Combinatorial bases for covariant representations of the Lie superalgebra $\mathfrak{gl}_{m|n}$

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Abstract

Covariant tensor representations of $\mathfrak{gl}_{m|n}$ occur as irreducible components of tensor powers of the natural $(m+n)$-dimensional representation. We construct a basis of each covariant representation and give explicit formulas for the action of the generators of $\mathfrak{gl}_{m|n}$ in this basis. The basis has the property that the natural Lie subalgebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ act by the classical Gelfand–Tsetlin formulas. The main role in the construction is played by the fact that the subspace of $\mathfrak{gl}_m$-highest vectors in any finite-dimensional irreducible representation of $\mathfrak{gl}_{m|n}$ carries a structure of an irreducible module over the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. One consequence is a new proof of the character formula for the covariant representations first found by Berele and Regev and by Sergeev.

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1 Introduction

Finite-dimensional irreducible representations of the Lie superalgebra $\mathfrak{gl}_{m|n}$ over $\mathbb{C}$ are parameterized by their highest weights. The highest weight of such a representation is a tuple λ of complex numbers of the form $\lambda = (\lambda_1, \dots, \lambda_m \,|\, \lambda_{m+1}, \dots, \lambda_{m+n})$ satisfying the conditions

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+, \qquad \text{for} \quad i = 1, \dots, m+n-1, \quad i \neq m. \tag{1.1}$$

We let $L(\lambda)$ denote the corresponding representation. It is isomorphic to the unique irreducible quotient of the *Kac module* $K(\lambda)$ which is defined as a universal induced module associated with the irreducible module over the Lie subalgebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ with the highest weight λ. Due to the work of Kac [10, 11], the module $K(\lambda)$ is irreducible if and only if λ is a *typical* weight which thus provides a character formula for the irreducible representations $L(\lambda)$ associated with typical highest weights.

The problem of finding the characters of the complementary family of *atypical* representations remained open until the work of Serganova [22]. She produced character formulas for $L(\lambda)$ in terms of the characters of the Kac modules involving generalized Kazhdan–Lusztig polynomials. These results were extended and made more explicit by Brundan [4] and Su and Zhang [25] which led to character and dimension formulas for all representations $L(\lambda)$ and allowed to prove conjectures on the characters stated by several authors; see [4] and [25] for more detailed discussions and references.

In this paper we develop a different approach allowing to employ the Yangian representation theory to construct representations of $\mathfrak{gl}_{m|n}$ and calculate their characters. We consider the *multiplicity space* $L(\lambda)^+_\mu$ spanned by the $\mathfrak{gl}_m$-highest vectors in $L(\lambda)$ of weight μ, which is isomorphic to the space of $\mathfrak{gl}_m$-homomorphisms $\mathrm{Hom}_{\mathfrak{gl}_m}(L'(\mu), L(\lambda))$, where $L'(\mu)$ denotes irreducible representation of the Lie algebra $\mathfrak{gl}_m$ with the highest weight μ. The multiplicity space is a natural module over the centralizer $\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$. As with the Lie algebra case, there exists an 'almost surjective' homomorphism from the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$ to the centralizer thus yielding an irreducible action of $\mathrm{Y}(\mathfrak{gl}_n)$ on $L(\lambda)^+_\mu$. This is analogous to the Olshanski *centralizer construction*, originally appeared in the context of the classical Lie algebras [18, 19] and which has led, in particular, to constructions of weight bases for representations of classical Lie algebras [15].

On the other hand, finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ are classified in terms of their highest weights or Drinfeld polynomials. Hence, identifying the parameters of the representation $L(\lambda)^+_\mu$ of $\mathrm{Y}(\mathfrak{gl}_n)$ we can, in principle, find its character by using the results on the characters of the Yangian modules; see [1, Theorem 15] and [5, Corollary 8.22].

We apply the Yangian approach to a particular family of *covariant tensor representations* $L(\lambda)$ of $\mathfrak{gl}_{m|n}$ (which we simply call the *covariant representations*). They occur as irreducible components of the tensor powers of the natural $(m+n)$-dimensional representation $\mathbb{C}^{m|n}$. The irreducible decompositions of the tensor powers and the characters of the covariant representations were first found by Berele and Regev [3] and Sergeev [23]. The character formulas are expressed in terms of the *supersymmetric Schur polynomials* which admit a combinatorial presentation in terms of *supertableaux*. The covariant representations of $\mathfrak{gl}_{m|n}$ include both typical and atypical representations and share many common properties with the polynomial representations of $\mathfrak{gl}_m$ arising in the decomposition of the tensor powers of $\mathbb{C}^m$.

Our main result is an explicit construction of all covariant representations $L(\lambda)$ of the Lie superalgebra $\mathfrak{gl}_{m|n}$. We construct a basis of $L(\lambda)$ parameterized by supertableaux and give explicit formulas for the action of the generators of $\mathfrak{gl}_{m|n}$. We do this by using the vector space isomorphism

$$L(\lambda) \cong \bigoplus_{\mu} L'(\mu) \otimes L(\lambda)^+_{\mu}, \tag{1.2}$$

summed over $\mathfrak{gl}_m$-highest weights μ. It turns out that if $L(\lambda)$ is a covariant representation, then the $\mathrm{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_{\mu}$ is isomorphic to the *skew module* $\overline{L}(\lambda')^+_{\mu'}$ arising from the Olshanski centralizer construction applied to the *Lie algebra* $\mathfrak{gl}_{r+n}$ and its subalgebra $\mathfrak{gl}_r$. The skew modules were introduced and studied by Cherednik [6]. Their Drinfeld polynomials were first calculated by Nazarov and Tarasov [17], and different proofs were also given later in [9] and [13]; see also [15, Sec. 8.5]. The skew module $\overline{L}(\lambda')^+_{\mu'}$ possesses a basis parameterized by trapezium-like Gelfand–Tsetlin patterns, and hence so does the multiplicity space $L(\lambda)^+_{\mu}$. Moreover, the $\mathfrak{gl}_m$-module $L'(\mu)$ admits the classical Gelfand–Tsetlin basis, so that a basis of $L(\lambda)$ can be naturally parameterized by pairs of patterns. An equivalent combinatorial description of the basis in terms of supertableaux is also given.

To calculate the Yangian highest weight of the representation $L(\lambda)^+_{\mu}$ and to derive the matrix element formulas for the generators of $\mathfrak{gl}_{m|n}$ in this basis we employ a relationship between the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ and the *Mickelsson–Zhelobenko algebra* $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$. Namely, we find the images of generators of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ under the composition

$$\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m} \to \mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m),$$

where the second map is a natural homomorphism. The elements of each of the natural subalgebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ act in the basis of $L(\lambda)$ by the classical Gelfand–Tsetlin formulas, and the action of the odd component of the Lie superalgebra is also expressed in an explicit form; cf. [15, Ch. 9] where a similar approach was used to construct weight bases for representations of the classical Lie algebras.

In our arguments we will avoid using the character formulas of [3] and [23] so we thus obtain a new proof of those formulas.

Note that analogues of the Gelfand–Tsetlin bases for a certain class of *essentially typical* representations of the Lie superalgebra $\mathfrak{gl}_{m|n}$ were given in [21] and [26]. The constructions rely on the fact that consecutive restrictions of an essentially typical representation to the subalgebras of the chain $\mathfrak{gl}_{m|1} \subset \mathfrak{gl}_{m|2} \subset \cdots \subset \mathfrak{gl}_{m|n}$ are completely reducible thus allowing to apply the approach of Gelfand and Tsetlin [7] to construct a basis and to derive explicit formulas for the action of the generators. In the special case $n = 1$ any finite-dimensional irreducible representation of the Lie superalgebra $\mathfrak{gl}_{m|1}$ can be realized via a basis of Gelfand–Tsetlin type by using the restriction to the natural Lie subalgebra $\mathfrak{gl}_m$; see [20]. In a recent work [24] these results were extended to all covariant representations of $\mathfrak{gl}_{m|n}$ by constructing a Gelfand–Tsetlin type basis and providing matrix element formulas of the generators of $\mathfrak{gl}_{m|n}$ in the basis. That basis appears to have properties quite different from ours.

