{ab}$.

For each $a = 1, \ldots, k$ the supertrace str_a with respect to the a-th copy of $\mathrm{End}\,\mathbb{C}^{m|n}$ in (2.1) is the linear map

$$\mathrm{str}_a : \big(\mathrm{End}\,\mathbb{C}^{m|n}\big)^{\otimes k} \otimes \mathcal{M}_{m|n} \to \big(\mathrm{End}\,\mathbb{C}^{m|n}\big)^{\otimes k-1} \otimes \mathcal{M}_{m|n},$$

defined by

$$\mathrm{str}_a : x_1 \otimes \ldots \otimes x_{a-1} \otimes e_{ij} \otimes x_{a+1} \otimes \ldots \otimes x_k \otimes y \\ \mapsto \delta_{ij}\, x_1 \otimes \ldots \otimes x_{a-1} \otimes x_{a+1} \otimes \ldots \otimes x_k \otimes y\, (-1)^{\bar{\imath}}.$$

In the case $k = 1$ this definition clearly agrees with (1.6). The following cyclic property of the supertrace will often be used: if $X = [x_{ij}]$ and $Y = [y_{ij}]$ are even matrices with pairwise super-commuting entries, $[x_{ij}, y_{kl}] = 0$, then

$$\mathrm{str}(XY - YX) = 0.$$

We will need the properties of Manin matrices given in the next proposition. In the even case ($n = 0$) they were formulated in [4, Proposition 18] without a proof. So we supply our own argument. Recall that H_k and A_k denote the respective images of the symmetrizer and antisymmetrizer in (2.1); see (1.4) and (1.5).

Proposition 2.1. *We have the identities in the superalgebra* (2.1)

$$A_k Z_1 \ldots Z_k A_k = A_k Z_1 \ldots Z_k \tag{2.3}$$

and

$$H_k Z_1 \ldots Z_k H_k = Z_1 \ldots Z_k H_k. \tag{2.4}$$

Proof. We use induction on k. For the proof of (2.3) note that

$$A_k = \frac{1}{k}\, A_{k-1} - \frac{k-1}{k}\, A_{k-1} P_{k-1,k}\, A_{k-1}.$$

The defining relations (1.9) imply that

$$(1 - P_{k-1,k})\, Z_{k-1} Z_k\, P_{k-1,k} = -(1 - P_{k-1,k}) Z_{k-1} Z_k.$$

Since $A_k A_{k-1} = A_k$ and $A_k (1 - P_{k-1,k}) = 2\, A_k$, using the induction hypothesis we get

$$A_k Z_1 \ldots Z_k A_k = A_k Z_1 \ldots Z_k \Big(\frac{1}{k}\, A_{k-1} - \frac{k-1}{k}\, A_{k-1} P_{k-1,k}\, A_{k-1}\Big) = A_k Z_1 \ldots Z_k,$$

as required. To verify (2.4) we write

$$H_k = \frac{1}{k}\, H_{k-1} + \frac{k-1}{k}\, H_{k-1} P_{k-1,k}\, H_{k-1}$$

and use another consequence of (1.9),

$$P_{k-1,k}\, Z_{k-1} Z_k\, (1 + P_{k-1,k}) = Z_{k-1} Z_k\, (1 + P_{k-1,k}),$$

to complete the proof in a similar way. □

We are now in a position to prove the MacMahon Master Theorem for the right quantum superalgebra $\mathcal{M}_{m|n}$. We use the notation (1.7).

Theorem 2.2. *We have the identity*

$$\mathrm{Bos} \times \mathrm{Ferm} = 1.$$

Proof. It is sufficient to show that for any integer $k \geqslant 1$ we have the identity in the superalgebra (2.1)

$$\sum_{r=0}^{k} (-1)^{k-r} \mathrm{str}_{1,\dots,r} H_r Z_1 \dots Z_r \times \mathrm{str}_{r+1,\dots,k} A_{\{r+1,\dots,k\}} Z_{r+1} \dots Z_k = 0,$$

where $A_{\{r+1,\dots,k\}}$ denotes the antisymmetrizer in (2.1) over the copies of $\mathrm{End}\,\mathbb{C}^{m|n}$ labeled by $r+1,\dots,k$. The identity can be written as

$$\sum_{r=0}^{k} (-1)^{r} \mathrm{str}_{1,\dots,k} H_r A_{\{r+1,\dots,k\}} Z_1 \dots Z_k = 0. \tag{2.5}$$

Our next step is to show that the product of the symmetrizer and antisymmetrizer in (2.5) can be replaced as follows:

$$H_r A_{\{r+1,\dots,k\}} \mapsto \frac{r(k-r+1)}{k}\, H_r A_{\{r,\dots,k\}} + \frac{(r+1)(k-r)}{k}\, H_{r+1} A_{\{r+1,\dots,k\}}. \tag{2.6}$$

Indeed, the right hand side of (2.6) equals

$$\begin{aligned} &\frac{r(k-r+1)}{k}\, H_r \Big(\frac{1}{k-r+1}\, A_{\{r+1,\dots,k\}} - \frac{k-r}{k-r+1}\, A_{\{r+1,\dots,k\}} P_{r,r+1} A_{\{r+1,\dots,k\}}\Big) \\ &+ \frac{(r+1)(k-r)}{k} \Big(\frac{1}{r+1}\, H_r + \frac{r}{r+1}\, H_r P_{r,r+1} H_r\Big) A_{\{r+1,\dots,k\}}. \end{aligned}$$

Since H_r commutes with $A_{\{r+1,\dots,k\}}$, using the cyclic property of the supertrace we get

$$\begin{aligned} &\mathrm{str}_{1,\dots,k} H_r A_{\{r+1,\dots,k\}} P_{r,r+1} A_{\{r+1,\dots,k\}} Z_1 \dots Z_k \\ &\qquad = \mathrm{str}_{1,\dots,k} H_r P_{r,r+1} A_{\{r+1,\dots,k\}} Z_1 \dots Z_k A_{\{r+1,\dots,k\}}. \end{aligned}$$

Now apply the first relation of Proposition 2.1 to write this element as

$$\mathrm{str}_{1,\dots,k} H_r P_{r,r+1} A_{\{r+1,\dots,k\}} Z_1 \dots Z_k.$$

Similarly, using the second relation of Proposition 2.1 and the cyclic property of the supertrace, we get

$$\mathrm{str}_{1,\dots,k} H_r P_{r,r+1} H_r A_{\{r+1,\dots,k\}} Z_1 \dots Z_k = \mathrm{str}_{1,\dots,k} H_r P_{r,r+1} A_{\{r+1,\dots,k\}} Z_1 \dots Z_k$$

thus showing that the left hand side of (2.5) remains unchanged after the replacement (2.6). Then the telescoping sum in (2.5) vanishes and the proof is complete. □

Taking the particular cases $n = 0$ and $m = 0$ we thus get a new proof of the MacMahon Master Theorem for the right quantum algebra; cf. [9], [13], [24]. Note that some other noncommutative versions of the theorem associated with classical Lie algebras can be found e.g. in [28, Ch. 7].

2.2 Affine right quantum superalgebras

In accordance to the definition we gave in the Introduction, the affine right quantum superalgebra $\widehat{\mathcal{M}}_{m|n}$ is generated by the elements $z_{ij}^{(r)}$ of parity $\bar{\imath} + \bar{\jmath}$, where r runs over the set of positive integers and $1 \leqslant i, j \leqslant m+n$. The defining relations (1.10) can be written more explicitly as follows: for all positive integers p we have

$$\sum_{r+s=p} \Big([z_{ij}^{(r)}, z_{kl}^{(s)}] - [z_{kj}^{(r)}, z_{il}^{(s)}](-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}} \Big) = 0, \tag{2.7}$$

summed over nonnegative integers r and s, where we set $z_{ij}^{(0)} = \delta_{ij}$. Equivalently, in terms of the formal power series (1.11) they take the form

$$[z_{ij}(u), z_{kl}(u)] = [z_{kj}(u), z_{il}(u)](-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}}, \qquad i, j, k, l \in \{1, \dots, m+n\}. \tag{2.8}$$

We will keep the notation $z'_{ij}(u)$ for the entries of the inverse matrix $Z^{-1}(u)$.

