THE LINKING PAIRINGS OF ORIENTABLE SEIFERT MANIFOLDS

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Abstract. We compute the p-primary components of the linking pairings of orientable 3-manifolds admitting a fixed-point free S^1-action. Using this, we show that any nonsingular linking pairing on a finite abelian group with homogeneous 2-primary summand is realized by such a manifold. However there are pairings on inhomogeneous 2-groups which are not realizable.

In [3] we computed the linking pairings of oriented 3-manifolds which are Seifert fibred over non-orientable base orbifolds. Here we shall consider the remaining case, when the base orbifold is also orientable. Thus the Seifert fibration is induced by a fixed-point free S^1-action on the manifold. (We shall henceforth call such a space a "Seifert manifold", for brevity.) We give presentations for the localization of the torsion at a prime p in §2, and use these to give explicit formulae for the localized linking pairings in §3. We then study the cases p odd and $p = 2$ separately, in §§3-6 and §§7-8, respectively. Every nonsingular pairing on a finite abelian group whose 2-primary subgroup is isomorphic to $(Z/2^kZ)^m$ (for some k, m) is the linking pairing of a Seifert manifold with geometry $\mathbb{H}^2 \times \mathbb{E}^1$, and also of one with geometry $\widetilde{\mathbb{SL}}$. However if the 2-primary subgroup has exponent 2^k but is inhomogeneous the restrictions of the pairing to direct summands of exponent properly dividing 2^k must be odd. The final section §9 summarizes briefly the earlier work of Oh on the Witt classes of such pairings [6].

1. Bilinear pairings

A *linking pairing* on a finite abelian group N is a symmetric bilinear function $\ell : N \times N \to \mathbb{Q}/\mathbb{Z}$ which is nonsingular in the sense that $\tilde{\ell} : n \mapsto \ell(-, n)$ defines an isomorphism from N to $Hom(N, \mathbb{Q}/\mathbb{Z})$. If L is a subgroup of N then $\tilde{\ell}$ induces an isomorphism $L^\perp = \{t \in N \mid \ell_M(t, l) = 0 \ \forall l \in L\} \cong N/L$. Such a pairing splits uniquely as the orthogonal sum (over primes p) of its restrictions to the p-primary

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subgroups of N. It is *metabolic* if there is a subgroup P with $P = P^{\perp}$, *split* [5] if also P is a direct summand and *hyperbolic* if N is the direct sum of two such subgroups. If ℓ is split N is a direct double. A linking pairing ℓ is *even* if $2^{k-1}\ell(x,x) \in \mathbb{Z}$ for all $x \in N$ such that $2^k x = 0$. Hyperbolic pairings are even. We shall say that ℓ is *odd* if it is not even.

If $w = \frac{p}{q} \in \mathbb{Q}^{\times}$ (where $(p,q) = 1$) let ℓ_w be the pairing on Z/qZ given by $\ell_w(m,n) = [mnw] \in \mathbb{Q}/\mathbb{Z}$. Then $\ell_w \cong \ell_{w'}$ if and only if $w' = n^2 w$ for some integer n with $(n,q) = 1$. In particular, if $q = 2^k$ then $\ell_w \cong \ell_{w'}$ if and only if $2^k w' \equiv 2^k w \ mod\ (2^k, 8)$. Every linking pairing on an abelian group of odd order is an orthogonal sum of pairings on cyclic groups. However, if the order is even we need also pairings on the groups $(Z/2^kZ)^2$. (See [5, 7]. Our strategy shall be to localize at a prime p, and we shall consider appropriate invariants later.

If M is a closed oriented 3-manifold Poincaré duality determines a linking pairing $\ell_M : T(M) \times T(M) \to \mathbb{Q}/\mathbb{Z}$, which may be described as follows. Let w, z be disjoint 1-cycles representing elements of $T(M)$ and suppose that $mz = \partial C$ for some 2-chain C which is transverse to w and some nonzero $m \in \mathbb{Z}$. Then $\ell_M([w],[z]) = (w \bullet C)/m \in \mathbb{Q}/\mathbb{Z}$. It follows easily from the Mayer-Vietoris theorem and duality that if M embeds in $\mathbb{R}^4$ then ℓ_M is hyperbolic. (If X and Y are the closures of the components of $\mathbb{R}^4 - M$ and T_X and T_Y are the kernels of the induced homomorphisms from $T(M)$ to $H_1(X;\mathbb{Z})$ and $H_1(Y;\mathbb{Z})$ (respectively) then $T(M) \cong T_X \oplus T_Y$ and the restriction of ℓ_M to each of these summands is trivial [5]).

The linking pairing has a dual formulation, in terms of cohomology. Let $\beta_{\mathbb{Q}/\mathbb{Z}} : H^1(M;\mathbb{Q}/\mathbb{Z}) \to H^2(M;\mathbb{Z})$ be the Bockstein homomorphism associated with the coefficient sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

and let $D : H_1(M;\mathbb{Z}) \to H^2(M;\mathbb{Z})$ be the Poincaré duality isomorphism. Then ℓ_M may be given by the equation

$$\ell(w,z) = (D(w) \cup \beta_{\mathbb{Q}/\mathbb{Z}}^{-1} D(z))([M]) \in \mathbb{Q}/\mathbb{Z}.$$

The cup-product and Bockstein structure on $H^*(M;\mathbb{F}_p)$ for M a Seifert manifold have been computed in [2]. Bryden and Zvengrowski also treat the case of coefficients $\mathbb{Z}/p^s\mathbb{Z}$, under the hypothesis that the p-adic valuations of the cone point orders are either s, 1 or 0.

2. THE TORSION SUBGROUP

Assume now that $M = M(g;S)$ is a Seifert manifold with Seifert data $S = ((\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$, where $r \geq 1$ and $\alpha_i > 1$ for all $i \leq r$.