2 Main results

Consider the standard basis E_{ij}, $1 \leqslant i, j \leqslant m + n$, of the Lie superalgebra $\mathfrak{gl}_{m|n}$ over $\mathbb{C}$. The $\mathbb{Z}_2$-grading on $\mathfrak{gl}_{m|n}$ is defined by setting $\deg E_{ij} = \bar{\imath} + \bar{\jmath}$, where we use the notation $\bar{\imath} = 0$ for $1 \leqslant i \leqslant m$ and $\bar{\imath} = 1$ for $m + 1 \leqslant i \leqslant m + n$. The commutation relations have the form

$$\big[E_{ij}, E_{kl}\big] = \delta_{kj}\, E_{il} - \delta_{il}\, E_{kj}(-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})},$$

where the square brackets denote the super-commutator.

Given a tuple $\lambda = (\lambda_1, \dots, \lambda_m \,|\, \lambda_{m+1}, \dots, \lambda_{m+n})$ of complex numbers, the irreducible representation $L(\lambda)$ of $\mathfrak{gl}_{m|n}$ is generated by a nonzero vector ζ (the highest vector) satisfying the conditions

$$\begin{aligned} &E_{ij}\, \zeta = 0 \qquad &&\text{for} \quad 1 \leqslant i < j \leqslant m + n \qquad \text{and} \\ &E_{ii}\, \zeta = \lambda_i\, \zeta \qquad &&\text{for} \quad 1 \leqslant i \leqslant m + n. \end{aligned}$$

The representation $L(\lambda)$ is finite-dimensional if and only if the conditions (1.1) hold. We will suppose that this is the case as we will only work with finite-dimensional representations.

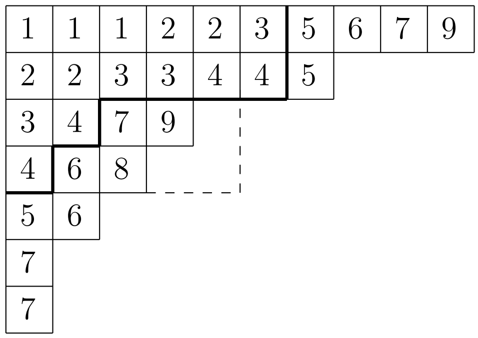
The family of the *covariant representations* is distinguished by the following conditions: all components $\lambda_1, \dots, \lambda_{m+n}$ of λ are nonnegative integers and the number ℓ of nonzero components among $\lambda_{m+1}, \dots, \lambda_{m+n}$ does not exceed λ_m; see [3] and [23]. To each highest weight λ satisfying these conditions, we will associate the Young diagram Γ_λ containing $\lambda_1 + \cdots + \lambda_{m+n}$ boxes which is determined by the conditions that the first m rows of Γ_λ are $\lambda_1, \dots, \lambda_m$ while the first ℓ columns are $\lambda_{m+1} + m, \dots, \lambda_{m+\ell} + m$. The condition $\ell \leqslant \lambda_m$ ensures that Γ_λ is the diagram of a partition. A *supertableau* Λ of shape Γ_λ is obtained by filling in the boxes of the diagram Γ_λ with the numbers $1, \dots, m + n$ in such a way that

the entries weakly increase from left to right along each row and down each column;

the entries in $\{1, \dots, m\}$ strictly increase down each column;

the entries in $\{m + 1, \dots, m + n\}$ strictly increase from left to right along each row.

Example 2.1. The following is a supertableau of shape Γ_λ associated with the highest weight $\lambda = (10, 7, 4, 3 \,|\, 3, 1, 0, 0, 0)$ of $\mathfrak{gl}_{4|5}$:



The thicker line indicates the subtableau formed by the entries in $\{1, 2, 3, 4\}$. $\square$

Given such a supertableau Λ, for any $1 \leqslant i \leqslant s \leqslant m$ denote by λ_{si} the number of entries in row i which do not exceed s. Furthermore, set $r = \lambda_{m1}$ and for any $0 \leqslant p \leqslant n$ and $1 \leqslant j \leqslant r + p$ denote by $\lambda'_{r+p,j}$ the number of entries in column j which do not exceed $m + p$.

The subtableau of the supertableau Λ occupied by the entries in $\{1, \dots, m\}$ is a *column-strict* tableau of the shape $\mu = (\lambda_{m1}, \dots, \lambda_{mm})$. The subtableau $\mathcal{T}$ of Λ occupied by the entries in $\{m + 1, \dots, m + n\}$ is a *row-strict* tableau of the skew shape Γ_λ/μ. We set

$$l_i = \lambda_i - i + 1, \quad l_{si} = \lambda_{si} - i + 1, \quad l'_{r+p,j} = \lambda'_{r+p,j} - j + 1.$$

For each $s = 1, \dots, m + n$ we denote by ω_s the number of entries in Λ equal to s. The symbols $\wedge_i$ in the denominators below indicate that the i-th factor should be skipped.

Our main theorem (Theorem 4.18) states that each covariant representation $L(\lambda)$ admits a basis ζ_Λ parameterized by all supertableaux Λ of shape Γ_λ. Moreover, the action of the generators of the Lie superalgebra $\mathfrak{gl}_{m|n}$ in this basis is given by the formulas

$$E_{ss}\,\zeta_\Lambda = \omega_s\,\zeta_\Lambda, \tag{2.1}$$

$$E_{s,s+1}\,\zeta_\Lambda = \sum_{\Lambda'} c_{\Lambda\Lambda'}\,\zeta_{\Lambda'}, \tag{2.2}$$

$$E_{s+1,s}\,\zeta_\Lambda = \sum_{\Lambda'} d_{\Lambda\Lambda'}\,\zeta_{\Lambda'}, \tag{2.3}$$

where the sums in (2.2) and (2.3) are taken over supertableaux Λ' obtained from Λ respectively by replacing an entry $s+1$ by s and by replacing an entry s by $s+1$. The coefficients $c_{\Lambda\Lambda'}$ and $d_{\Lambda\Lambda'}$ are found by explicit formulas which depend on s, as well as on the row or column number of Λ where the replacement occurs. For the values $s \neq m$ the coefficients are given by

$$c_{\Lambda\Lambda'} = -\frac{(l_{si} - l_{s+1,1}) \dots (l_{si} - l_{s+1,s+1})}{(l_{si} - l_{s1}) \dots \wedge_i \dots (l_{si} - l_{ss})},$$
$$d_{\Lambda\Lambda'} = \frac{(l_{si} - l_{s-1,1}) \dots (l_{si} - l_{s-1,s-1})}{(l_{si} - l_{s1}) \dots \wedge_i \dots (l_{si} - l_{ss})},$$

if $1 \leqslant s \leqslant m-1$ and the replacement occurs in row i, and by

$$c_{\Lambda\Lambda'} = -\frac{(l'_{r+p,j} - l'_{r+p+1,1}) \dots (l'_{r+p,j} - l'_{r+p+1,r+p+1})}{(l'_{r+p,j} - l'_{r+p,1}) \dots \wedge_j \dots (l'_{r+p,j} - l'_{r+p,r+p})},$$
$$d_{\Lambda\Lambda'} = \frac{(l'_{r+p,j} - l'_{r+p-1,1}) \dots (l'_{r+p,j} - l'_{r+p-1,r+p-1})}{(l'_{r+p,j} - l'_{r+p,1}) \dots \wedge_j \dots (l'_{r+p,j} - l'_{r+p,r+p})},$$

if $s = m + p$ for $1 \leqslant p \leqslant n-1$ and the replacement occurs in column j. In the case $s = m$ the coefficients $c_{\Lambda\Lambda'}$ and $d_{\Lambda\Lambda'}$ are given by more complicated formulas; see Theorem 4.18. This includes the case where the parameter r is changed to $r+1$ or $r-1$ which results in additional factors in the expressions for $c_{\Lambda\Lambda'}$ and $d_{\Lambda\Lambda'}$.