Proposition 2.3. *The mapping*

$$z_{ij}(u) \mapsto \delta_{ij} + z_{ij}\, u \tag{2.9}$$

defines a surjective homomorphism $\widehat{\mathcal{M}}_{m|n} \to \mathcal{M}_{m|n}$. *Moreover, the mapping*

$$z_{ij} \mapsto z_{ij}^{(1)} \tag{2.10}$$

defines an embedding $\mathcal{M}_{m|n} \hookrightarrow \widehat{\mathcal{M}}_{m|n}$.

Proof. This follows easily from the defining relations of $\mathcal{M}_{m|n}$ and $\widehat{\mathcal{M}}_{m|n}$. The injectivity of the map (2.10) follows from the observation that the composition of (2.10) and (2.9) yields the identity map on $\mathcal{M}_{m|n}$. □

We will identify $\mathcal{M}_{m|n}$ with a subalgebra of $\widehat{\mathcal{M}}_{m|n}$ via the embedding (2.10).

Definition 2.4. The *Berezinian* of the matrix $Z(u)$ is a formal power series in u with coefficients in $\widehat{\mathcal{M}}_{m|n}$ defined by the formula

$$\begin{aligned} \operatorname{Ber} Z(u) = &\sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn} \sigma \cdot z_{\sigma(1)1}(u) \dots z_{\sigma(m)m}(u) \\ &\times \sum_{\tau \in \mathfrak{S}_n} \operatorname{sgn} \tau \cdot z'_{m+1,m+\tau(1)}(u) \dots z'_{m+n,m+\tau(n)}(u). \end{aligned} \tag{2.11}$$

The image of the Berezinian $\operatorname{Ber} Z(u)$ under the homomorphism (2.9) will be denoted by $\operatorname{Ber}(1 + uZ)$. This is a formal power series in u with coefficients in the right quantum superalgebra $\mathcal{M}_{m|n}$. □

An alternative formula for the Berezinian $\operatorname{Ber} Z(u)$ will be given in Corollary 2.14 below.

Our goal now is to derive a quasideterminant factorization of $\operatorname{Ber} Z(u)$ and then use it in the proof of the identities (1.13)–(1.15). We start by providing some symmetries of the superalgebra $\widehat{\mathcal{M}}_{m|n}$.

Lemma 2.5. *If Z is a Manin matrix, then for any positive integer r the following identity holds:*

$$(1 - P_{12}) \sum_{k+l=r} [Z_1^k, Z_2^l] = 0, \tag{2.12}$$

where k and l run over nonnegative integers.

Proof. We use induction on r. The identity is trivial for $r = 1$, while for $r = 2$ it is equivalent to the definition of a Manin matrix; see (1.9). Suppose that $r \geqslant 3$. Note first that using the induction hypothesis we get

$$(1 - P_{12}) Z_2 Z_1 \sum_{k+l=r-2} [Z_1^k, Z_2^l] = (1 - P_{12}) Z_1 Z_2 \sum_{k+l=r-2} [Z_1^k, Z_2^l]$$
$$= (1 - P_{12}) Z_1 Z_2 P_{12} \sum_{k+l=r-2} [Z_1^k, Z_2^l] = -(1 - P_{12}) Z_2 Z_1 \sum_{k+l=r-2} [Z_1^k, Z_2^l]$$

and so

$$(1 - P_{12}) Z_2 Z_1 \sum_{k+l=r} [Z_1^{k-1}, Z_2^{l-1}] = 0. \tag{2.13}$$

Now for $k \geqslant 1$ write

$$[Z_1^k, Z_2^l] = [Z_1, Z_2^l]\, Z_1^{k-1} + Z_1 [Z_1^{k-1}, Z_2^l].$$

If $l \geqslant 1$ then we also have

$$[Z_1, Z_2^l]\, Z_1^{k-1} = [Z_1, Z_2]\, Z_2^{l-1} Z_1^{k-1} + Z_2 [Z_1, Z_2^{l-1}]\, Z_1^{k-1}.$$

Therefore, due to (1.9) we obtain

$$(1 - P_{12}) \sum_{k+l=r} [Z_1^k, Z_2^l] = (1 - P_{12}) \sum_{k+l=r} \Big(Z_2 [Z_1, Z_2^{l-1}]\, Z_1^{k-1} + Z_1 [Z_1^{k-1}, Z_2^l] \Big),$$

where k and l run over positive integers. By the induction hypothesis,

$$(1 - P_{12}) \sum_{k+l=r} Z_1 [Z_1^{k-1}, Z_2^l] = (1 - P_{12}) Z_1 P_{12} \sum_{k+l=r} [Z_1^{k-1}, Z_2^l]$$
$$= -(1 - P_{12}) \sum_{k+l=r} Z_2 [Z_1^{k-1}, Z_2^l].$$

Now for $k \geqslant 2$ write

$$[Z_1^{k-1}, Z_2^l] = [Z_1, Z_2^l]\, Z_1^{k-2} + Z_1 [Z_1^{k-2}, Z_2^l].$$

Hence, taking appropriate restrictions on the summation indices k and l we get

$$\begin{aligned}&(1 - P_{12}) \sum_{k+l=r} [Z_1^k, Z_2^l] \\ &= (1 - P_{12}) \sum_{k+l=r} \Big(Z_2 [Z_1, Z_2^{l-1}]\, Z_1^{k-1} - Z_2 [Z_1, Z_2^l]\, Z_1^{k-2} - Z_2 Z_1 [Z_1^{k-2}, Z_2^l] \Big)\end{aligned}$$

which is zero by (2.13). □

Proposition 2.6. *The mapping*

$$\omega : Z(u) \mapsto Z^{-1}(-u) \tag{2.14}$$

defines an involutive automorphism of the superalgebra $\widehat{\mathcal{M}}_{m|n}$ which is identical on the subalgebra $\mathcal{M}_{m|n}$.

Proof. Set $\check{Z} = 1 - Z(-u)$ so that $\check{Z}$ is a formal power series in u without a constant term. Then $Z^{-1}(-u)$ can be written in the form

$$Z^{-1}(-u) = \sum_{k=0}^{\infty} \check{Z}^k.$$

Note that the right hand side is a well-defined power series in u with coefficients in $\widehat{\mathcal{M}}_{m|n}$. By Lemma 2.5, for any positive integer r we have

$$(1 - P_{12}) \sum_{k+l=r} [\check{Z}_1^k, \check{Z}_2^l] = 0,$$

which implies

$$(1 - P_{12})[Z_1^{-1}(-u), Z_2^{-1}(-u)] = 0$$

thus proving that ω is a homomorphism.

Applying ω to both sides of $\omega\big(Z(u)\big)\, Z(-u) = 1$ we get $\omega^2\big(Z(u)\big)\, Z^{-1}(u) = 1$, so that ω is an involutive automorphism of $\widehat{\mathcal{M}}_{m|n}$. The second statement is clear. □

We also introduce the *affine left quantum superalgebra* $\widehat{\mathcal{M}}^{\circ}_{m|n}$. It is generated by a countable set of elements $y_{ij}^{(r)}$ of parity $\bar{\imath} + \bar{\jmath}$, where r runs over the set of positive integers. The defining relations of $\widehat{\mathcal{M}}^{\circ}_{m|n}$ take the form

$$[Y_1(u), Y_2(u)]\,(1 - P_{12}) = 0, \tag{2.15}$$

where u is a formal variable, and the matrix elements of the matrix $Y(u) = [y_{ij}(u)]$ are the formal power series

$$y_{ij}(u) = \delta_{ij} + y_{ij}^{(1)}\, u + y_{ij}^{(2)}\, u^2 + \cdots . \tag{2.16}$$

The defining relations (2.15) can be written as

$$[y_{ij}(u), y_{kl}(u)] + [y_{kj}(u), y_{il}(u)](-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}} = 0, \qquad i,j,k,l \in \{1,\dots,m+n\}, \tag{2.17}$$

or, equivalently, for all positive integers p we have

$$\sum_{r+s=p} \Big([y_{ij}^{(r)}, y_{kl}^{(s)}] + [y_{kj}^{(r)}, y_{il}^{(s)}](-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}} \Big) = 0, \tag{2.18}$$

summed over nonnegative integers r and s, where we set $y_{ij}^{(0)} = \delta_{ij}$.