Then $H_1(M;\mathbb{Z}) \cong Z^{2g} \oplus H$, where H has a presentation

$$\langle q_1, \ldots, q_r, h \mid \Sigma q_i = 0,\ \alpha_i q_i + \beta_i h = 0,\ \forall i \geq 1\rangle.$$

The torsion subgroup $T(M)$ is a subgroup of H. Let $\varepsilon_S = -\Sigma\frac{\beta_i}{\alpha_i}$ be the generalized Euler invariant of the Seifert fibration.

We shall modify this presentation to obtain one with more convenient generators. Our approach involves localizing at a prime p. After reordering the Seifert data, if necessary, we may assume that α_{i+1} divides α_i in $\mathbb{Z}_{(p)}$, for all $i \geq 1$. (Note that if $\varepsilon_S = 0$ then $\frac{\alpha_1}{\alpha_2}$ is invertible in $\mathbb{Z}_{(p)}$.) Localization loses nothing, since ℓ_M is uniquely the orthogonal sum of pairings on the p-primary summands of $T(M)$. (We shall often write ℓ_M rather than $\mathbb{Z}_{(p)} \otimes \ell_M$, for simplicity of notation.)

Using the relation $\Sigma q_i = 0$ to eliminate the generator q_1, we see that $\mathbb{Z}_{(p)} \otimes T(M)$ has the equivalent presentation

$$\langle q_2, \ldots, q_r, h \mid \alpha_1 \varepsilon_S h = 0,\ \alpha_i q_i + \beta_i h = 0,\ \forall i \geq 2\rangle.$$

Since α_2 and β_2 are relatively prime there are integers m, n such that $m\alpha_2 + n\beta_2 = 1$. Let $\gamma_i = \frac{\alpha_2}{\alpha_i}\beta_i$ and $q_i' = \gamma_2 q_i - \gamma_i q_2$, for all i. (Then $q_2' = 0$.) Let $s = -mh + nq_2$ and $t = \alpha_2 q_2 + \beta_2 h$. Then $h = -\alpha_2 s + nt$ and $q_2 = \beta_2 s + mt$. Since $t = 0$ in H this simplifies to

$$\langle q_3', \ldots, q_r', s \mid \alpha_1 \alpha_2 \varepsilon_S s = 0,\ \alpha_i q_i' = 0,\ \forall i \geq 3\rangle.$$

In particular, if exactly r_p of the cone point orders α_i are divisible by p and $\varepsilon_S = 0$ then $T(M)$ has nontrivial p-torsion if and only if $r_p \geq 3$, in which case $\mathbb{Z}_{(p)} \otimes T(M)$ is the direct sum of $r_p - 2$ cyclic submodules, while if $\varepsilon_S \neq 0$ then $T(M)$ has nontrivial p-torsion if and only if $r_p \geq 2$ and then $\mathbb{Z}_{(p)} \otimes T(M)$ is the direct sum of $r_p - 1$ cyclic submodules.

3. THE LINKING PAIRING

The Seifert structure gives natural 2-chains relating the 1-cycles representing the generators of H. For let N_i be a torus neighborhood of the i^{th} exceptional fibre, and let B_o be a section of the restriction of the Seifert fibration to $M* = M \setminus \cup int N_i$. Let ξ_i and θ_i be simple closed curves in ∂N_i which represent q_i and h, respectively. Then $\partial B_o = \Sigma \xi_i$, and there are singular 2-chains D_i in N_i such that $\partial D_i = \alpha_i \xi_i + \beta_i \theta_i$, since $\alpha_i q_i + \beta_i h = 0$ in $H_1(N_i;\mathbb{Z})$. We may choose disjoint annuli A_i in M^* with $\partial A_i = \theta_2 - \theta_i$, for $i \neq 2$. For convenience in our formulae, we shall also let $A_2 = 0$. Then $C_i = \beta_2 D_i - \beta_i D_2 + \beta_2 \beta_i A_i$ is a singular 2-chain with $\partial C_i = \alpha_i \beta_2 \xi_i - \alpha_2 \beta_i \xi_2$.

Let $\xi_i' = \gamma_2 \xi_i - \gamma_i \xi_2$, for $i \geq 3$, $\sigma = -m\theta_2 + n\xi_2$ and

$$U = \alpha_1 B_o + \alpha_1 \varepsilon_S n D_2 - \Sigma \frac{\alpha_1}{\alpha_i}(D_i + \beta_i A_i).$$

Then ξ_i' is a singular 1-chain representing q_i' and $\partial C_i = \alpha_i\xi_i'$, for all $i \geq 3$, σ is a singular 1-chain representing s and U is a singular 2-chain with $\partial U = \alpha_1\alpha_2\varepsilon_S\sigma$.

We may assume that $\xi_i \bullet \theta_i = 1$ in ∂N_i. In order to calculate intersections and self-intersections of the 1-cycles ξ_i with the 2-chains C_i in M, we may push each ξ_i off N_i. Then ξ_i and D_j are disjoint, for all i, j, while $\xi_2 \bullet A_i = 1$, $\xi_i \bullet A_i = -1$ and $\xi_j \bullet A_i = 0$, if $i, j \neq 2$ and $j \neq i$. Similarly, we may assume that θ_2 is disjoint from the discs D_j (for all j) and the annuli A_k (for all $k \neq 2$). Since B_o is oriented so that $\partial B_o = \Sigma\xi_i$, we must have $\theta_2 \bullet B_o = -1$. Hence

$$\xi_i' \bullet C_i = -\beta_2\beta_i(\gamma_2 + \gamma_i),$$

$$\xi_i' \bullet C_j = -\beta_2\beta_j\gamma_i,$$

and

$$\xi_i' \bullet U = \alpha_1\varepsilon_S\gamma_i$$

for all $i, j \geq 3$ with $j \neq i$, while

$$\sigma \bullet U = \frac{\alpha_1}{\alpha_2} - n\alpha_1\varepsilon_S$$

and

$$\sigma \bullet C_i = n\beta_2\beta_i.$$

Let $\tilde{q}_i = c_i q_i'$, where $c_i\beta_i \equiv 1$ modulo α_2, for all $i \geq 3$. Then

$$\ell_M(\tilde{q}_i, \tilde{q}_i) = [-\beta_2\frac{c_i\alpha_i\beta_2 + \alpha_2}{\alpha_i^2}] \in \mathbb{Q}/\mathbb{Z}$$

and

$$\ell_M(\tilde{q}_i, \tilde{q}_j) = [-\beta_2\frac{\alpha_2}{\alpha_i\alpha_j}] \in \mathbb{Q}/\mathbb{Z}.$$