These matrix element formulas can be interpreted by using an equivalent combinatorial description of the basis vectors. Namely, to each supertableau Λ of shape Γ_λ we can associate the pair of arrays of nonnegative integers $(\mathcal{U}, \mathcal{V})$ of the form

$$\mathcal{U} = \begin{array}{ccccccc} \lambda_{m1} & & \lambda_{m2} & & \cdots & & \lambda_{mm} \\ & \lambda_{m-1,1} & & \cdots & & \lambda_{m-1,m-1} & \\ & & \cdots & \cdots & \cdots & & \\ & & \lambda_{21} & & \lambda_{22} & & \\ & & & \lambda_{11} & & & \end{array}$$

and

$$\mathcal{V} = \begin{array}{cccccc} \lambda'_{r+n,1} & \lambda'_{r+n,2} & \cdots & & \cdots & \lambda'_{r+n,r+n} \\ \ddots & \ddots & \cdots & & \cdots & \vdots \\ \lambda'_{r+1,1} & \lambda'_{r+1,2} & & \cdots & & \lambda'_{r+1,r+1} \\ \lambda'_{r,1} & \lambda'_{r,2} & & \cdots & \lambda'_{r,r} & \end{array}$$

By the properties of the supertableau Λ, both $\mathcal{U}$ and $\mathcal{V}$ are *patterns* as the following *betweenness* (or *interlacing*) conditions hold:

$$\lambda_{k+1,i} \geqslant \lambda_{ki} \geqslant \lambda_{k+1,i+1} \qquad \text{for} \quad 1 \leqslant i \leqslant k \leqslant m-1$$

and

$$\lambda'_{r+p,j} \geqslant \lambda'_{r+p-1,j} \geqslant \lambda'_{r+p,j+1} \qquad \text{for} \quad p = 1, \dots, n \qquad \text{and} \quad j = 1, \dots, r+p-1.$$

The basis elements E_{ij} of $\mathfrak{gl}_{m|n}$ with $1 \leqslant i, j \leqslant m$ span a subalgebra isomorphic to $\mathfrak{gl}_m$. The action of the elements of this subalgebra affects only the pattern $\mathcal{U}$ leaving $\mathcal{V}$ unchanged, while the action of the subalgebra isomorphic to $\mathfrak{gl}_n$ which is spanned by the E_{ij} with $m+1 \leqslant i, j \leqslant m+n$, affects only the pattern $\mathcal{V}$ leaving $\mathcal{U}$ unchanged. The above formulas for the action of the generators of both of these subalgebras coincide with the classical Gelfand–Tsetlin formulas. The action of the odd generators $E_{m,m+1}$ and $E_{m+1,m}$ affects both $\mathcal{U}$ and $\mathcal{V}$.

For any complex number a the mapping

$$E_{ij} \mapsto E_{ij} + \delta_{ij} (-1)^{\bar{\imath}} a$$

defines an automorphism of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_{m|n})$. Twisting $L(\lambda)$ by such an automorphism amounts to the shift $\lambda_i \mapsto \lambda_i + (-1)^{\bar{\imath}} a$ of the components of λ. Hence, the basis provided by Theorem 4.18 and the matrix element formulas are also valid for any representation $L(\lambda)$ which is isomorphic to the composition of a covariant representation with such an automorphism.

The tuple of complex numbers $\omega = (\omega_1, \dots, \omega_m \,|\, \omega_{m+1}, \dots, \omega_{m+n})$ is a *weight* of $L(\lambda)$ if the subspace

$$L(\lambda)_\omega = \{\eta \in L(\lambda) \mid E_{ii}\eta = \omega_i \eta, \quad i = 1, \dots, m+n\}$$

is nonzero. Let $x_1, \dots, x_{m+n}$ be indeterminates. Then the *character* of $L(\lambda)$ is the polynomial

$$\operatorname{ch} L(\lambda) = \sum_\omega \dim L(\lambda)_\omega \, x_1^{\omega_1}, \dots, x_{m+n}^{\omega_{m+n}}. \tag{2.4}$$

By Theorem 4.18, $\dim L(\lambda)_\omega$ equals the number of supertableaux Λ containing ω_s entries equal to s for each $s = 1, \dots, m+n$. Hence, we recover the formula for $\operatorname{ch} L(\lambda)$ originally obtained in [3] and [23]; see Corollary 4.20 below.

3 Mickelsson–Zhelobenko superalgebra and Yangian

We start by applying the standard methods of the Mickelsson algebra theory developed by Zhelobenko (see [29, 30]) to the Lie superalgebra $\mathfrak{gl}_{m|n}$ and its natural subalgebra $\mathfrak{gl}_m$. The corresponding Mickelsson–Zhelobenko superalgebra will be described in a way similar to its even counterpart; cf. [13].

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{gl}_m$ spanned by the elements $E_{11}, \dots, E_{mm}$ and let $\mathrm{R}(\mathfrak{h})$ denote the field of fractions of the commutative algebra $\mathrm{U}(\mathfrak{h})$. Consider the extension of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_{m|n})$ defined by

$$\mathrm{U}'(\mathfrak{gl}_{m|n}) = \mathrm{U}(\mathfrak{gl}_{m|n}) \otimes_{\mathrm{U}(\mathfrak{h})} \mathrm{R}(\mathfrak{h}).$$

Let J denote the left ideal of $\mathrm{U}'(\mathfrak{gl}_{m|n})$ generated by the elements E_{ij} with $1 \leqslant i < j \leqslant m$. Then J is a two-sided ideal of the normalizer

$$\operatorname{Norm} \mathrm{J} = \{x \in \mathrm{U}'(\mathfrak{gl}_{m|n}) \mid \mathrm{J}x \subseteq \mathrm{J}\}. \tag{3.1}$$

The *Mickelsson–Zhelobenko superalgebra* $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ is defined as the quotient algebra

$$\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m) = \operatorname{Norm} \mathrm{J}/\mathrm{J}.$$

Then $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ is a superalgebra over $\mathbb{C}$ and a natural left and right $\mathrm{R}(\mathfrak{h})$-module.

Generators of the superalgebra $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ can be constructed by using the *extremal projector* p for the Lie algebra $\mathfrak{gl}_m$; see [2]. The projector p is an element of an algebra $\mathrm{F}(\mathfrak{gl}_m)$ of formal series of elements of $\mathrm{U}(\mathfrak{gl}_m)$ and can be defined as follows. The positive roots of $\mathfrak{gl}_m$ with respect to $\mathfrak{h}$ are naturally enumerated by the pairs of indices (i, j) such that $1 \leqslant i < j \leqslant m$ so that the element E_{ij} is the corresponding root vector. Call a linear ordering of the positive roots *normal* if any composite root lies between its components. Set

$$p_{ij} = \sum_{k=0}^{\infty} (E_{ji})^k (E_{ij})^k \, \frac{(-1)^k}{k!\,(h_i - h_j + 1) \dots (h_i - h_j + k)},$$

where $h_i = E_{ii} - i + 1$. Then p is given by the product

$$p = \prod_{i<j} p_{ij}$$

taken in any normal ordering on the pairs (i, j). The formal series p does not depend on the normal ordering and has the properties

$$E_{ij}\, p = p\, E_{ji} = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant m.$$

Moreover, p satisfies the conditions $p^2 = p$ and $p^* = p$, where $x \mapsto x^*$ is the involutive anti-automorphism of the algebra $\mathrm{F}(\mathfrak{gl}_m)$ such that $(E_{ij})^* = E_{ji}$.

The extremal projector p can be regarded as a natural operator on the quotient space $\mathrm{U}'(\mathfrak{gl}_{m|n})/\mathrm{J}$. The Mickelsson–Zhelobenko superalgebra $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ can be identified with the image of $\mathrm{U}'(\mathfrak{gl}_{m|n})/\mathrm{J}$:

$$\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m) = p\big(\mathrm{U}'(\mathfrak{gl}_{m|n})/\mathrm{J}\big).$$

For $i = 1, \dots, m$ and $a = m + 1, \dots, m + n$ introduce elements of $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ by

$$z_{ia} = pE_{ia}(h_i - h_1) \dots (h_i - h_{i-1}),$$
$$z_{ai} = pE_{ai}(h_i - h_{i+1}) \dots (h_i - h_m).$$

The $\mathbb{Z}_2$-grading on the superalgebra $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ is inherited from that of the superalgebra $\mathrm{U}(\mathfrak{gl}_{m|n})$ so that all the elements z_{ia}, z_{ai} are odd. Together with the even elements E_{ab} with $a, b \in \{m+1, \dots, m+n\}$ they generate the Mickelsson–Zhelobenko superalgebra in the sense that monomials in the z_{ia}, z_{ai} and E_{ab} span $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ as a left (or right) $\mathrm{R}(\mathfrak{h})$-module.