It is straightforward to verify that the superalgebra $\widehat{\mathcal{M}}^{\circ}_{m|n}$ is isomorphic to the affine right quantum superalgebra $\widehat{\mathcal{M}}_{m|n}$. An isomorphism can be given by the supertransposition map $Y(u) \mapsto Z(u)^t$ so that $y_{ij}(u) \mapsto z_{ji}(u)(-1)^{\bar{\imath}(\bar{\jmath}+1)}$.

Proposition 2.7. *The mapping*

$$\zeta : y_{ij}(u) \mapsto z'_{m+n-i+1,m+n-j+1}(u) \tag{2.19}$$

defines an isomorphism $\widehat{\mathcal{M}}^{\circ}_{n|m} \to \widehat{\mathcal{M}}_{m|n}$.

Proof. Observe that the mapping

$$y_{ij}(u) \mapsto z_{m+n-i+1,m+n-j+1}(-u)$$

defines an isomorphism $\widehat{\mathcal{M}}^{\circ}_{n|m} \to \widehat{\mathcal{M}}_{m|n}$, which follows easily from the defining relations of the affine left and right quantum superalgebras. It remains to note that ζ is the composition of this isomorphism and the automorphism ω defined in Proposition 2.6. □

We will now adapt the arguments used by Gow [16, 17] for the Yangian of the Lie superalgebra $\mathfrak{gl}_{m|n}$ to derive a quasideterminant decomposition of the Berezinian Ber $Z(u)$; see Definition 2.4.

If $A = [a_{ij}]$ is a square matrix over a ring with 1, then its *ij-th quasideterminant* is defined if A is invertible and the ji-th entry $(A^{-1})_{ji}$ is an invertible element of the ring. The ij-th quasideterminant is then given by

$$\begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1N} \\ & \dots & & \dots & \\ a_{i1} & \dots & \boxed{a_{ij}} & \dots & a_{iN} \\ & \dots & & \dots & \\ a_{N1} & \dots & a_{Nj} & \dots & a_{NN} \end{vmatrix} = \big((A^{-1})_{ji}\big)^{-1}.$$

We refer the reader to [14] and references therein for the properties and applications of quasideterminants.

We will need the Gauss decompositions of the matrices $Z(u)$ and $Y(u)$. There exist unique matrices $D(u)$, $E(u)$ and $F(u)$ whose entries are formal power series in u with coefficients in $\widehat{\mathcal{M}}_{m|n}$ such that $D(u) = \operatorname{diag}\big[d_1(u), \dots, d_{m+n}(u)\big]$ and

$$E(u) = \begin{bmatrix} 1 & e_{12}(u) & \dots & e_{1,m+n}(u) \\ 0 & 1 & \dots & e_{2,m+n}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \qquad F(u) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ f_{21}(u) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n,1}(u) & f_{m+n,2}(u) & \dots & 1 \end{bmatrix}$$

satisfying $Z(u) = F(u)\, D(u)\, E(u)$. Explicit formulas for the entries of the matrices $D(u)$, $E(u)$ and $F(u)$ can be given in terms of quasideterminants; see [14]. In particular,

$$d_i(u) = \begin{vmatrix} z_{11}(u) & \dots & z_{1,i-1}(u) & z_{1i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ z_{i-1,1}(u) & \dots & z_{i-1,i-1}(u) & z_{i-1,i}(u) \\ z_{i1}(u) & \dots & z_{i,i-1}(u) & \boxed{z_{ii}(u)} \end{vmatrix}, \tag{2.20}$$

for $i = 1, \dots, m+n$.

We write the Gauss decomposition of the matrix $Y(u)$ as $Y(u) = F^{\circ}(u)\, D^{\circ}(u)\, E^{\circ}(u)$ so that the corresponding entries $d_i^{\circ}(u), e_{ij}^{\circ}(u), f_{ij}^{\circ}(u)$ of the matrices $D^{\circ}(u)$, $E^{\circ}(u)$ and $F^{\circ}(u)$ are formal power series in u with coefficients in $\widehat{\mathcal{M}}^{\circ}_{m|n}$. These entries are found by the same formulas as above with the $z_{ij}(u)$ respectively replaced by $y_{ij}(u)$.

Lemma 2.8. *Under the isomorphism* $\zeta : \widehat{\mathcal{M}}^{\circ}_{n|m} \to \widehat{\mathcal{M}}_{m|n}$ *defined in* (2.19) *we have*

$$\zeta : d_k^{\circ}(u) \mapsto d_{m+n-k+1}(u)^{-1}, \qquad k = 1, \dots, n+m.$$

Proof. The entries of the inverse matrix $Z^{-1}(u)$ are found from the decomposition $Z^{-1}(u) = E^{-1}(u)\, D^{-1}(u)\, F^{-1}(u)$ so that

$$z'_{i'j'}(u) = \sum_{k \leqslant i,j} e'_{i'k'}(u)\, d_{k'}(u)^{-1}\, f'_{k'j'}(u), \tag{2.21}$$

where $e'_{ij}(u)$ and $f'_{ij}(u)$ denote the entries of the matrices $E^{-1}(u)$ and $F^{-1}(u)$, respectively, and we set $i' = m+n-i+1$. On the other hand, the entries of $Y(u)$ are found by

$$y_{ij}(u) = \sum_{k \leqslant i,j} f^{\circ}_{ik}(u)\, d^{\circ}_k(u)\, e^{\circ}_{kj}(u). \tag{2.22}$$

Hence, by (2.19) we have

$$\zeta(d_1^{\circ}(u)) = \zeta(y_{11}(u)) = z'_{1'1'}(u) = d_{1'}(u)^{-1}.$$

Now, comparing (2.21) and (2.22), and arguing by induction we find that

$$\zeta : f^{\circ}_{ik}(u) \mapsto e'_{i'k'}(u), \qquad e^{\circ}_{kj}(u) \mapsto f'_{k'j'}(u), \qquad d^{\circ}_{k}(u) \mapsto d_{k'}(u)^{-1},$$

as required. □

Theorem 2.9. *The Berezinian* Ber $Z(u)$ *admits the quasideterminant factorization in the superalgebra* $\widehat{\mathcal{M}}_{m|n}[[u]]$:

$$\operatorname{Ber} Z(u) = d_1(u) \dots d_m(u)\, d^{-1}_{m+1}(u) \dots d^{-1}_{m+n}(u).$$

Proof. The upper-left $m \times m$ submatrix of $Z(u)$ is a matrix with even entries satisfying (1.1). A quasideterminant decomposition of the corresponding determinant was obtained in [4, Lemma 8] and it takes the form

$$\sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn} \sigma \cdot z_{\sigma(1)1}(u) \dots z_{\sigma(m)m}(u) = d_1(u) \dots d_m(u). \tag{2.23}$$

The second factor on the right hand side of (2.11) is the image of the determinant

$$\sum_{\tau \in \mathfrak{S}_n} \operatorname{sgn} \tau \cdot y_{n,\tau(n)}(u) \dots y_{1,\tau(1)}(u) \in \widehat{\mathcal{M}}^{\circ}_{n|m}[[u]]$$

under the isomorphism (2.19). However, the upper-left $n \times n$ submatrix of $Y(u)$ is a matrix with even entries whose transpose satisfies (1.1). The corresponding quasideterminant decomposition is proved in the same way as (2.23) (see [4]), so that

$$\sum_{\tau \in \mathfrak{S}_n} \operatorname{sgn} \tau \cdot y_{n,\tau(n)}(u) \dots y_{1,\tau(1)}(u) = d^{\circ}_n(u) \dots d^{\circ}_1(u).$$

By Lemma 2.8, the image of this determinant under ζ is $d^{-1}_{m+1}(u) \dots d^{-1}_{m+n}(u)$. □

An alternative factorization of the Berezinian involving different quasideterminants is provided by Corollary 2.14 below.