If $\varepsilon_S \neq 0$ then we also have

$$\ell_M(s, s) = [\frac{\alpha_1 - n\alpha_1\alpha_2\varepsilon_S}{\alpha_1\alpha_2^2\varepsilon_S}] \in \mathbb{Q}/\mathbb{Z}$$

and

$$\ell_M(s, \tilde{q}_i) = [\frac{1}{\alpha_i}] \in \mathbb{Q}/\mathbb{Z}.$$

In particular, the linking pairings depend only on S and not on g. (We could arrange that the deminators are powers of p, after further rescaling the basis elements. However that would tend to obscure the dependence on the Seifert data.)

Let S and S' be two systems of Seifert data, with concatenation S'', and let $M'' = M\#_f M' = M(0; S'')$ be the fibre-sum of $M = M(0; S)$ and $M' = M(0; S')$. If all the cone point orders of S' are relatively prime to all the cone point orders of S and either $\varepsilon_{S'} = \varepsilon_S = 0$ or

$\varepsilon_{S'}\varepsilon_S \neq 0$ then $\varepsilon_{S''} = \varepsilon_S + \varepsilon_{S'}$ and $\ell_{M''} = \ell_M \perp \ell_{M'}$. Thus if every p-primary summand of a linking pairing ℓ can be realized by some $M(0; S)$ with all cone point orders powers of p then ℓ can also be realized by a Seifert manifold. (It is not clear how the linking pairings of M, M' and M'' are related when some of the cone point orders in S and S' have a common factor.)

4. THE HOMOGENEOUS CASE: p ODD

Every nonsingular torsion pairing on a finite abelian p-group is the orthogonal sum of pairings on homogeneous summands, and when p is odd the decomposition is essentially unique [7].

The group $\mathbb{Z}_{(p)} \otimes T(M)$ is homogeneous of exponent p^k if $u_i = \frac{\alpha_i}{p^k}$ is invertible in $\mathbb{Z}_{(p)}$ for $3 \leq i \leq r_p$ and either $\varepsilon_S = 0$ or $\alpha_1\alpha_2\varepsilon_S/p^k$ is invertible in $\mathbb{Z}_{(p)}$. We shall assume that this is so, for some odd prime p, throughout this section. For convenience, we shall also write $u_i = \frac{\alpha_i}{p^k}$ for $i = 1$ and 2.

Let ℓ be a linking pairing on $N \cong (Z/p^kZ)^\rho$ and $L \in \mathrm{GL}(\rho, Z/p^kZ)$ be the matrix with (i, j) entry $p^k\ell(e_i, e_j)$, where $e_1, \dots, e_\rho$ is some basis for N. The *rank* of ℓ is $rk(\ell) = \dim_{\mathbb{F}_p} N/pN = \rho$. If p is odd then a linking pairing ℓ on a free Z/p^kZ-module N is determined up to isomorphism by $rk(\ell)$ and the image $d(\ell)$ of $\det(L)$ in $\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2 = Z/2Z$. (This is independent of the choice of basis for N.) In particular, ℓ is hyperbolic if and only if $\rho = rk(\ell)$ is even and $d(\ell) = [(-1)^{\frac{\rho}{2}}]$.

Suppose first that $\varepsilon_S = 0$. Then $\frac{\alpha_1}{\alpha_2} = \frac{u_1}{u_2}$ is also invertible in $\mathbb{Z}_{(p)}$, and $\mathbb{Z}_{(p)} \otimes T(M) \cong (Z/p^kZ)^{r_p-2}$, with basis $e_i = \tilde{q}_{i+2}$, for $1 \leq i \leq r_p - 2$. On applying row operations

$$row_i \mapsto row_i - \frac{u_3}{u_{i+2}} row_1$$

for $1 < j \leq r_p - 2$ and then

$$row_1 \mapsto row_1 - \frac{u_2}{\beta_2 u_3} \Sigma_{1<i\leq r_p-2} \beta_{i+2} row_i$$

to L we obtain a lower triangular matrix. Hence

$$\det(L) = (-\beta_2)^{r_p-2}(-\beta_2^{-1}L_{11} + \Sigma_{3<i\leq r_p} \frac{\beta_i u_2}{\beta_3 u_3 u_i}) \Pi_{3<j\leq r_p}((\beta_j u_j)^{-1}\beta_2).$$

Now

$$-\beta_2^{-1}L_{11} + \Sigma_{3<i\leq r_p} \frac{\beta_i u_2}{\beta_3 u_3 u_i} = \frac{\alpha_2}{\beta_3 u_3} \Sigma_{1<i\leq r_p} \frac{\beta_i}{\alpha_i}$$

in $\mathbb{Z}_{(p)}$. This is congruent to $-\frac{\alpha_2}{\beta_3 u_3}\frac{\beta_1}{\alpha_1}$ mod (p), since $\varepsilon_S = 0$, and so

$$\det(L) \equiv (-1)^{r_p-2}\beta_2^{2r_p-5}(-\beta_1\frac{\alpha_2}{\alpha_1})\Pi_{3\leq j\leq r_p}(\beta_j u_j)^{-1} \quad mod\ (p).$$

Hence

$$d(\ell_M) = [(-1)^{r_p-1}\frac{\alpha_1}{\alpha_2}(\Pi_{1\leq i\leq r}\beta_i)(\Pi_{3\leq j\leq r}u_j)].$$

When u_1 and u_2 are also invertible in $\mathbb{Z}_{(p)}$ (i.e., all the cone point orders have the same p-adic valuation) then $[\frac{\alpha_1}{\alpha_2}] = [u_1u_2]$ in $\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$, and so the formula is invariant under permutation of the indices.