The explicit formulas for the elements z_{ia} and z_{ai} (modulo J) have the form

$$z_{ia} = \sum_{i>i_1>\dots>i_s\geqslant 1} E_{ii_1} E_{i_1 i_2} \dots E_{i_{s-1} i_s} E_{i_s a} (h_i - h_{j_1}) \dots (h_i - h_{j_r}), \tag{3.2}$$

$$z_{ai} = \sum_{i<i_1<\dots<i_s\leqslant m} E_{i_1 i} E_{i_2 i_1} \dots E_{i_s i_{s-1}} E_{a i_s} (h_i - h_{j_1}) \dots (h_i - h_{j_r}), \tag{3.3}$$

where $s = 0, 1, \dots$ and $\{j_1, \dots, j_r\}$ is the complementary subset to $\{i_1, \dots, i_s\}$ respectively in the set $\{1, \dots, i-1\}$ or $\{i+1, \dots, m\}$.

In the following proposition we let the indices i, j and a, b, c run over the sets $\{1, \dots, m\}$ and $\{m+1, \dots, m+n\}$, respectively.

Proposition 3.1. *The following relations hold in* $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$:

$$[E_{ab}, z_{ci}] = \delta_{bc} z_{ai}, \qquad [E_{ab}, z_{ic}] = -\delta_{ac} z_{ib}. \tag{3.4}$$

Moreover, if $i \neq j$ *then*

$$z_{ai} z_{bj} = -z_{bj} z_{ai} \, \frac{h_i - h_j + 1}{h_i - h_j} + z_{bi} z_{aj} \, \frac{1}{h_i - h_j},$$

while

$$z_{ai} z_{bi} = -z_{bi} z_{ai}. \tag{3.5}$$

Finally, if $i \neq j$ *then*

$$z_{ia} z_{bj} = -z_{bj} z_{ia}, \tag{3.6}$$

while

$$z_{ia} z_{bi} = \big(\delta_{ba}(h_i + m - 1) - E_{ba}\big) \prod_{j=1,\, j\neq i}^{m} (h_i - h_j - 1) - \sum_{j=1}^{m} z_{bj} z_{ja} \prod_{k=1,\, k\neq j}^{m} \frac{h_i - h_k - 1}{h_j - h_k}. \tag{3.7}$$

Proof. All relations are verified by a standard calculation involving the extremal projector p; cf. [27]. They differ only by signs from the relations in the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{gl}_{m+n}, \mathfrak{gl}_m)$; see [13]. □

For all $a, b \in \{m+1, \dots, m+n\}$ introduce the polynomials $Z_{ab}(u)$ in a variable u with coefficients in $\mathrm{Z}(\mathfrak{gl}_{m+n}, \mathfrak{gl}_m)$ by

$$Z_{ab}(u) = \Big(\delta_{ab}(u + m) + E_{ab}\Big) \prod_{i=1}^{m} (u - h_i) + \sum_{i=1}^{m} z_{ai} z_{ib} \prod_{j=1,\, j\neq i}^{m} \frac{u - h_j}{h_i - h_j}. \tag{3.8}$$

We will need to evaluate $Z_{ab}(u)$ at $u = h$, where h is an element of $\mathrm{U}(\mathfrak{h})$. In order to make such evaluations unambiguous, we will always assume that the coefficients of the polynomial are written to the left of the powers of u. In particular, we have

$$Z_{ab}(h_i) = z_{ai} z_{ib}, \qquad i \in \{1, \dots, m\}. \tag{3.9}$$

Moreover, the relation (3.7) implies that

$$Z_{ab}(h_i - 1) = -z_{ib} z_{ai}, \qquad i \in \{1, \dots, m\}. \tag{3.10}$$

This implies an alternative formula for the polynomials $Z_{ab}(u)$,

$$Z_{ab}(u) = \Big(\delta_{ab} u + E_{ab}\Big) \prod_{i=1}^{m} (u - h_i + 1) - \sum_{i=1}^{m} z_{ib} z_{ai} \prod_{j=1,\, j\neq i}^{m} \frac{u - h_j + 1}{h_i - h_j}. \tag{3.11}$$

The mapping $E_{ij} \mapsto (E_{ij})^* = E_{ji}$ defines an involutive anti-automorphism of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_{m|n})$. We will use the same notation for its natural extension to $\mathrm{U}'(\mathfrak{gl}_{m|n})$ and to the Mickelsson–Zhelobenko superalgebra $\mathrm{Z}(\mathfrak{gl}_{m+n}, \mathfrak{gl}_m)$. The elements of $\mathrm{R}(\mathfrak{h})$ are fixed points of this anti-automorphism, and the images of the generators of the superalgebra are easy to calculate. It is also easy to verify directly that the formulas in the next proposition define an involutive anti-automorphism of $\mathrm{Z}(\mathfrak{gl}_{m+n}, \mathfrak{gl}_m)$.

Proposition 3.2. *For any $i \in \{1, \dots, m\}$ and $a, b \in \{m+1, \dots, m+n\}$ we have*

$$(z_{ia})^* = z_{ai} \, \frac{(h_i - h_1 - 1) \dots (h_i - h_{i-1} - 1)}{(h_i - h_{i+1}) \dots (h_i - h_m)}$$
$$(z_{ai})^* = z_{ia} \, \frac{(h_i - h_{i+1} + 1) \dots (h_i - h_m + 1)}{(h_i - h_1) \dots (h_i - h_{i-1})}$$

and $(E_{ab})^* = E_{ba}$. Moreover, $\big(Z_{ab}(u)\big)^* = Z_{ba}(u)$. □

Proposition 3.3. *For any $i \in \{1, \dots, m\}$ and $a, b, c \in \{m+1, \dots, m+n\}$ the following relations hold in the Mickelsson–Zhelobenko superalgebra $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$,*

$$Z_{ab}(u) z_{ci} = z_{ci} Z_{ab}(u) \, \frac{u - h_i + 1}{u - h_i} + z_{ai} Z_{cb}(u) \, \frac{1}{u - h_i} \tag{3.12}$$

$$Z_{ab}(u) z_{ci} = z_{ci} Z_{ab}(u) \, \frac{u - h_i + 2}{u - h_i + 1} + Z_{cb}(u) z_{ai} \, \frac{1}{u - h_i + 1}. \tag{3.13}$$

Proof. The first relation is verified by a straightforward calculation with the use of the relations in $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ given in Proposition 3.1. The second relation is obtained by writing another form of (3.12) with a and c swapped and solving the system of two equations simultaneously for the unknowns $z_{ci} Z_{ab}(u)$ and $z_{ai} Z_{cb}(u)$. □

Note that analogous relations involving the raising operators z_{ia} can be obtained by applying the anti-automorphism of Proposition 3.2 to the relations of Proposition 3.3.

Recall that the *Yangian for* $\mathfrak{gl}_n$ is a unital associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ over $\mathbb{C}$ with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where i and j run over the set $\{1,\dots,n\}$. The defining relations of $\mathrm{Y}(\mathfrak{gl}_n)$ have the form

$$[t_{ij}^{(r+1)},t_{kl}^{(s)}]-[t_{ij}^{(r)},t_{kl}^{(s+1)}]=t_{kj}^{(r)}t_{il}^{(s)}-t_{kj}^{(s)}t_{il}^{(r)}, \tag{3.14}$$

where $r,s\geqslant 0$ and $t_{ij}^{(0)}:=\delta_{ij}$. Using the formal generating series

$$t_{ij}(u)=\delta_{ij}+t_{ij}^{(1)}u^{-1}+t_{ij}^{(2)}u^{-2}+\dots$$

the defining relations (3.14) can be written in the equivalent form

$$(u-v)\,[t_{ij}(u),t_{kl}(v)]=t_{kj}(u)\,t_{il}(v)-t_{kj}(v)\,t_{il}(u).$$

A detailed exposition of the algebraic structure and representation theory of the Yangian can be found in [15].