Remark 2.10. The quantum Berezinian Ber $T(u)$ of the generator matrix of the Yangian for the Lie superalgebra $\mathfrak{gl}_{m|n}$ was introduced by Nazarov [31]. The quasideterminant decomposition of Ber $T(u)$ found by Gow [16, Theorem 1] can be obtained as a particular case of Theorem 2.9 by taking $Z(u) = e^{-\partial_u} T(u)$. The latter is a Manin matrix which follows easily from the defining relations of the Yangian; cf. [3, 4]. Hence, all identities for the Berezinian of the matrix $Z(u)$ obtained in this paper imply the corresponding counterparts for Ber $T(u)$. □

2.3 Berezinian identities

Consider the Berezinian $\mathrm{Ber}\,(1+uZ)$, where Z is a Manin matrix; see Definition 2.4. The expression $\mathrm{Ber}\,(1+uZ)$ is a formal power series in u with coefficients in the right quantum superalgebra $\mathcal{M}_{m|n}$. The next theorem provides some identities for the coefficients of this series, including a noncommutative analogue of the Newton identities (2.26); cf. [3], [23], [32]. We will use the matrix notation of Sec. 2.1. The subscripts like $\mathrm{str}_{1,\dots,k}$ of the supertrace will indicate that it is taken over the copies $1,2,\dots,k$ of the superalgebra $\mathrm{End}\,\mathbb{C}^{m|n}$ in (2.1).

Theorem 2.11. *We have the identities*

$$\mathrm{Ber}\,(1+uZ)=\sum_{k=0}^{\infty}u^k\,\mathrm{str}_{1,\dots,k}\,A_kZ_1\dots Z_k, \tag{2.24}$$

$$\big[\mathrm{Ber}\,(1-uZ)\big]^{-1}=\sum_{k=0}^{\infty}u^k\,\mathrm{str}_{1,\dots,k}\,H_kZ_1\dots Z_k, \tag{2.25}$$

$$\big[\mathrm{Ber}\,(1+uZ)\big]^{-1}\,\partial_u\,\mathrm{Ber}\,(1+uZ)=\sum_{k=0}^{\infty}(-u)^k\,\mathrm{str}\,Z^{k+1}. \tag{2.26}$$

Proof. Due to the MacMahon Master Theorem (Theorem 2.2), identities (2.24) and (2.25) are equivalent. Moreover, (2.24) is clear for $n=0$ as the Berezinian turns into a determinant. We will be proving (2.25) by induction on n, assuming that $n\geqslant 1$. Let $\widetilde{Z}$ be the matrix obtained from Z by deleting the row and column $m+n$. Set

$$h_k(Z)=\mathrm{str}_{1,\dots,k}\,H_kZ_1\dots Z_k=\mathrm{str}_{1,\dots,k}\,Z_1\dots Z_kH_k;$$

the second equality holds by the cyclic property of the supertrace. Applying Theorem 2.9 we derive that

$$\mathrm{Ber}\,(1-uZ)=\mathrm{Ber}\,(1-u\widetilde{Z})\big[(1-uZ)^{-1}\big]_{m+n,m+n}$$

and so

$$\big[(1-uZ)^{-1}\big]_{m+n,m+n}\big[\mathrm{Ber}\,(1-uZ)\big]^{-1}=\big[\mathrm{Ber}\,(1-u\widetilde{Z})\big]^{-1}.$$

Hence, (2.25) will follow if we show that

$$\big[(1-uZ)^{-1}\big]_{m+n,m+n}\cdot\sum_{k=0}^{\infty}u^k\,h_k(Z)=\sum_{k=0}^{\infty}u^k\,h_k(\widetilde{Z}),$$

or, equivalently, that for any $r\geqslant 1$ we have

$$\sum_{k+l=r}(Z^k)_{m+n,m+n}\cdot h_l(Z)=h_r(\widetilde{Z}). \tag{2.27}$$

In order to verify (2.27) we use a relation in the superalgebra (2.1),

$$Z_1^k = \mathrm{str}_{2,\dots,k} Z_1 \dots Z_k P_{k-1,k} \dots P_{23} P_{12}, \tag{2.28}$$

which follows easily by induction with the use of the relation $\mathrm{str}_2 Z_2 P_{12} = Z_1$. The next lemma is a Manin matrix version of the corresponding identities obtained in [21].

Lemma 2.12. *We have the identity*

$$\sum_{k=1}^{r} Z_1^k \, h_{r-k}(Z) = r \, \mathrm{str}_{2,\dots,r} Z_1 \dots Z_r H_r. \tag{2.29}$$

Proof. By (2.28), the left hand side can be written as

$$\sum_{k=1}^{r} \mathrm{str}_{2,\dots,r} Z_1 \dots Z_r H_{\{k+1,\dots,r\}} P_{k-1,k} \dots P_{23} P_{12}. \tag{2.30}$$

Write

$$H_{\{k+1,\dots,r\}} = (r-k+1)\, H_{\{k,\dots,r\}} - (r-k)\, H_{\{k+1,\dots,r\}} P_{k,k+1} H_{\{k+1,\dots,r\}}.$$

By the second relation of Proposition 2.1 and the cyclic property of the supertrace, we have

$$\begin{aligned} &\mathrm{str}_{2,\dots,r} Z_1 \dots Z_r H_{\{k+1,\dots,r\}} P_{k,k+1} H_{\{k+1,\dots,r\}} P_{k-1,k} \dots P_{23} P_{12} \\ &= \mathrm{str}_{2,\dots,r} H_{\{k+1,\dots,r\}} Z_1 \dots Z_r H_{\{k+1,\dots,r\}} P_{k,k+1} P_{k-1,k} \dots P_{23} P_{12} \\ &= \mathrm{str}_{2,\dots,r} Z_1 \dots Z_r H_{\{k+1,\dots,r\}} P_{k,k+1} \dots P_{23} P_{12}. \end{aligned}$$

Hence, (2.30) takes the form of a telescoping sum which simplifies to become the right hand side of (2.29). □

Taking into account Lemma 2.12, we can represent (2.27) in the equivalent form

$$\mathrm{str}_{1,\dots,r} Z_1 \dots Z_r H_r + r \left[\mathrm{str}_{2,\dots,r} Z_1 \dots Z_r H_r \right]_{m+n,m+n} = \mathrm{str}_{1,\dots,r} \widetilde{Z}_1 \dots \widetilde{Z}_r H_r. \tag{2.31}$$

Write

$$Z_1 \dots Z_r H_r = \sum e_{i_1 j_1} \otimes \dots \otimes e_{i_r j_r} \otimes z^{i_1,\dots,i_r}_{j_1,\dots,j_r},$$

summed over all indices $i_a, j_a \in \{1, \dots, m+n\}$, where the $z^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ are certain elements of the right quantum superalgebra $\mathcal{M}_{m|n}$. Then

$$\mathrm{str}_{1,\dots,r} Z_1 \dots Z_r H_r = \sum z^{i_1,\dots,i_r}_{i_1,\dots,i_r} (-1)^{\bar{\imath}_1 + \dots + \bar{\imath}_r}. \tag{2.32}$$

Due to the presence of the symmetrizer H_r, the index $m+n$ may occur at most once amongst the summation indices $i_1, \dots, i_r$. The sum in (2.32) over the indices restricted to $i_1, \dots, i_r \in \{1, \dots, m+n-1\}$ coincides with the expression $\mathrm{str}_{1,\dots,r} \widetilde{Z}_1 \dots \widetilde{Z}_r H_r$, while the sum over the multisets of indices containing $m+n$ equals $-r\,\big[\mathrm{str}_{2,\dots,r} Z_1 \dots Z_r H_r\big]_{m+n,m+n}$, thus proving (2.31). This completes the proof of (2.25).