A similar argument applies if $\alpha_1\alpha_2\varepsilon_S/p^k$ is invertible in $\mathbb{Z}_{(p)}$. In this case the integers $u_2 = \frac{\alpha_2}{p^k}$ and $v = \alpha_1\varepsilon_S$ are also invertible in $\mathbb{Z}_{(p)}$, since $uv = \alpha_1\alpha_2\varepsilon_S/p^k$. However $u_1 = \frac{\alpha_1}{p^k}$ may be divisible by p. Let $\Sigma^* = \Sigma_{3\leq i\leq r_p}\frac{\beta_i}{u_i}$. Then $\Sigma^* \in \mathbb{Z}_{(p)}$ and $v - \beta_1 - \frac{\beta_2 u_1}{u_2} \equiv \Sigma^*$ mod (p). We now have $\mathbb{Z}_{(p)} \otimes T(M) \cong (Z/p^kZ)^{r_p-1}$, with basis $e_i = \tilde{q}_{i+2}$, for $1 \leq i \leq r_p - 2$, and $e_{r_p-1} = s$.

Let $d_* = \beta_2^{-2}(\frac{1}{u_2} - \frac{\beta_2 u_1}{u_2^2 v})$ be the element in the bottom right corner of the matrix L. On applying row operations

$$row_i \mapsto row_i + \frac{\beta_2 u_2}{u_{i+2}} row_{r_p-1}$$

for $1 \leq j \leq r_p - 2$ and then

$$row_{r_p-1} \mapsto row_{r_p-1} + \beta_2^{-2}\Sigma_{1\leq i\leq r_p-2}\beta_{i+2}row_i$$

to L we obtain a lower triangular matrix. Hence

$$\det(L) = (-\beta_2)^{r_p-1}(\Pi_{3\leq i\leq r_p}((\beta_i u_i)^{-1}\beta_2).(d_*(1+\frac{u_2}{\beta_2}\Sigma^*) - \beta_2^{-3}\Sigma^*)$$

$$= (-1)^{r_p-1}\beta_2^{2r_p-6}((1-\frac{\beta_2 u_1}{u_2 v})(\frac{\beta_2}{u_2}\Sigma^*) - \Sigma^*)\Pi_{3\leq i\leq r_p}(\beta_i u_i)^{-1}$$

$$\equiv (-1)^{r_p-1}\beta_2^{2r_p-6}\frac{\beta_1\beta_2}{u_2 v}\Pi_{3\leq i\leq r_p}(\beta_i u_i)^{-1} \quad mod\ (p).$$

Hence

$$d(\ell_M) = [(-1)^{r_p-1}(\Pi_{1\leq i\leq r_p}\beta_i)(\Pi_{2\leq j\leq r_p}u_j)v].$$

When u_1 is also invertible in $\mathbb{Z}_{(p)}$ we have $v = u_1p^k\varepsilon_S$, and so the formula is again invariant under permutation of the indices.

The localized pairing $\mathbb{Z}_{(p)} \otimes \ell_M$ is hyperbolic if and only if either $\varepsilon_S = 0$, $r_p = \rho + 2$ is even and $[\frac{\alpha_1}{\alpha_2}(\Pi_{1\leq i\leq r_p}\beta_i)(\Pi_{3\leq j\leq r_p}u_j)] = [(-1)^{\frac{r_p}{2}-1}]$ or $\varepsilon_S \neq 0$, $r_p = \rho+1$ is odd and $[(\Pi_{1\leq i\leq r_p}\beta_i)(\Pi_{2\leq j\leq r_p}u_j)v] = [(-1)^{\frac{r_p-1}{2}}]$.

5. REALIZATION OF HOMOGENEOUS p-PRIMARY PAIRINGS: p ODD

Let p be an odd prime and ℓ a linking pairing on $(Z/p^kZ)^\rho$, where $\rho \geq 1$, and let $d(\ell) = [w]$. We shall show that $\ell \cong \ell_M$ and $\ell \cong \ell_{M'}$, where $M = M(0; S)$ and $M' = M(0; S')$ are Seifert manifolds and $\varepsilon_S = 0$ and $\varepsilon_{S'} \neq 0$. If $p \geq 5$ or $\rho > 2$ we may assume that $\alpha_i = p^k$ for all $i \leq r$, where $r = \rho + 2$ if $\varepsilon_S = 0$ and $r = \rho + 1$ if $\varepsilon_S \neq 0$. In this case $u_i = 1$ for all i and so $d(\ell_M) = [(-1)^{\rho+1}\Pi\beta_i]$, if $\varepsilon_S = 0$, and $d(\ell_M) = [(-1)^\rho(\Pi\beta_i)(\Sigma\beta_i)]$, if $\varepsilon_S \neq 0$. If $p = 3$ and $\rho = 2$ a similar result holds, but we must now allow cone point orders dividing 3^{k+1}.