Proposition 3.4. *The mapping*

$$t_{ij}(u)\mapsto Z_{m+i,m+j}(u)\,\frac{1}{(u+m)(u-h_1)\dots(u-h_m)}$$

defines an algebra homomorphism

$$\varphi:\mathrm{Y}(\mathfrak{gl}_n)\to \mathrm{Z}(\mathfrak{gl}_{m|n},\mathfrak{gl}_m). \tag{3.15}$$

Proof. This is a super-analogue of the homomorphism from the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ to the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{gl}_{m+n},\mathfrak{gl}_m)$; see [13, Theorem 3.1]. Similar to the proof of that result, we note first that the normalizer Norm J defined in (3.1) contains the centralizer $\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$ of $\mathfrak{gl}_m$ as a natural subalgebra. We will then obtain the homomorphism (3.15) as the composition of the super version of the *Olshanski homomorphism* $\mathrm{Y}(\mathfrak{gl}_n)\to\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$ and the natural homomorphism $\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}\to\mathrm{Z}(\mathfrak{gl}_{m|n},\mathfrak{gl}_m)$.

In more detail, let E denote the $m\times m$ matrix whose (i,j) entry is the basis element E_{ij} of $\mathfrak{gl}_m$. Then the mapping $\psi:\mathrm{Y}(\mathfrak{gl}_n)\to\mathrm{U}(\mathfrak{gl}_{m|n})$ given by

$$\begin{aligned} t_{ij}^{(1)}&\mapsto E_{m+i,m+j},\\ t_{ij}^{(r)}&\mapsto \sum_{k,l=1}^{m}E_{m+i,k}(E^{r-2})_{kl}E_{l,m+j},\qquad r\geqslant 2,\end{aligned} \tag{3.16}$$

defines an algebra homomorphism. It can be verified directly that the images of the generators $t_{ij}^{(r)}$ satisfy the defining relations of the Yangian. Alternatively, we can write this map as the composition of an embedding of $\mathrm{Y}(\mathfrak{gl}_n)$ into the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and the evaluation homomorphism $\mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{U}(\mathfrak{gl}_{m|n})$; see [8, Formulas (7) and (12)].

Furthermore, it is easy to verify that each element $\psi(t_{ij}^{(r)})$ commutes with the elements E_{kl}, $1 \leqslant k, l \leqslant m$, so that the image of the homomorphism ψ is contained in the centralizer $\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$. The next step is to find the images of the elements $\psi(t_{ij}^{(r)})$ in the Mickelsson–Zhelobenko superalgebra by using calculations similar to the even case; cf. [13, Sec. 3] and [15, Sec. 9.3]. The final formula for the images of the coefficients of the series $t_{ij}(u)$ is obtained by twisting the homomorphism ψ by the shift automorphism of the Yangian sending $t_{ij}(u)$ to $t_{ij}(u+m)$. □

4 Yangian action on the multiplicity space

For any tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_m \,|\, \lambda_{m+1}, \dots, \lambda_{m+n})$ satisfying the conditions (1.1) consider the corresponding finite-dimensional irreducible representation $L(\lambda)$ of $\mathfrak{gl}_{m|n}$. Denote by $L(\lambda)^+$ the subspace of $\mathfrak{gl}_m$-highest vectors in $L(\lambda)$:

$$L(\lambda)^+ = \{\eta \in L(\lambda) \,|\, E_{ij}\eta = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant m\}.$$

Given an m-tuple of complex numbers $\mu = (\mu_1, \dots, \mu_m)$ such that $\mu_i - \mu_{i+1} \in \mathbb{Z}_+$ for all i we denote by $L(\lambda)^+_\mu$ the corresponding weight subspace of $L(\lambda)^+$:

$$L(\lambda)^+_\mu = \{\eta \in L(\lambda)^+ \,|\, E_{ii}\eta = \mu_i\, \eta \qquad \text{for} \quad i = 1, \dots, m\}.$$

We have the weight space decomposition

$$L(\lambda)^+ = \bigoplus_{\mu} L(\lambda)^+_\mu.$$

It follows from the super-extension of the general results of the Mickelsson algebra theory ([29], [30, Theorem 4.3.8]), that given a total weight-consistent order on the set of elements E_{ba} with $m+1 \leqslant a < b \leqslant m+n$ and z_{ai} with $a = m+1, \dots, m+n$ and $i = 1, \dots, m$, the subspace $L(\lambda)^+$ is spanned by the vectors $M\zeta$, where M runs over the set of ordered monomials in these elements and ζ is the highest vector of $L(\lambda)$. This implies that the multiplicity space $L(\lambda)^+_\mu$ is nonzero only if all components of the m-tuple $\mu = (\mu_1, \dots, \mu_m)$ satisfy the inequalities $0 \leqslant \lambda_i - \mu_i \leqslant n$. However, we will not need to rely on this result as its independent proof will follow from the explicit construction of the representation $L(\lambda)$. Namely, assuming in addition that μ is a partition, we will construct a basis of each space $L(\lambda)^+_\mu$ and use these bases to produce a $\mathfrak{gl}_{m|n}$-submodule K of $L(\lambda)$. Since $L(\lambda)$ is irreducible, we will conclude that $K = L(\lambda)$ so that the space $L(\lambda)^+_\mu$ is nonzero if and only if μ satisfies the above conditions.

The dimension of $L(\lambda)^+_\mu$ coincides with the multiplicity of the $\mathfrak{gl}_m$-module $L'(\mu)$ in the restriction of $L(\lambda)$ to $\mathfrak{gl}_m$. The multiplicity space $L(\lambda)^+_\mu$ is a representation of the centralizer $\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$. Therefore, using the Olshanski homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$ we can equip $L(\lambda)^+_\mu$ with an action of the Yangian. As with the case of the skew representations of the Yangian associated with the pair of Lie algebras $\mathfrak{gl}_{m+n}$ and $\mathfrak{gl}_m$ (see e.g. [15, Sec. 8.5]) one can show by extending the arguments to the super case that if the space $L(\lambda)^+_\mu$ is nonzero then the resulting representation of $\mathrm{Y}(\mathfrak{gl}_n)$ in $L(\lambda)^+_\mu$ is irreducible. We do not bring all details here as this would lengthen the paper significantly and also because the irreducibility of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$ could be established in a more direct way by using the character formula for $L(\lambda)$ obtained in [3] and [23]; see also Corollary 4.20 below. As we want to demonstrate how this formula follows from the basis construction, in our exposition we will rely on the properties of the super-version of the Olshanski homomorphism.

4.1 Covariant representations

We will now suppose that all components λ_i of λ are nonnegative integers and the number ℓ of nonzero components among $\lambda_{m+1}, \dots, \lambda_{m+n}$ does not exceed λ_m. The cyclic $\mathrm{U}(\mathfrak{gl}_m)$-span of every nonzero element of the multiplicity space $L(\lambda)^+_\mu$ is a finite-dimensional representation of $\mathfrak{gl}_m$ isomorphic to $L'(\mu)$.

The elements z_{ia} and z_{ai} which are given by explicit formulas (3.2) and (3.3) preserve the subspace of $\mathfrak{gl}_m$-highest vectors in $L(\lambda)$. Moreover, they raise and lower the $\mathfrak{gl}_m$-weights, respectively:

$$z_{ia} : L(\lambda)^+_\mu \to L(\lambda)^+_{\mu+\delta_i}, \qquad z_{ai} : L(\lambda)^+_\mu \to L(\lambda)^+_{\mu-\delta_i},$$

where $\mu \pm \delta_i$ is obtained from μ by replacing the component μ_i by $\mu_i \pm 1$. We will call the z_{ia} and z_{ai} the *raising* and *lowering operators*, respectively.

Suppose that $\mu = (\mu_1, \dots, \mu_m)$ is a partition satisfying the conditions $0 \leqslant \lambda_i - \mu_i \leqslant n$ for all $i = 1, \dots, m$. Introduce the element $\zeta_\mu \in L(\lambda)^+_\mu$ by

$$\zeta_\mu = \prod_{j=1}^{m} \big(z_{m+\lambda_j-\mu_j,j} \dots z_{m+2,j}\, z_{m+1,j} \big)\, \zeta,$$

with the product taken in the increasing order of j.