Now we prove the Newton identity (2.26) which can be written in the equivalent form

$$-\partial_u \big[\mathrm{Ber}\,(1+uZ)\big]^{-1} = \sum_{k=0}^{\infty} (-u)^k \,\mathrm{str}\, Z^{k+1} \cdot \big[\mathrm{Ber}\,(1+uZ)\big]^{-1}.$$

Equating the coefficients of the same powers of u we can also write this as

$$\sum_{k=1}^{r} \big(\mathrm{str}\, Z^k\big)\, h_{r-k}(Z) = r\, h_r(Z), \qquad r = 1, 2, \dots .$$

However, this relation is immediate from Lemma 2.12 by taking the supertrace str_1 over the first copy of the superalgebra $\mathrm{End}\,\mathbb{C}^{m|n}$. □

Remark 2.13. In the case $n = 0$ we thus get a new proof of the Newton identities for Manin matrices based on Lemma 2.12; cf. [3]. This argument essentially follows [21]. □

Using Theorem 2.11 we can obtain alternative expressions for the Berezinians $\mathrm{Ber}\, Z(u)$ and $\mathrm{Ber}\,(1+uZ)$ and different quasideterminant factorizations; cf. Definition 2.4 and Theorem 2.9.

Corollary 2.14. *The following relations hold*

$$\begin{aligned}\big[\mathrm{Ber}\, Z(u)\big]^{-1} = & \sum_{\sigma\in\mathfrak{S}_m} \mathrm{sgn}\,\sigma \cdot z'_{\sigma(1)1}(u) \dots z'_{\sigma(m)m}(u) \\ & \times \sum_{\tau\in\mathfrak{S}_n} \mathrm{sgn}\,\tau \cdot z_{m+1,m+\tau(1)}(u) \dots z_{m+n,m+\tau(n)}(u)\end{aligned} \tag{2.33}$$

and

$$\big[\mathrm{Ber}\, Z(u)\big]^{-1} = \bar{d}_1^{-1}(u) \dots \bar{d}_m^{-1}(u)\, \bar{d}_{m+1}(u) \dots \bar{d}_{m+n}(u), \tag{2.34}$$

where $\bar{d}_i(u)$ is the quasideterminant

$$\bar{d}_i(u) = \begin{vmatrix} \boxed{z_{ii}(u)} & \dots & z_{i,m+n}(u) \\ \vdots & \ddots & \vdots \\ z_{m+n,i}(u) & \dots & z_{m+n,m+n}(u) \end{vmatrix}.$$

In particular, the corresponding relations hold for the Berezinian $\mathrm{Ber}\,(1+uZ)$.

Proof. The arguments are quite similar to those used in the proof of Theorems 2.9 and 2.11 so we only sketch the main steps. We show first that the right hand side of (2.33) coincides with the product of quasideterminants on the right hand side of (2.34). Using the same calculation as in [4, Lemma 8], we obtain the following quasideterminant decomposition

$$\sum_{\tau\in\mathfrak{S}_n} \operatorname{sgn}\tau \cdot z_{m+1,m+\tau(1)}(u)\dots z_{m+n,m+\tau(n)}(u) = \bar{d}_{m+1}(u)\dots\bar{d}_{m+n}(u).$$

Furthermore, the determinant

$$\sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \cdot z'_{\sigma(1)1}(u)\dots z'_{\sigma(m)m}(u)$$

coincides with the image of a certain determinant of a submatrix of $Y(u)$ under the isomorphism (2.19) which leads to the desired factorization via a natural dual analogue of Lemma 2.8.

Observe that the matrix $Z(u)$ can be written as $Z(u) = 1 + \widetilde{Z}(u)$ and $\widetilde{Z}(u)$ satisfies (1.10). It is therefore sufficient to verify relations (2.33) and (2.34) for matrices of the form $Z(u) = 1 + uZ$, where Z satisfies (1.9). Since we have verified that the right hand sides of (2.33) and (2.34) coincide, it is enough to show that (2.34) holds. We will use induction on m, assuming that $m \geqslant 1$. Let $\overline{Z}$ be the matrix obtained from Z by deleting the row and column 1. Then we need to verify that

$$\big[\operatorname{Ber}(1+uZ)\big]^{-1} = \big[(1+uZ)^{-1}\big]_{11}\big[\operatorname{Ber}(1+u\overline{Z})\big]^{-1},$$

or, equivalently,

$$\operatorname{Ber}(1+uZ)\cdot\big[(1+uZ)^{-1}\big]_{11} = \operatorname{Ber}(1+u\overline{Z}). \tag{2.35}$$

Set

$$\sigma_k(Z) = \operatorname{str}_{1,\dots,k} A_k Z_1\dots Z_k.$$

Due to (2.24), the relation (2.35) will follow if we show that for any $r \geqslant 1$

$$\sum_{k+l=r} \sigma_k(Z)\cdot(Z^l)_{11} = \sigma_r(\overline{Z}).$$

However, this follows by the same argument as for the proof of (2.27). □

3 Sugawara operators for $\widehat{\mathfrak{gl}}_{m|n}$

Consider the Lie superalgebra $\widehat{\mathfrak{gl}}_{m|n}\oplus\mathbb{C}\tau$ with its commutation relations (1.16) and (1.17). We assume that m and n are positive integers and $m \neq n$. Recall also that $V = V_\kappa(\mathfrak{gl}_{m|n})$ is the affine vertex algebra at the level $\kappa\in\mathbb{C}$. This means that V is equipped with the additional data $(Y, D, 1)$, where $1\in V$ is the vacuum vector, the state-field correspondence Y is a map

$$Y : V \to \operatorname{End} V[[z, z^{-1}]],$$

the infinitesimal translation D is an operator $D : V \to V$. These data satisfy the vertex algebra axioms; see e.g. [12], [22]. For $a \in V$ we write

$$Y(a,z) = \sum_{r\in\mathbb{Z}} a_{(r)} z^{-r-1}, \quad a_{(r)} \in \operatorname{End} V.$$

In particular, for all $a, b \in V$ we have $a_{(r)}\, b = 0$ for $r \gg 0$. The span in $\operatorname{End} V$ of all *Fourier coefficients* $a_{(r)}$ of all vertex operators $Y(a,z)$ is a Lie superalgebra $\mathrm{U}_\kappa(\widehat{\mathfrak{gl}}_{m|n})_{\mathrm{loc}}$ with the super-commutator

$$[a_{(r)}, b_{(s)}] = \sum_{k\geqslant 0} \binom{r}{k} \big(a_{(k)}\, b\big)_{(r+s-k)}, \tag{3.1}$$

which is called the *local completion* of the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{gl}}_{m|n})$ by the ideal generated by $K - \kappa$; see [12, Sec. 3.5].

The translation operator is determined by

$$D : 1 \mapsto 0 \qquad \text{and} \qquad \big[D, e_{ij}[r]\big] = -r\, e_{ij}[r-1] \tag{3.2}$$

so that $D = \operatorname{ad} \tau$. The state-field correspondence Y is defined by setting $Y(1,z) = \mathrm{id}$,

$$Y(e_{ij}[-1], z) = e_{ij}(z) := \sum_{r\in\mathbb{Z}} e_{ij}[r]\, z^{-r-1}, \tag{3.3}$$

and then extending the map to the whole of V with the use of *normal ordering*. Namely, the normally ordered product of homogeneous fields

$$a(z) = \sum_{r\in\mathbb{Z}} a_{(r)} z^{-r-1} \qquad \text{and} \qquad b(w) = \sum_{r\in\mathbb{Z}} b_{(r)} w^{-r-1}$$

is the formal power series

$$:a(z)b(w): = a(z)_+ b(w) + (-1)^{\deg a\, \deg b}\, b(w)\, a(z)_-, \tag{3.4}$$

where

$$a(z)_+ = \sum_{r<0} a_{(r)} z^{-r-1} \qquad \text{and} \qquad a(z)_- = \sum_{r\geqslant 0} a_{(r)} z^{-r-1}.$$

This definition extends to an arbitrary number of fields with the convention that the normal ordering is read from right to left. Then

$$Y(e_{i_1 j_1}[-r_1-1] \dots e_{i_m j_m}[-r_m-1], z) = \frac{1}{r_1! \dots r_m!} :\partial_z^{r_1} e_{i_1 j_1}(z) \dots \partial_z^{r_m} e_{i_m j_m}(z): .$$