If $\varepsilon_S = 0$ and ρ is odd the equation $\Sigma\beta_i = 0$ always has solutions with all $\beta_i \in (Z/p^kZ)^\times$. If ξ is a nonsquare in $(Z/p^kZ)^\times$ setting $\beta_i' = \xi\beta_i$ for all i gives another solution, and $[\Pi\beta_i'] = [\xi][\Pi\beta_i]$. (If $p \equiv 3$ *mod* (4) we may take $\xi = -1$, which corresponds to a change of orientation of the 3-manifold.) Thus the two isomorphism classes of linking pairings on $(Z/p^kZ)^\rho$ are realized by $M(0; (p^k, \beta_1), \ldots, (p^k, \beta_{\rho+2}))$ and $M(0; (p^k, \beta_1'), \ldots, (p^k, \beta_{\rho+2}'))$.

If $\rho = 4t - 2$ and $w \not\equiv 1$ *mod* (p) there is an integer x such that $x \equiv \frac{1}{2}(w-1)$ *mod* (p). The images of x and $w - 1 - x$ are invertible in Z/p^kZ. Let $\beta_1 = 1$, $\beta_2 = -w$, $\beta_3 = x$ and $\beta_4 = w - 1 - x$ and $\beta_{2i+1} = 1$ and $\beta_{2i+2} = -1$ for $2 \leq i < 2t$. Then $\Sigma\beta_i = 0$, $\beta_4 \equiv \beta_3$ *mod* (p) and $[(-1)^{r-1}\Pi\beta_i] = [w]$.

If $\rho = 4t$ and $w \not\equiv 1$ *mod* (p) let $\beta_1 = 1$, $\beta_2 = w$, $\beta_3 = \beta_4 = \beta_5 = y$, $\beta_6 = w - 1 - 3y$ and $\beta_{2i+1} = 1$ and $\beta_{2k+2} = -1$ for $3 \leq i \leq 2t$, where $y \equiv -\frac{1}{4}(1 + w)$ *mod* (p). Then $\Sigma\beta_i = 0$, $\beta_6 \equiv \beta_3$ *mod* (p) and $[(-1)^{r-1}\Pi\beta_i] = [w]$.

These choices work equally well for all $p \geq 3$, if $[w] \neq 1$. If ρ is even, $w \equiv 1$ *mod* (p) and $p > 3$ there is an integer n such that $n^2 \neq 0$ or 1 *mod* (p), and we solve as before, after replacing w by $\hat{w} = n^2w$.

However if $p = 3$ and $[w] = 1$ we must vary our choices. If $\rho = 4t - 2$ with $t > 1$ let $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$, $\beta_5 = \beta_6 = -2$ and $\beta_{2i+1} = 1$ and $\beta_{2i+2} = -1$ for $3 \leq i < 2t$. If $\rho = 4t$ let $\beta_{2i-1} = 1$ and $\beta_{2i} = -1$ for $1 \leq i \leq 2t + 1$. In the remaining case (when $\rho = 2$) we find that if $\Sigma_{1 \leq i \leq 4}\beta_i = 0$ then $[-\Pi\beta_i] = [-1]$. In this case we must use instead $S = ((3^{k+1}, 1), (3^{k+1}, 5), (3^k, -1), (3^k, -1))$ to realize the pairing with $[w] = [1]$.

The manifolds with Seifert data as above are $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds, except when $\rho = 1$ and $p = 3$, in which case they are the flat manifold G_3 (with its two possible orientations). The manifold $M = M(1; (3, 1), (3, 1), (3, -2))$ is an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold with $T(M) \cong Z/3Z$ and base orbifold a torus with cone points.

All such pairings may also be realized by by $\widetilde{\mathbb{SL}}$-manifolds.

If $r \not\equiv 2+(-1)^{r-1}w \ mod\ (p)$ let $\beta_1=\beta_2=\frac{1}{2}(2+(-1)^{r-1}w-r)$ and $\beta_i=1$ for $2<i\le r$. Then $\Pi\beta_i$ is a square and $v=\Sigma\beta_i=(-1)^{r-1}w$, so $[(-1)^{r-1}(\Pi\beta_i)v]=[w]$.

If $r\equiv 2+(-1)^{r-1}w \ mod\ (p)$ and $r\ge 4$ let $\beta_1=\beta_2=\frac{p+1}{2}$, $\beta_3=\beta_4=-\frac{p-1}{2}$ and $\beta_i=1$ for $4<i\le r$.

The remaining case (when $\rho=2$, $p=3$ and $[w]=1$) is realized by $M(0;(3^{k+1},1),(3^k,1),(3^k,1))$.

When $\rho=1$ these constructions give lens spaces ($\mathbb{S}^3$-manifolds), while $M(0;(3,1),(3,1),(3,-1))$ is a $\mathbb{N}il^3$-manifold. Forming the fibre-sum with $S^1\times S^1\times S^1$ (adding a handle to the base orbifold) in these cases gives $\widetilde{\mathbb{SL}}$-manifolds.

6. The inhomogeneous case: p odd

We now consider the inhomogeneous case, assuming for simplicity of notation that the cone point orders are powers of the odd prime p. Let $\alpha_i=p^k$ for $1\le i\le m_1$, $\alpha_i=p^{k-\lambda_1}$ for $m_1<i\le m_2$, $\dots$, and $\alpha_i=p^{k-\lambda_t}$ for $m_t<i\le r=m_{t+1}$, where $0<\lambda_1<\cdots<\lambda_t<k$.

If $\varepsilon_S=0$ and $m_1=2$, let $m=m_2$, while if $m_1>2$, let $m=m_1-2$. If $\varepsilon_S\ne 0$ and $\alpha_1\alpha_2\varepsilon_S/\alpha_3$ is divisible by p, let $m=1$. Otherwise, let $m=m_1-1$.