Proposition 4.1. *Under the action of* $\mathrm{Y}(\mathfrak{gl}_n)$ *on* $L(\lambda)^+_\mu$ *we have*

$$t_{ij}(u)\,\zeta_\mu = 0 \qquad \textit{for} \quad 1 \leqslant i < j \leqslant n$$

and

$$t_{pp}(u)\,\zeta_\mu = \lambda_{m+p}(u)\,\zeta_\mu \qquad \textit{for} \quad p = 1, \dots, n,$$

where

$$\lambda_{m+p}(u) = \frac{u+\lambda_{m+p}+m}{u+m} \prod_{i=1,\ \lambda_i-\mu_i\geqslant p}^{m} \frac{u-\mu_i+i}{u-\mu_i+i-1}. \tag{4.1}$$

Proof. Using Proposition 3.4 we can rewrite the statements in terms of the action of the coefficients of the polynomials $Z_{ab}(u)$. Each element h_i acts in $L(\lambda)^+_\mu$ as multiplication by the scalar $\sigma_i = \mu_i - i + 1$ so that the relations can be written as

$$Z_{ab}(u)\zeta_\mu = 0 \qquad \text{for} \quad m+1 \leqslant a < b \leqslant m+n \tag{4.2}$$

and

$$Z_{m+p,m+p}(u)\zeta_\mu = (u+\lambda_{m+p}+m) \prod_{j=1,\ \lambda_j-\mu_j\geqslant p}^{m} (u-\sigma_j+1) \prod_{j=1,\ \lambda_j-\mu_j<p}^{m} (u-\sigma_j)\,\zeta_\mu \tag{4.3}$$

for $p = 1,\dots,n$. We will prove them by induction on the sum $\sum_{j=1}^m(\lambda_j-\mu_j)$. If the sum is zero, then $\zeta_\mu = \zeta$ is the highest vector of $L(\lambda)$. Then the relations follow from (3.8) as $z_{ia}\,\zeta = 0$ for all i and a. Now let i be the minimum index such that $\lambda_i - \mu_i > 0$. Then $\zeta_\mu = z_{ci}\,\zeta_{\mu+\delta_i}$ with $c = m + \lambda_i - \mu_i$. Suppose that $a < b$. If $a \geqslant c$, then we use (3.12) to write

$$Z_{ab}(u)\,z_{ci}\,\zeta_{\mu+\delta_i} = z_{ci}Z_{ab}(u)\,\frac{u-h_i+1}{u-h_i}\,\zeta_{\mu+\delta_i} + z_{ai}Z_{cb}(u)\,\frac{1}{u-h_i}\,\zeta_{\mu+\delta_i}. \tag{4.4}$$

If $a < c$, then using (3.13) we get

$$Z_{ab}(u)\,z_{ci}\,\zeta_{\mu+\delta_i} = z_{ci}Z_{ab}(u)\,\frac{u-h_i+2}{u-h_i+1}\,\zeta_{\mu+\delta_i} + Z_{cb}(u)z_{ai}\,\frac{1}{u-h_i+1}\,\zeta_{\mu+\delta_i}. \tag{4.5}$$

Note that $z_{ai}^2 = 0$ by (3.5), so that $z_{ai}\zeta_{\mu+\delta_i} = 0$ if $a < c$. Therefore, applying the induction hypothesis we derive (4.2).

Now let $a = b = m+p$. Applying (4.4) and (4.5) together with (4.2), we get

$$\begin{aligned}
Z_{aa}(u)\,z_{ci}\,\zeta_{\mu+\delta_i} &= z_{ci}Z_{aa}(u)\,\frac{u-\sigma_i}{u-\sigma_i-1}\,\zeta_{\mu+\delta_i}, &&\text{if} \quad \lambda_i-\mu_i<p,\\
Z_{aa}(u)\,z_{ci}\,\zeta_{\mu+\delta_i} &= z_{ci}Z_{aa}(u)\,\frac{u-\sigma_i+1}{u-\sigma_i}\,\zeta_{\mu+\delta_i} &&\text{if} \quad \lambda_i-\mu_i>p,\\
Z_{aa}(u)\,z_{ai}\,\zeta_{\mu+\delta_i} &= z_{ai}Z_{aa}(u)\,\frac{u-\sigma_i+1}{u-\sigma_i-1}\,\zeta_{\mu+\delta_i} &&\text{if} \quad \lambda_i-\mu_i=p.
\end{aligned}$$

This proves (4.3). □

Recall the notation $l_j = \lambda_j - j + 1$ for $j = 1,\dots,m$. Fix an index $i \in \{1,\dots,m\}$ and set $k = \lambda_i - \mu_i$.

Corollary 4.2. *If $k=0$ then $z_{i,m+p}\,\zeta_\mu=0$ for all $p=1,\dots,n$. If $k\geqslant 1$ then*

$$z_{i,m+k}\,\zeta_\mu=(\sigma_i+\lambda_{m+k}+m)\prod_{j=1}^{i-1}(-1)^{\lambda_j-\mu_j}\,(\sigma_i-l_j)$$
$$\times\prod_{j=i+1,\ \lambda_j-\mu_j\geqslant k}^{m}(\sigma_i-\sigma_j+1)\prod_{j=i+1,\ \lambda_j-\mu_j<k}^{m}(\sigma_i-\sigma_j)\,\zeta_{\mu+\delta_i}. \tag{4.6}$$

Moreover, for $0\leqslant k\leqslant n-1$ we have

$$z_{m+k+1,i}\,\zeta_\mu=\prod_{j=1}^{i-1}\frac{(-1)^{\lambda_j-\mu_j}}{\sigma_i-l_j-1}\prod_{j=1,\ \lambda_j-\mu_j>k}^{i-1}(\sigma_i-\sigma_j)\prod_{j=1,\ \lambda_j-\mu_j\leqslant k}^{i-1}(\sigma_i-\sigma_j-1)\,\zeta_{\mu-\delta_i}. \tag{4.7}$$

Proof. Since $z_{i,m+p}\,\zeta=0$, the first claim is immediate from (3.6). Furthermore, if $k\geqslant 1$, then applying (3.6), we get

$$z_{i,m+k}\,\zeta_\mu=\prod_{j=1}^{i-1}(-1)^{\lambda_j-\mu_j}\prod_{j=1}^{i-1}\big(z_{m+\lambda_j-\mu_j,j}\dots z_{m+2,j}\,z_{m+1,j}\big)\,z_{i,m+k}\,\zeta_{\widetilde{\mu}},$$

where $\widetilde{\mu}=(\lambda_1,\dots,\lambda_{i-1},\mu_i,\dots,\mu_m)$. By (3.10), we have $z_{i,m+k}\,z_{m+k,i}=-Z_{m+k,m+k}(h_i-1)$ so that

$$z_{i,m+k}\,\zeta_{\widetilde{\mu}}=z_{i,m+k}\,z_{m+k,i}\,\zeta_{\widetilde{\mu}+\delta_i}=-Z_{m+k,m+k}(\sigma_i)\,\zeta_{\widetilde{\mu}+\delta_i},$$

and (4.6) follows from (4.3).