The *center* of the vertex algebra $V_{n-m}(\mathfrak{gl}_{m|n})$ at the critical level $\kappa = n - m$ is

$$\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n}) = \{ b \in V_{n-m}(\mathfrak{gl}_{m|n}) \mid \mathfrak{gl}_{m|n}[t]\, b = 0 \}$$

which can be identified with a commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$. Elements of $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ are called *Segal–Sugawara vectors.* Due to the commutator formula (3.1), if $b \in \mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$, then all Fourier coefficients of the corresponding field $b(z) = Y(b,z)$ belong to the center of the Lie superalgebra $\mathrm{U}_\kappa(\widehat{\mathfrak{gl}}_{m|n})_{\mathrm{loc}}$. These Fourier coefficients are called the *Sugawara operators* for $\widehat{\mathfrak{gl}}_{m|n}$. In particular, they commute with the elements of $\widehat{\mathfrak{gl}}_{m|n}$ and thus form a commuting family of $\widehat{\mathfrak{gl}}_{m|n}$-endomorphisms of Verma modules over $\widehat{\mathfrak{gl}}_{m|n}$ at the critical level; cf. [7], [15], [20]. We will apply the results of Sec. 2 to construct several families of Sugawara operators for $\widehat{\mathfrak{gl}}_{m|n}$.

3.1 Segal–Sugawara vectors

Consider the square matrix

$$\tau + \widehat{E}[-1] = \big[\delta_{ij}\tau + e_{ij}[-1](-1)^{\bar\imath}\big] \tag{3.5}$$

with the entries in the universal enveloping algebra for $\widehat{\mathfrak{gl}}_{m|n} \oplus \mathbb{C}\tau$. The following observation will play a key role in what follows.

Lemma 3.1. *The matrix $\tau + \widehat{E}[-1]$ is a Manin matrix.*

Proof. We have

$$\begin{aligned}
&\big[\delta_{ij}\tau + e_{ij}[-1](-1)^{\bar\imath}, \delta_{kl}\tau + e_{kl}[-1](-1)^{\bar k}\big]\\
&\qquad = \delta_{ij}\, e_{kl}[-2](-1)^{\bar k} - \delta_{kl}\, e_{ij}[-2](-1)^{\bar\imath}\\
&\qquad + \delta_{kj}\, e_{il}[-2](-1)^{\bar\imath+\bar k} - \delta_{il}\, e_{kj}[-2](-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)+\bar\imath+\bar k}.
\end{aligned}$$

This expression remains unchanged after swapping i and k and multiplying by $(-1)^{\bar\imath\bar\jmath+\bar\imath\bar k+\bar\jmath\bar k}$. Thus, the matrix elements of $\tau + \widehat{E}[-1]$ satisfy (1.2). ☐

Theorem 3.2. *For any $k \geqslant 0$ all coefficients s_{kl} in the expansion*

$$\mathrm{str}(\tau + \widehat{E}[-1])^k = s_{k0}\,\tau^k + s_{k1}\,\tau^{k-1} + \dots + s_{kk}$$

are Segal–Sugawara vectors.

Proof. It is sufficient to verify that for all i, j

$$e_{ij}[0]\,\mathrm{str}(\tau + \widehat{E}[-1])^k = e_{ij}[1]\,\mathrm{str}(\tau + \widehat{E}[-1])^k = 0 \tag{3.6}$$

in the $\widehat{\mathfrak{gl}}_{m|n}$-module $V_{n-m}(\mathfrak{gl}_{m|n}) \otimes \mathbb{C}[\tau]$. We will employ matrix notation of Sec. 2.1 and consider the tensor product superalgebra

$$\mathrm{End}\,\mathbb{C}^{m|n} \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^{m|n} \otimes \mathrm{U}$$

with $k+1$ copies of $\operatorname{End}\mathbb{C}^{m|n}$ labeled by $0,1,\dots,k$, where U stands for the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{gl}}_{m|n}\oplus\mathbb{C}\tau)$. Set $T=\tau+\widehat{E}[-1]$ and for any integer r introduce the matrix $\widehat{E}[r]=\big[e_{ij}[r](-1)^{\bar{i}}\big]$. Relations (3.6) can now be written in the equivalent form

$$\widehat{E}_0[0]\operatorname{str}_1 T_1^k=0\qquad\text{and}\qquad\widehat{E}_0[1]\operatorname{str}_1 T_1^k=0 \tag{3.7}$$

modulo the left ideal of U generated by $\mathfrak{gl}_{m|n}[t]$ and $K+m-n$. In order to verity them, note that by the commutation relations in the Lie superalgebra $\widehat{\mathfrak{gl}}_{m|n}\oplus\mathbb{C}\tau$ we have

$$[\widehat{E}_0[0],T_1]=P_{01}T_1-T_1P_{01}, \tag{3.8}$$

$$[\widehat{E}_0[1],T_1]=\widehat{E}_0[0]+P_{01}\widehat{E}_1[0]-\widehat{E}_1[0]P_{01}+KP_{01}-\frac{K}{m-n}. \tag{3.9}$$

The following identity is well-known:

$$[\widehat{E}_0[0],T_1^k]=P_{01}T_1^k-T_1^kP_{01},\qquad k=0,1,2,\dots. \tag{3.10}$$

It follows immediately from (3.8):

$$[\widehat{E}_0[0],T_1^k]=\sum_{r=1}^{k}T_1^{r-1}[\widehat{E}_0[0],T_1]T_1^{k-r}=\sum_{r=1}^{k}T_1^{r-1}[P_{01},T_1]T_1^{k-r}=[P_{01},T_1^k].$$

Now the first relation in (3.7) follows by taking the supertrace str_1 on both sides of (3.10).

For the proof of the second relation in (3.7) use (3.9) to write

$$[\widehat{E}_0[1],T_1^k]=\sum_{i=1}^{k}T_1^{i-1}\Big(\widehat{E}_0[0]+P_{01}\widehat{E}_1[0]-\widehat{E}_1[0]P_{01}+KP_{01}-\frac{K}{m-n}\Big)T_1^{k-i}. \tag{3.11}$$

Applying (3.10) and the relation $\operatorname{str}_2 T_2^{k-i}P_{12}=T_1^{k-i}$ we can rewrite (3.11) modulo the left ideal of U generated by $\mathfrak{gl}_{m|n}[t]$ as

$$[\widehat{E}_0[1],T_1^k]=\sum_{i=1}^{k}\Big(T_1^{i-1}[P_{01},T_1^{k-i}]+KT_1^{i-1}P_{01}T_1^{k-i}\Big)-\frac{kK}{m-n}T_1^{k-1}$$
$$+\operatorname{str}_2\sum_{i=1}^{k}T_1^{i-1}\big(P_{01}\widehat{E}_1[0]-\widehat{E}_1[0]P_{01}\big)T_2^{k-i}P_{12}.$$

Now transform the last summand using (3.10) to get

$$\operatorname{str}_2\sum_{i=1}^{k}T_1^{i-1}\big(P_{01}\widehat{E}_1[0]-\widehat{E}_1[0]P_{01}\big)T_2^{k-i}P_{12}$$
$$=\operatorname{str}_2\sum_{i=1}^{k}T_1^{i-1}P_{01}[P_{12},T_2^{k-i}]P_{12}-\operatorname{str}_2\sum_{i=1}^{k}T_1^{i-1}[P_{12},T_2^{k-i}]P_{01}P_{12}.$$

Taking into account the relation $\mathrm{str}_2 P_{02} = 1$, we can simplify this to

$$\sum_{i=1}^{k-1} \Big((m-n) T_1^{i-1} P_{01} T_1^{k-i} - T_1^{i-1} P_{01}\, \mathrm{str}\, T^{k-i} + T_1^{i-1} T_0^{k-i} \Big) - (k-1) T_1^{k-1}.$$