Then the matrix L has the block form $L=\begin{pmatrix} A & p^{\lambda_1}B \\ p^{\lambda_1}B^{tr} & p^{\lambda_1}D\end{pmatrix}$, where A is an $m\times m$ block with $\det(A)\not\equiv 0 \ mod\ (p)$, and where the blocks B and D may be further partitioned into blocks divisible by higher powers of p. Let $Q=\begin{pmatrix} I_m & -A^{-1}p^{\lambda_1}B \\ 0 & I_{\rho-m}\end{pmatrix}$. Then $Q^{tr}LQ=\begin{pmatrix} A & 0 \\ 0 & p^{\lambda_1}D'\end{pmatrix}$, where $D'=D-p^{\lambda_1}B^{tr}A^{-1}B$. Block-diagonalizing L in this fashion does not change the residue $mod\ (p)$ of D. Thus on iterating this process we see that ℓ_M is an orthogonal sum of pairings on homogeneous groups $(Z/p^kZ)^m$, $(Z/p^{k-\lambda_2}Z)^{m_2-m_1}$ $\dots$, $(Z/p^{k-\lambda_t}Z)^{\rho-m_t}$. We may read off the determinantal invariants of each summand from the corresponding diagonal block of the original matrix L.

With these reductions in mind, we may now contruct Seifert manifolds realizing given pairings. Let ℓ be a linking pairing on a p-primary group. Then ℓ is the orthogonal sum $\perp_{0\le j\le t}\ell_j$, where ℓ_j is a pairing on $(Z/p^{k_j}Z)^{\rho_j}$, with $\rho_j>0$ for $0\le j\le t$ and $0<k_j<k_{j-1}$ for $1\le j\le t$. Let $d(\ell_j)=[w_j]$ for $0\le j\le t$, and let $k=k_0$. We shall adapt the constructions of §5 to obtain a Seifert manifold M with $\ell_M\cong\ell$.

If $p\ge 5$ and $\varepsilon_S=0$ we let $\alpha_i=p^k$ for $1\le i\le m_1=\rho_0+2$, $\alpha_i=p^{k_1}$ for $m_1<i\le m_2=m_1+\rho_1$, $\dots$, and $\alpha_i=p^{k_t}$ for $m_t<i\le(\Sigma\rho_j)+2$. For each $1\le j\le t$ we let $\beta_i=1$ for $m_j<i<m_{j+1}$ and $\beta_{m_{j+1}}=w_j$.

We must then choose β_i for $1 \leq i \leq m_1$ so that $[\Pi_{1\leq i\leq m_1}\beta_i] = [w_1]$ and $\Sigma_{1\leq i\leq m_1}\beta_i = p^k\varepsilon - \Sigma_{m_1<i\leq r}p^k\frac{\beta_i}{\alpha_i}$. This is straightforward.

If $p = 3$ we may have to modify α_1 and α_2 as in §5 in order to realize all possible pairings.

A similar strategy applies when $\varepsilon_S \neq 0$. We set $m_1 = \rho_0 + 1$ and proceed as before. (The prime $p = 3$ again needs separate treatment.)

7. REALIZATION OF HOMOGENEOUS 2-PRIMARY PAIRINGS

The situation is more complicated when $p = 2$. A linking pairing ℓ on $(Z/2^kZ)^\rho$ is determined by its rank ρ and certain invariants $\sigma_j(\ell) \in Z/8Z \cup \{\infty\}$, for $\rho - 2 \leq j \leq \rho$. (See §3 of [5], and [4].) We shall not calculate these invariants here. Instead, we shall take advantage of the particular form of the pairings given in §3.

Let ℓ be a linking pairing on $N = (Z/2^kZ)^2$, and let e, f be the standard basis for N. Suppose that $a = 2^k\ell(e,e)$ is odd, and let $b = 2^k\ell(e,f)$ and $d = 2^k\ell(f,f)$. (Then b is even and d is odd, by nonsingularity of the pairing.) Let $f' = -a^{-1}be + f$. Then $\ell(e,f') = 0$ and $\ell(f',f') = [\frac{d'}{2^k}] \in \mathbb{Q}/\mathbb{Z}$, where $d' \equiv d - a^{-1}b^2 \ mod\ (2^k)$. Therefore $\ell \cong \ell_{\frac{a}{2^k}} \perp \ell_{\frac{d'}{2^k}}$. In particular, if $b \equiv 0 \ mod\ (4)$ then $\ell \cong \ell_{\frac{a}{2^k}} \perp \ell_{\frac{d}{2^k}}$.

Let $M = M(0;S)$ be a Seifert manifold. The pairing $\mathbb{Z}_{(2)} \otimes \ell_M$ is even if and only if

(1) there is a $k \geq 1$ such that $u_i = \frac{\alpha_i}{2^k}$ is odd, for all $1 \leq i \leq r_2$; and
(2) either r_2 is odd or $\varepsilon_S = 0$ (in which case r_2 is even).

In particular, $\mathbb{Z}_{(2)} \otimes T(M) \cong (Z/2^kZ)^{2s}$ for some $s \geq 0$. (Conversely, if $\mathbb{Z}_{(2)} \otimes T(M)$ is homogeneous of exponent 2^k, the localized pairing is odd if and only if either $\varepsilon_S = 0$, $u_i = \frac{\alpha_i}{2^k}$ is even for $i = 1$ and 2, and is odd for $2 < i \leq r_2$, or $\varepsilon_S \neq 0$, u_1 is even and u_i is odd for $1 < i \leq r_2$.)

Suppose that $\varepsilon_S = 0$ and $\mathbb{Z}_{(2)} \otimes \ell_M$ is odd. Then the diagonal entries of L are odd and the off-diagonal elements are even. In particular, $A = \left(\begin{smallmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{smallmatrix}\right)$ is invertible. We may partition L as $L = \left(\begin{smallmatrix} A & C \\ C^{tr} & B \end{smallmatrix}\right)$, where C is a $2 \times (r_2 - 4)$ block with even entries and B is a $(r_2-4) \times (r_2-4)$ matrix. Let $Q = \left(\begin{smallmatrix} I_2 & -A^{-1}C \\ 0 & I_{r_2-4} \end{smallmatrix}\right)$. Then $\det(Q) = 1$ and $Q^{tr}LQ = \left(\begin{smallmatrix} A & 0 \\ 0 & B' \end{smallmatrix}\right)$, where $B' = B - C^{tr}A^{-1}C$. The columns of C are proportional, and the ratio u_3/u_4 is odd. Since the entries of C are even and since $A - I_2$ has even entries, $B' \equiv B \ mod\ (8)$. Iterating this process, we may replace L by a block-diagonal matrix, where the blocks are either 2×2 or 1×1, and are congruent $mod\ (8)$ to the corresponding blocks of L. We may diagonalize each such 2×2 block as above, and so we may easily represent ℓ_M as an orthogonal sum of pairings of rank 1.