Now apply $z_{m+k,i}$ to both sides of (4.6). Using (3.9) and (4.3), for the left hand side we obtain

$$z_{m+k,i}\,z_{i,m+k}\,\zeta_\mu=Z_{m+k,m+k}(h_i)\,\zeta_\mu=Z_{m+k,m+k}(\sigma_i)\,\zeta_\mu$$
$$=(\sigma_i+\lambda_{m+k}+m)\prod_{j=1,\ \lambda_j-\mu_j\geqslant k}^{m}(\sigma_i-\sigma_j+1)\prod_{j=1,\ \lambda_j-\mu_j<k}^{m}(\sigma_i-\sigma_j)\,\zeta_\mu.$$

Rewriting the resulting relation for $\mu-\delta_i$ instead of μ, we get (4.7). □

Corollary 4.3. *Suppose that $\mu=(\mu_1,\dots,\mu_m)$ is a partition such that the conditions $0\leqslant\lambda_i-\mu_i\leqslant n$ hold for all $i=1,\dots,m$. Then the vector ζ_μ is nonzero.*

Proof. As in the proof of Proposition 4.1, we argue by induction on the sum $\sum_{j=1}^{m}(\lambda_j-\mu_j)$ and suppose that $\zeta_\mu=z_{m+k,i}\,\zeta_{\mu+\delta_i}$ with $k=\lambda_i-\mu_i$, where i is the minimum index such that $\lambda_i-\mu_i>0$. Then $\sigma_j=l_j$ for $j=1,\dots,i-1$ and using (4.6) we obtain

$$z_{i,m+k}\,\zeta_\mu=(\sigma_i+\lambda_{m+k}+m)\prod_{j=1,\ \lambda_j-\mu_j\geqslant k}^{m}(\sigma_i-\sigma_j+1)\prod_{j=1,\ \lambda_j-\mu_j<k}^{m}(\sigma_i-\sigma_j)\,\zeta_{\mu+\delta_i}.$$

We have $\sigma_i + \lambda_{m+k} + m \geqslant \sigma_i + m > 0$. Moreover, $\sigma_1 > \cdots > \sigma_m$ so that the only factor in the coefficient of $\zeta_{\mu+\delta_i}$ which could be equal to zero is $\sigma_i - \sigma_{i-1} + 1$. However, in this case $\mu_i = \mu_{i-1} = \lambda_{i-1}$ which contradicts the assumption $\lambda_i - \mu_i > 0$. Thus, the coefficient is nonzero. By the induction hypothesis, the vector $\zeta_{\mu+\delta_i}$ is nonzero and hence so is ζ_μ. □

We will keep the assumptions of Proposition 4.1 and Corollary 4.3. Due to these statements, the irreducible $\mathrm{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$ is a highest weight representation whose highest weight is the n-tuple $(\lambda_{m+1}(u), \dots, \lambda_{m+n}(u))$. The corresponding *Drinfeld polynomials* $P_1(u), \dots, P_{n-1}(u)$ are monic polynomials in u which are defined by the relations

$$\frac{\lambda_{m+k}(u)}{\lambda_{m+k+1}(u)} = \frac{P_k(u+1)}{P_k(u)} \tag{4.8}$$

for $k = 1, \dots, n-1$; see e.g. [15, Sec. 3.2]. In Sec. 2 we associated a Young diagram Γ_λ to each covariant highest weight λ. To give formulas for the polynomials $P_k(u)$, consider the skew diagram Γ_λ/μ obtained from Γ_λ by removing the first μ_i boxes in row i for each $i = 1, \dots, m$. By the *content* of any box α of Γ_λ/μ we will mean the number $c(\alpha) = j - i$ if α is the intersection of row i and column j of the diagram.

Corollary 4.4. *The Drinfeld polynomials associated with the* $\mathrm{Y}(\mathfrak{gl}_n)$*-module* $L(\lambda)^+_\mu$ *are given by the formulas*

$$P_k(u) = \prod_\alpha \big(u - c(\alpha)\big),$$

where α *runs over the leftmost boxes of the rows of length* k *in the diagram* Γ_λ/μ.

Proof. The formulas of Proposition 4.1 imply that

$$P_k(u) = \prod_{i=1,\ \lambda_i-\mu_i=k}^{m} (u - \sigma_i) \prod_{j=1}^{\lambda_{m+k}-\lambda_{m+k+1}} (u + \lambda_{m+k+1} + m + j - 1).$$

Writing the factors in terms of the contents of the boxes of the diagram Γ_λ/μ gives the desired formulas. □

Now we introduce some parameters of the diagram conjugate to Γ_λ/μ. Set $r = \mu_1$ and let $\mu' = (\mu'_1, \dots, \mu'_r)$ be the diagram conjugate to μ so that μ'_j equals the number of boxes in column j of μ. Furthermore, set $\lambda' = (\lambda'_1, \dots, \lambda'_{r+n})$, where λ'_j equals the number of boxes in column j of the diagram Γ_λ.

Consider the general linear Lie algebra $\mathfrak{gl}_{r+n}$ and its natural subalgebra $\mathfrak{gl}_r$. Denote by $\overline{L}(\lambda')$ the finite-dimensional irreducible representation of $\mathfrak{gl}_{r+n}$ with the highest weight λ'. The subspace $\overline{L}(\lambda')^+_{\mu'}$ of $\mathfrak{gl}_r$-highest vectors in $\overline{L}(\lambda')$ of weight μ' is equipped with a structure of irreducible representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. This is a *skew representation* of $\mathrm{Y}(\mathfrak{gl}_n)$; see e.g. [15, Sec. 8.5] for a more detailed description of these representations. To define the action, consider the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_{r+n})$ defined by

$$\begin{aligned} t_{ij}^{(1)} &\mapsto \overline{E}_{r+i,r+j}, \\ t_{ij}^{(p)} &\mapsto (-1)^{p-1} \sum_{k,l=1}^{r} \overline{E}_{r+i,k} (\widehat{E}^{\,p-2})_{kl} \overline{E}_{l,r+j}, \qquad p \geqslant 2, \end{aligned} \tag{4.9}$$

where $\widehat{E}$ denotes the $r \times r$ matrix whose (i,j) entry is the basis element $\overline{E}_{ij}$ of $\mathfrak{gl}_r$. The image of the homomorphism is contained in the centralizer $\mathrm{U}(\mathfrak{gl}_{r+n})^{\mathfrak{gl}_r}$ which allows to define the Yangian action on the vector space $\overline{L}(\lambda')^+_{\mu'}$ via this homomorphism. We will work with the twisted action of the Yangian on this space which is obtained by taking its composition with the automorphism sending $t_{ij}(u)$ to $t_{ij}(u-r)$.

Theorem 4.5. *The* $\mathrm{Y}(\mathfrak{gl}_n)$*-modules* $L(\lambda)^+_\mu$ *and* $\overline{L}(\lambda')^+_{\mu'}$ *are isomorphic.*

Proof. The Drinfeld polynomials $\overline{P}_k(u)$ of the skew $\mathrm{Y}(\mathfrak{gl}_n)$-module $\overline{L}(\lambda')^+_{\mu'}$ were first calculated in [17] for a slightly different action of the Yangian. In our setting they can be written in the form

$$\overline{P}_k(u) = \prod_{\alpha} \big(u + c(\alpha)\big), \qquad k = 1, \dots, n-1,$$

where α runs over the top boxes of the columns of height k in the diagram λ'/μ'; see also [15, Sec. 8.5]. Hence, by Corollary 4.4 the Drinfeld polynomials of $\overline{L}(\lambda')^+_{\mu'}$ coincide with those of the module $L(\lambda)^+_\mu$. We want to verify that the highest weight of this module is the same as the highest weight $\nu(u) = (\nu_1(u), \dots, \nu_n(u))$ of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $\overline{L}(\lambda')^+_{\mu'}$. We will use the formula for $\nu(u)$ given in [15, Theorem 8.5.4]: the components $\nu_k(u)$ are found by

$$\nu_k(u) = \frac{(u + \nu_k^{(1)})(u + \nu_k^{(2)} - 1) \dots (u + \nu_k^{(r+1)} - r)}{(u + \mu'_1)(u + \mu'_2 - 1) \dots (u + \mu'_r - r + 1)(u - r)},$$

where

$$\nu_k^{(s)} = \mathrm{mid}\{\mu'_{s-1}, \mu'_s, \lambda'_{k+s-1}\},$$

assuming μ'_0 is sufficiently large, $\mu'_{r+1} = 0$, and $\mathrm{mid}\{a,b,c\}$ denotes the middle of the three integers. Since the components of the highest weight and the Drinfeld polynomials are related by (4.8), it will be sufficient to demonstrate that $\nu_1(u) = \lambda_{m+1}(u)$. Write $\mu = (\mu_1, \dots, \mu_m)$ in the form

$$\mu = (\underbrace{r, \dots, r}_{\mu'_r}, \underbrace{r-1, \dots, r-1}_{\mu'_{r-1} - \mu'_r}, \dots, \underbrace{1, \dots, 1}_{\mu'_1 - \mu'_2}, \underbrace{0, \dots, 0}_{m - \mu'_1}).$$