Combining all the terms and taking the supertrace str_1 in (3.11) we derive

$$\begin{aligned}[]
[\widehat{E}_0[1], \mathrm{str}_1 T_1^k] = (K+m-n)\Big(T_0^{k-1} - \frac{k}{m-n} \mathrm{str}\, T^{k-1} \Big) + (K+m-n-k+2)\, T_0^{k-1} \\
+ (K+m-n+1)\, \mathrm{str}_1 \sum_{i=2}^{k-1} P_{01} T_0^{i-1} T_1^{k-i} - \mathrm{str}_1 \sum_{i=2}^{k-1} [T_0^{i-1}, T_1^{k-i}].
\end{aligned}$$

Finally, we use Lemmas 2.5 and 3.1 to write

$$\sum_{i=2}^{k-1} [T_0^{i-1}, T_1^{k-i}] = \sum_{i=2}^{k-1} P_{01} [T_0^{i-1}, T_1^{k-i}]$$

which brings the previous formula to the form

$$[\widehat{E}_0[1], \mathrm{str}_1 T_1^k] = (K+m-n)\Big(2T_0^{k-1} - \frac{k}{m-n} \mathrm{str}\, T^{k-1} + \mathrm{str}_1 \sum_{i=2}^{k-1} P_{01} T_0^{i-1} T_1^{k-i} \Big).$$

This expression vanishes in the vacuum module $V_{n-m}(\mathfrak{gl}_{m|n})$ as $K+m-n=0$. □

The following corollary is immediate from Theorems 2.11 and 3.2.

Corollary 3.3. *All the coefficients* $\sigma_{k\,l}, h_{k\,l}, b_{k\,l} \in \mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$ *in the expansions*

$$\begin{aligned}
\mathrm{str}_{1,\dots,k}\, A_k T_1 \dots T_k &= \sigma_{k\,0} \tau^k + \sigma_{k1} \tau^{k-1} + \dots + \sigma_{kk}, \\
\mathrm{str}_{1,\dots,k}\, H_k T_1 \dots T_k &= h_{k\,0} \tau^k + h_{k1} \tau^{k-1} + \dots + h_{kk}, \\
\mathrm{Ber}\, \big(1 + uT\big) &= \sum_{k=0}^{\infty} \sum_{l=0}^{k} b_{k\,l}\, u^k \tau^{k-l}
\end{aligned}$$

are Segal–Sugawara vectors. Moreover, $b_{k\,l} = \sigma_{k\,l}$ *for all* k *and* l. □

Remark 3.4. Denote by $\bar{s}_{k\,k}$, $\bar{h}_{k\,k}$ and $\bar{b}_{k\,k}$ the respective images of the elements $s_{k\,k}$, $h_{k\,k}$ and $b_{k\,k}$ in the associated graded algebra $\mathrm{gr}\, \mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}]) \cong \mathrm{S}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$. Each of the families $\bar{s}_{k\,k}$, $\bar{h}_{k\,k}$ and $\bar{b}_{k\,k}$ with $k \geqslant 1$ is the image of a generating set of the algebra of invariants $\mathrm{S}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_{m|n}}$ under the embedding $\mathrm{S}(\mathfrak{gl}_{m|n}) \hookrightarrow \mathrm{S}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$ defined by the assignment $X \mapsto X[-1]$. In the case $n=0$ the families of Segal–Sugawara vectors with such a property are known as *complete sets of Segal–Sugawara vectors* (see [5], [7], [15], [20]) so that this terminology can be extended to the super case. It is natural to suppose that each of the families $D^r s_{k\,k}$, $D^r h_{k\,k}$ and $D^r b_{k\,k}$ with $r \geqslant 0$ and $k \geqslant 1$ generates the center $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ of the vertex algebra $V_{n-m}(\mathfrak{gl}_{m|n})$; cf. [11]. However, if both m and n are positive integers then none of the families is algebraically independent. □

The application of the state-field correspondence map Y to the Segal–Sugawara vectors produces elements of the center of the local completion $\mathrm{U}_{n-m}(\widehat{\mathfrak{gl}}_{m|n})_{\mathrm{loc}}$ at the critical level $\kappa = n - m$. Hence, Theorem 3.2 and Corollary 3.3 provide explicit formulas for the corresponding Sugawara operators for $\widehat{\mathfrak{gl}}_{m|n}$.

Recall the matrix $\widehat{E}(z) = [e_{ij}(z)(-1)^{\bar{\imath}}]$, where the fields $e_{ij}(z)$ are defined in (3.3). Set $T(z) = \partial_z + \widehat{E}(z)$, where ∂_z is understood as a scalar matrix of size $m + n$.

Corollary 3.5. *All Fourier coefficients of the fields $s_{kl}(z)$, $b_{kl}(z)$, $\sigma_{kl}(z)$ and $h_{kl}(z)$ defined by the decompositions*

$$\begin{aligned}
:\mathrm{str}\, T(z)^k: \ &= s_{k0}(z)\,\partial_z^k + s_{k1}(z)\,\partial_z^{k-1} + \dots + s_{kk}(z),\\
:\mathrm{Ber}\,\big(1 + u\,T(z)\big): \ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} b_{kl}(z)\,u^k\partial_z^{k-l},\\
:\mathrm{str}_{1,\dots,k}\, A_k T_1(z)\dots T_k(z): \ &= \sigma_{k0}(z)\,\partial_z^k + \sigma_{k1}(z)\,\partial_z^{k-1} + \dots + \sigma_{kk}(z),\\
:\mathrm{str}_{1,\dots,k}\, H_k T_1(z)\dots T_k(z): \ &= h_{k0}(z)\,\partial_z^k + h_{k1}(z)\,\partial_z^{k-1} + \dots + h_{kk}(z)
\end{aligned}$$

are Sugawara operators for $\widehat{\mathfrak{gl}}_{m|n}$. Moreover, $s_{kl}(z) = b_{kl}(z)$ for all k and l. □

3.2 Commutative subalgebras and super Gaudin Hamiltonians

By the vacuum axiom of a vertex algebra, the application of any of the fields $s_{kl}(z)$, $b_{kl}(z)$, $\sigma_{kl}(z)$ and $h_{kl}(z)$ introduced in Corollary 3.5 to the vacuum vector yields formal power series in z with coefficients in the universal enveloping algebra $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$. Moreover, since the Fourier coefficients of the fields belong to the center of the local completion $\mathrm{U}_{n-m}(\widehat{\mathfrak{gl}}_{m|n})_{\mathrm{loc}}$, all coefficients of the power series will belong to the center $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ of the vertex algebra $V_{n-m}(\mathfrak{gl}_{m|n})$. In particular, these coefficients generate a commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$. Explicitly, they can be given by the same expansions as in Corollary 3.5 by omitting the normal ordering signs and by replacing $T(z)$ with the matrix $T(z)_+ = \partial_z + \widehat{E}(z)_+$ with $\widehat{E}(z)_+ = [(-1)^{\bar{\imath}} e_{ij}(z)_+]$ and

$$e_{ij}(z)_+ = \sum_{r=0}^{\infty} e_{ij}[-r-1]\,z^r. \tag{3.12}$$

Now we produce the corresponding families of commuting elements of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$. The commutation relations of the Lie superalgebra $t^{-1}\mathfrak{gl}_{m|n}[t^{-1}]$ can be written in terms of the series (3.12) as

$$\begin{aligned}
(z-w)\,[e_{ij}(z)_+, e_{kl}(w)_+] = \delta_{kj}\big(e_{il}(z)_+ - e_{il}(w)_+\big)&\\
- \delta_{il}\big(e_{kj}(z)_+ - e_{kj}(w)_+\big)(-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}&.
\end{aligned} \tag{3.13}$$

Setting

$$e_{ij}(z)_- = \sum_{r=0}^{\infty} e_{ij}[r]\, z^{-r-1},$$

we find that

$$\begin{aligned}(z-w)\,[e_{ij}(z)_-, e_{kl}(w)_-] = {} & -\delta_{kj}\big(e_{il}(z)_- - e_{il}(w)_-\big) \\ & + \delta_{il}\big(e_{kj}(z)_- - e_{kj}(w)_-\big)(-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)}. \end{aligned} \tag{3.14}$$