Let $\ell = \perp_{1\le i\le\rho} \ell_{w_i}$ be an odd linking pairing on $(Z/2^kZ)^\rho$. Let $S = ((\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$, where $\alpha_1=\alpha_2=2^{k+2}$, $\alpha_i=2^k$ for $3\le i\le r=\rho+2$, $\beta_2=1$, β_i is an integer such that $\beta_i w_i \equiv 1 \ mod\ (8)$ for $3\le i\le r$ and $\beta_1=-1-4\Sigma_{i\ge3}\beta_i$. Then $\varepsilon_S=0$ and $\ell_M\cong\ell$.

When $\varepsilon_S\ne0$ and ℓ_M is odd we first replace each $\tilde{q}_i$ by $u_i\tilde{q}_i+z^{-1}s$, where $z=2^k\ell_M(s,s)$. We then see that $\ell_M\cong\tilde{\ell}\perp\ell_z$, where the matrix for $\tilde{\ell}$ has odd diagonal entries and even off-diagonal entries, and we may continue as before. (If r is odd we may assume that $\alpha_i=2^k$ for all i. If r is even we may assume that $\alpha_i=2^k$ for $i>1$, but homogeneity of $\mathbb{Z}_{(2)}\otimes T(M)$ requires that $\frac{\alpha_1}{\alpha_2}$ be even.)

If a linking pairing ℓ on $(Z/2^kZ)^\rho$ is even then ρ is also even, and either ℓ is hyperbolic (and is the orthogonal sum of $\frac{\rho}{2}$ copies of the pairing E_0^k) or it is the orthogonal direct sum of a hyperbolic pairing of rank $\rho-2$ with the pairing E_1^k (if $k>1$). When $k=1$ all even pairings are hyperbolic. Otherwise, ℓ is determined by the image of the matrix $L=2^k\ell(e_i,e_j)$ in $GL(2s,Z/4Z)$. In particular, if $L=\left(\begin{smallmatrix}0&c\\c&d\end{smallmatrix}\right)$ with c odd and d even the pairing is hyperbolic.

Suppose that $k>1$, and that $\alpha_i=2^k$ for $1\le i\le r$. (If r is even we assume also that $\varepsilon_S=0$.) Then the diagonal entries of L are all even and the off-diagonal entries are are all odd. We may reorder the basis of $T(M)$ so that $L_{ii}\equiv0 \ mod\ (4)$, for all $i\le t$ and $L_{ii}\equiv0$ it mod (4) for $t<i\le\rho$. We may again partition L as $L=\left(\begin{smallmatrix}A&C\\C^{tr}&B\end{smallmatrix}\right)$, where $A\in GL(2,Z/2^kZ)$, C is a $2\times(\rho-2)$ block. and B is a $(\rho-2)\times(\rho-2)$ matrix. If we conjugate by $Q=\left(\begin{smallmatrix}I_2&-A^{-1}C\\0&I_{\rho-2}\end{smallmatrix}\right)$ to obtain $Q^{tr}LQ=\left(\begin{smallmatrix}A&0\\0&B'\end{smallmatrix}\right)$, then $B'\equiv B \ mod\ (2)$. In particular, the off-diagonal entries are still odd, but the residues mod (4) of the diagonal entries are changed. Iterating this process, and using the fact that $E_1^k\perp E_1^k\cong E_0^k\perp E_0^k$, we find that ℓ is hyperbolic if and only if $\frac{\rho}{2}-t\equiv0$ or $3 \ mod\ (4)$.

In terms of the Seifert data, if $\varepsilon_S=0$ or $\beta_1\equiv\Sigma_{i\ge3}\beta_i \ mod\ (4)$ then

$$t=\#\{i\ge3 \mid \beta_i+\beta_2\equiv0 \ mod\ (4)\}.$$

Otherwise $\varepsilon_S\ne0$ and $\Sigma_{i\ne2}\beta_i\equiv0 \ mod\ (4)$, and then

$$t=\#\{i\ge3 \mid \beta_i+\beta_2\equiv0 \ mod\ (4)\}+1.$$

In particular, if $S=((2^k,\beta_1),\dots,(2^k,\beta_r))$ with $\beta_i=(-1)^i$ for $1\le i\le r$ then $\varepsilon_S=-\frac{1}{2^k}$ if r is odd and $\varepsilon_S=0$ if r is even, and $\ell_M\cong(E_0^k)^{\frac{\rho}{2}}$ is hyperbolic. We may realize the non-hyperbolic pairing $(E_0^k)^{\frac{\rho}{2}-1}\perp E_1^k$ using either $\beta_1=-3$, $\beta_2=\beta_3=1$ and $\beta_i=(-1)^i$ for $4\le i\le\rho+2$, with $\varepsilon_S=0$, or $\beta_1=1$ and $\beta_i=(-1)^i$ for $2\le i\le\rho+1$, with $\varepsilon_S=\frac{1}{2^k}$.

The manifolds constructed in this section are either $\mathbb{H}^2\times\mathbb{E}^1$-manifolds (if $\varepsilon_S=0$), or $\widetilde{\mathbb{SL}}$-manifolds (if $\varepsilon_S\ne0$), with the exceptions of the

Hantzsche-Wendt flat manifold $M(0;(2,-1),(2,1),(2,-1),(2,1))$ and the $\mathbb{S}^3$-manifolds $M(0;(2,1),(2,1),(2,\beta))$ and $M(0;(4,1),(2,1),(2,\beta))$.