The commuting families of elements of $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$ obtained by the application of Corollary 3.5 are expressed as coefficients of certain differential polynomials in the series $e_{ij}(z)_+$ (i.e., polynomials in $\partial_z^k e_{ij}(z)_+$). Therefore, comparing (3.13) and (3.14), we can conclude that the same differential polynomials with the $e_{ij}(z)_+$ respectively replaced by $-e_{ij}(z)_-$, generate a commutative subalgebra of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$. Indeed, suppose that $A(z)$ is a differential polynomial in the $e_{ij}(z)_+$ and $B(w)$ is a differential polynomial in the $e_{ij}(w)_+$. A relation of the form $[A(z), B(w)] = 0$ is a consequence of the commutation relations between the series (3.13) where the actual expansions of the $e_{ij}(z)_+$ and $e_{ij}(w)_+$ as power series in z and w are not used. This relies on the easily verified property that given any total ordering on the set of series $\partial_z^k e_{ij}(z)_+$, the corresponding ordered monomials are linearly independent over the polynomial ring $\mathbb{C}[z]$. Thus we arrive at the following corollary, where we use the notation

$$L(z) = \partial_z - \widehat{E}(z)_-, \qquad \widehat{E}(z)_- = [(-1)^{\bar\imath} e_{ij}(z)_-].$$

Corollary 3.6. *All coefficients of the series $S_{kl}(z)$, $B_{kl}(z)$, $\Sigma_{kl}(z)$ and $H_{kl}(z)$ defined by the decompositions*

$$\begin{aligned} \operatorname{str} L(z)^k &= S_{k0}(z)\,\partial_z^k + S_{k1}(z)\,\partial_z^{k-1} + \dots + S_{kk}(z), \\ \operatorname{Ber}\big(1 + u\,L(z)\big) &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} B_{kl}(z)\, u^k \partial_z^{k-l}, \\ \operatorname{str}_{1,\dots,k} A_k L_1(z)\dots L_k(z) &= \Sigma_{k0}(z)\,\partial_z^k + \Sigma_{k1}(z)\,\partial_z^{k-1} + \dots + \Sigma_{kk}(z), \\ \operatorname{str}_{1,\dots,k} H_k L_1(z)\dots L_k(z) &= H_{k0}(z)\,\partial_z^k + H_{k1}(z)\,\partial_z^{k-1} + \dots + H_{kk}(z) \end{aligned}$$

generate a commutative subalgebra of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$. *Moreover,* $S_{kl}(z) = B_{kl}(z)$ *for all* k, l. □

Note also that given any set of complex parameters $\lambda_1, \dots, \lambda_{m+n}$ the commutation relations (3.14) remain valid after the replacement

$$e_{ij}(z)_- \mapsto \delta_{ij}\lambda_i + e_{ij}(z)_- \tag{3.15}$$

Thus, we obtain more general families of commutative subalgebras of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$.

Corollary 3.7. *Given a set of complex parameters $\lambda_1, \dots, \lambda_{m+n}$, all coefficients of the series $S_{kl}(z)$, $B_{kl}(z)$, $\Sigma_{kl}(z)$ and $H_{kl}(z)$ defined by the decompositions of Corollary 3.6 with the series $e_{ij}(z)_-$ replaced as in* (3.15) *generate a commutative subalgebra of* $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$. $\square$

Remark 3.8. (i) A more general replacement $e_{ij}(z)_- \mapsto K_{ij} + e_{ij}(z)_-$ with appropriately defined elements K_{ij} preserves the commutation relations (3.14) as well, thus leading to even more general family of commutative subalgebras of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$.

(ii) In the particular case $n = 0$ the elements of the commutative subalgebras of $\mathrm{U}(\mathfrak{gl}_m[t])$ from Corollaries 3.6 and 3.7 were originally constructed in [33]; see also [5], [6], [29], [30]. In particular, the Berezinian turns into a determinant, and generators of the commutative subalgebra are found by the decomposition of the column determinant

$$\mathrm{cdet}\begin{bmatrix} \partial_z - e_{11}(z)_- & -e_{12}(z)_- & \dots & -e_{1m}(z)_- \\ e_{21}(z)_- & \partial_z - e_{22}(z)_- & \dots & -e_{2m}(z)_- \\ \vdots & \vdots & \ddots & \vdots \\ -e_{m1}(z)_- & -e_{m2}(z)_- & \dots & \partial_z - e_{mm}(z)_- \end{bmatrix}.$$

The commutative subalgebras of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$ can be used to construct higher Gaudin Hamiltonians following the same scheme as in the even case; see [8], [29], [33]. More precisely, given a finite-dimensional $\mathfrak{gl}_{m|n}$-module M and a complex number a we can define the corresponding evaluation $\mathfrak{gl}_{m|n}[t]$-module M_a. As a vector superspace, M_a coincides with M, while the action of the elements of the Lie superalgebra is given by $e_{ij}[r] \mapsto e_{ij}\, a^r$ for $r \geqslant 0$, or equivalently,

$$e_{ij}(z)_- \mapsto \frac{e_{ij}}{z - a}.$$

Now consider certain finite-dimensional $\mathfrak{gl}_{m|n}$-modules $M^{(1)}, \dots, M^{(k)}$ and let $a_1, \dots, a_k$ be complex parameters. Then the tensor product of the evaluation modules

$$M^{(1)}_{a_1} \otimes \dots \otimes M^{(k)}_{a_k} \tag{3.16}$$

becomes a $\mathfrak{gl}_{m|n}[t]$-module via the iterated comultiplication map (1.21). Then the images of the matrix elements of the matrix $L(z)$ are found by

$$\ell_{ij}(z) = \delta_{ij}\partial_z - (-1)^{\bar{\imath}} \sum_{r=1}^{k} \frac{e^{(r)}_{ij}}{z - a_r},$$

where $e^{(r)}_{ij}$ denotes the image of e_{ij} in the $\mathfrak{gl}_{m|n}$-module $M^{(r)}$. So replacing $L(z)$ by the matrix $\mathcal{L}(z) = [\ell_{ij}(z)]$ in the formulas of Corollary 3.6 we obtain a family of commuting operators in the module (3.16), thus producing higher Gaudin Hamiltonians associated with $\mathfrak{gl}_{m|n}$. In particular, such families are provided by the coefficients of the series defined by the expansion of the Berezinian $\mathrm{Ber}\big(1 + u\,\mathcal{L}(z)\big)$ and the supertrace

$$\mathrm{str}\,\mathcal{L}(z)^k = \mathcal{S}_{k0}(z)\,\partial_z^k + \mathcal{S}_{k1}(z)\,\partial_z^{k-1} + \dots + \mathcal{S}_{kk}(z).$$

The quadratic Gaudin Hamiltonian $\mathcal{H}(z) = \mathcal{S}_{22}(z)$ can be written explicitly as

$$\mathcal{H}(z) = \sum_{r,s=1}^{k} \frac{1}{(z-a_r)(z-a_s)} \sum_{i,j} e_{ij}^{(r)} e_{ji}^{(s)} (-1)^{\bar{j}} + \sum_{r=1}^{k} \frac{1}{(z-a_r)^2} \sum_{i} e_{ii}^{(r)}.$$

Assuming further that the parameters a_i are all distinct and setting

$$\mathcal{H}^{(r)} = \sum_{s\neq r} \frac{1}{a_r - a_s} \sum_{i,j} e_{ij}^{(r)} e_{ji}^{(s)} (-1)^{\bar{j}}$$

we can also write the Hamiltonian as

$$\mathcal{H}(z) = 2 \sum_{r=1}^{k} \frac{\mathcal{H}^{(r)}}{z-a_r} + \sum_{r=1}^{k} \frac{\Delta^{(r)}}{(z-a_r)^2},$$

where $\Delta^{(r)}$ denotes the eigenvalue of the Casimir element $\sum e_{ij}e_{ji}(-1)^{\bar{j}} + \sum e_{ii}$ of $\mathfrak{gl}_{m|n}$ in the representation $M^{(r)}$; cf. [8], [25].

More general families of commuting elements in $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$ and higher Gaudin Hamiltonians can be constructed by using extra parameters λ_i or K_{ij} as in Corollary 3.7 and Remark 3.8(i).

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