8. THE INHOMOGENEOUS CASE: $p=2$

In the inhomogeneous case a reduction as in §6 is possible when the $(\rho-m)\times(\rho-m)$ matrix $B^{tr}A^{-1}B\equiv 0$ *mod* (4). For instance, this is so if $\varepsilon_S=0$ and either

(1) the gaps λ_1, $\lambda_2-\lambda_1$, $\dots$, $\lambda_t-\lambda_{t-1}$ in the 2-adic valuations of the cone point orders are all at least 3;
(2) $\frac{\alpha_2}{\alpha_3}$ is even, m_i is even for all $1\le i\le t+1$ and the gaps are all at least 2; or
(3) $\frac{\alpha_2}{\alpha_3}$ is even and $m_i\equiv 2$ *mod* (4) for all $1\le i\le t+1$.

However the general situation is less clear.

If $\mathbb{Z}_{(2)}\otimes T(M)$ has exponent 2^k, but is not homogeneous, then there are generators $\tilde{q}_i$ of order $2^{k-\lambda}$, for some $0<\lambda<k$. If, moreover, either $\varepsilon_S=0$ or $T(M)/2^{k-1}T(M)$ is not cyclic then $\frac{\alpha_2}{\alpha_i}$ must be even and so $2^{k-\lambda}\ell_M(\tilde{q}_i,\tilde{q}_i)$ is odd for all such i.

We may derive from this observation a criterion for recognizing pairings which are not realizable by Seifert manifolds that is independent of the choice of generators. Let $N\cong N'\oplus N''$ be a finite abelian group, where the homogeneous summands of N' have exponent ≥ 4 and $2N''=0$, and let $\ell=\ell'\perp\ell''$, where ℓ' is a pairing on N' and and ℓ'' is a hyperbolic pairing on N''. Then $\ell(x,x)=0$ for all $x\in N$ such that $2x=0$. In particular, if N' is not cyclic then ℓ is not realizable by a Seifert manifold.

For example, the pairings $\ell_{\frac14}\perp\ell_{\frac14}\perp E_0^1$ and $E_0^2\perp E_0^1$ are not realized by any Seifert manifolds. (These provide counter-examples to a conjecture raised in [1].) The pairing $\ell_{\frac14}\perp E_0^1$ is not realizable by a Seifert manifold with $\varepsilon_S=0$. It is however realized by the $\mathbb{N}il^3$-manifold $M(0;(2,1),(2,1),(2,1),(2,-1))$.

9. WITT CLASSES

Two pairings ℓ and ℓ' are *Witt equivalent* if there are metabolic pairings μ and μ' such that $\ell\perp\mu\cong\ell'\perp\mu'$. The set of Witt equivalence classes is an abelian group $W(\mathbb{Q}/\mathbb{Z})$ with respect to orthogonal sum of pairings. The canonical decomposition into primary summands gives an isomorphism

$$W(\mathbb{Q}/\mathbb{Z})\cong\oplus_{p\ prime}W(\mathbb{F}_p)$$

where $W(\mathbb{F}_2)=Z/2Z$, $W(\mathbb{F}_p)\cong Z/4Z$ if $p\equiv 3$ *mod* (4) and $W(\mathbb{F}_p)\cong(Z/2Z)^2$ if $p\equiv 1$ *mod* (4). If a,b are relatively prime nonzero integers let

$w(\frac{b}{a})$ be the Witt class of the pairing $\ell_{\frac{b}{a}}$. The summands are generated by the classes of such pairings.

In [6] bordism arguments are used to compute the Witt class of ℓ_M, for $M = M(0; S)$ a Seifert manifold. If $\varepsilon_S = 0$ the Witt class of ℓ_M is $-\Sigma w(\frac{\beta_i}{\alpha_i})$, while if $\varepsilon_S = \frac{p}{q}$ (in lowest form) then it is $-w(\frac{1}{pq}) - \Sigma w(\frac{\beta_i}{\alpha_i})$ [6]. In particular, the image of ℓ_M in $W(\mathbb{F}_p)$ is nontrivial if $\varepsilon_S = 0$ and r_p is odd or if $\varepsilon_S \neq 0$ and r_p is even.

Remark. The definition of Witt equivalence given here is appropriate for obtaining bordism invariants, as in [6]. However the Witt groups defined in [5] use a finer equivalence relation, involving stabilization by split pairings (rather than by metabolic pairings).

References

[1] Bryden, J. and Deloup, F. A linking form conjecture for 3-manifolds, in *Advances in topological quantum field theory*, NATO Sci. Ser. II Math. Phys. Chem., 179, Kluwer Acad. Publ., Dordrecht (2004), 253–265.

[2] Bryden, J. and Zvengrowski, P. The cohomology ring of the orientable Seifert manifolds, II, Topology Appl. 127 (2003), 213–257.

[3] Crisp, J.S. and Hillman, J.A. Embedding Seifert fibred and Sol^3-manifolds in 4-space, Proc. London Math. Soc. 76 (1998), 685–710.

[4] Deloup, F. Monoide des enlacements at facteurs orthogonaux, Alg. Geom. Top. 5 (2005), 419–442.

[5] Kawauchi, A. and Kojima, S. Algebraic classification of linking pairings on 3-manifolds, Math. Ann. 253 (1980), 29–42.

[6] Oh, H.S. Witt classes of torsion linking forms, in it Combinatorial Methods in Topology and Algebraic Geometry, CONM 44, American Mathematical Society (1985), 179–188.

[7] Wall, C. T. C. Quadratic forms on finite groups, and related topics, Topology 2 (1964), 471–495.

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