Now calculate $\lambda_{m+1}(u)$ by (4.1). The part of the product

$$\prod_{i=1,\ \lambda_i-\mu_i\geqslant 1}^{m} \frac{u-\mu_i+i}{u-\mu_i+i-1}$$

corresponding to the values $\mu_i = s$ simplifies to the expression

$$\frac{u+\nu_1^{(s+1)}-s}{u+\mu'_{s+1}-s}$$

for each $s \in \{1, \dots, r\}$. The same expression equals the part of this product with $\mu_i = s = 0$ multiplied by the first factor in (4.1). □

4.2 Construction of the basis vectors

As we pointed out in the proof of Theorem 4.5, the representation $\overline{L}(\lambda')^+_{\mu'}$ of $\mathrm{Y}(\mathfrak{gl}_n)$ admits a basis parameterized by column-strict tableaux of shape λ'/μ'. The explicit action of the Drinfeld generators of the Yangian in this basis was given in [17] by using a combinatorially equivalent description of the basis vectors in terms of trapezium-like patterns. Using Theorem 4.5, we get a basis of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$ parameterized by the row-strict tableaux of shape Γ_λ/μ with entries in $\{m+1, \dots, m+n\}$, together with an explicit action of $\mathrm{Y}(\mathfrak{gl}_n)$ in this basis. Furthermore, taking the Gelfand–Tsetlin basis of each $\mathfrak{gl}_m$-module $L'(\mu)$ we get a basis of the direct sum K of the vector spaces in (1.2) taken over the highest weights μ satisfying the assumptions of Corollary 4.3. Our goal now is to obtain explicit formulas for the action of generators of $\mathfrak{gl}_{m|n}$ in this basis of K. This will show that K is stable under the action and hence, due to the irreducibility of $L(\lambda)$, that $K = L(\lambda)$. The first step will be to present the basis vectors explicitly in terms of the lowering operators.

Given elements $a_1, \dots, a_k$ and $b_1, \dots, b_k$ of the set $\{m+1, \dots, m+n\}$, introduce the corresponding *quantum minors*

$$Z^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u) = \sum_{p\in\mathfrak{S}_k} \operatorname{sgn} p \cdot Z_{a_1 b_{p(1)}}(u-k+1)\dots Z_{a_k b_{p(k)}}(u), \tag{4.10}$$

where the polynomials $Z_{ab}(u)$ are defined by (3.8). By Proposition 3.4, the polynomials (4.10) inherit the properties of the respective quantum minors $t^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u)$ in the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$; see e.g. [15, Sec. 1.6]. In particular, the polynomials are skew-symmetric with respect to permutations of the upper indices a_i or lower indices b_i, and they can be expressed in a form alternative to (4.10):

$$Z^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u) = \sum_{p\in\mathfrak{S}_k} \operatorname{sgn} p \cdot Z_{a_{p(1)} b_1}(u)\dots Z_{a_{p(k)} b_k}(u-k+1). \tag{4.11}$$

Moreover, their images with respect to the anti-automorphism of Proposition 3.2 are given by

$$\big(Z^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u)\big)^* = Z^{b_1,\dots,b_k}_{a_1,\dots,a_k}(u). \tag{4.12}$$

Lemma 4.6. *The following relations hold in* $\mathrm{Z}(\mathfrak{gl}_{m|n},\mathfrak{gl}_m)$*: for any* $c\in\{a_1,\dots,a_k\}$ *we have*

$$z_{ci}\, Z^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u) = Z^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u)\, z_{ci}\, \frac{u-h_i-k+1}{u-h_i+2}, \tag{4.13}$$

$$z_{ic}\, Z^{b_1,\dots,b_k}_{a_1,\dots,a_k}(u) = Z^{b_1,\dots,b_k}_{a_1,\dots,a_k}(u)\, z_{ic}\, \frac{u-h_i+1}{u-h_i-k}. \tag{4.14}$$

Proof. Due to Proposition 3.2 and (4.12), the relations are equivalent. Therefore it suffices to prove (4.13). By the skew-symmetry of the quantum minors, we may assume that $c=a_k$. We will argue by induction on k. If $k=1$, then (4.13) holds by (3.12) with $a=c$. For $k\geqslant 2$ write

$$Z^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u) = \sum_{j=1}^{k} (-1)^{j-1} Z_{a_1 b_j}(u-k+1)\, Z^{a_2,\dots,a_k}_{b_1,\dots,\hat{b}_j,\dots,b_k}(u).$$

Hence, using again (3.12) we get

$$z_{ci}\, Z^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u) = \sum_{j=1}^{k} (-1)^{j-1} \Big(Z_{a_1 b_j}(u-k+1)\, z_{ci}\, \frac{u-h_i-k+1}{u-h_i-k+2} \\ - z_{a_1 i}\, Z_{c b_j}(u-k+1)\, \frac{1}{u-h_i-k+2}\Big)\, Z^{a_2,\dots,a_k}_{b_1,\dots,\hat{b}_j,\dots,b_k}(u).$$

Since $Z^{c,a_2,\dots,a_k}_{b_1,\dots,b_k}(u)=0$, applying the induction hypothesis we arrive at (4.13). □

For each $p=1,\dots,n$ set

$$A_p(u) = Z^{m+1,\dots,m+p}_{m+1,\dots,m+p}(u)\, Z^{m+1,\dots,m+p-1}_{m+1,\dots,m+p-1}(u)^{-1} \prod_{i=2}^{m+1} \frac{1}{u-h_i-p+1}, \tag{4.15}$$

and for $p=1,\dots,n-1$ set

$$B_p(u) = -Z^{m+1,\dots,m+p}_{m+1,\dots,m+p-1,m+p+1}(u)\, Z^{m+1,\dots,m+p+1}_{m+1,\dots,m+p+1}(u)^{-1} \prod_{i=2}^{m+1} (u-h_i-p)$$

and

$$C_p(u) = Z^{m+1,\dots,m+p-1,m+p+1}_{m+1,\dots,m+p}(u)\, Z^{m+1,\dots,m+p-1}_{m+1,\dots,m+p-1}(u)^{-1} \prod_{i=2}^{m+1} \frac{1}{u-h_i-p+1},$$

where $h_{m+1} = -m$ and the second quantum minor in the formulas for $A_1(u)$ and $C_1(u)$ is understood as being equal to 1.

As before, we also regard the $Z_{ab}(u)$ as polynomials in u whose coefficients are operators in $L(\lambda)^+_\mu$. We will see below that the basis vectors of $L(\lambda)^+_\mu$ are eigenvectors for the coefficients of all quantum minors $Z^{m+1,\dots,m+l}_{m+1,\dots,m+l}(u)$. Therefore, the application of $A_p(u)$, $B_p(u)$ or $C_p(u)$ to a basis vector produces a linear combination of the basis vectors whose coefficients are rational functions in u.

Recalling the parametrization of the basis vectors of the skew representations of the Yangian (see [15, Sec. 8.5]) and using the isomorphism of Theorem 4.5, we find that the highest vector ζ_μ of the $\mathrm{Y}(\mathfrak{gl}_n)$-module corresponds to the *initial* Γ_λ/μ*-tableau* $\mathcal{T}^0$ which is obtained by filling in the boxes of each row by the consecutive numbers $m+1, m+2, \dots$ from left to right. The entries $\lambda^0_{r+p,j}$ of the corresponding pattern $\mathcal{V}^0$ are given by

$$\lambda^0_{r+p,j} = \min\{\lambda'_j, \mu'_{j-p}\}, \tag{4.16}$$

where we assume that μ'_i is sufficiently large for $i \leqslant 0$. Equivalently, the parameters $\lambda^0_{r+p,j}$ of $\mathcal{T}^0$ are defined by first extending $\mathcal{T}^0$ to an arbitrary supertableau of shape Γ_λ so that $\lambda^0_{r+p,j}$ is the number of entries in column j of this supertableau which do not exceed $m+p$. We will use the notation

$$l^0_{r+p,j} = \lambda^0_{r+p,j} - j + 1. \tag{4.17}$$

Given an arbitrary row-strict Γ_λ/μ-tableau $\mathcal{T}$, set

$$\zeta_{\mathcal{T}} = \prod_{(p,j)} \Big(C_p(-l'_{r+p,j} - 1) \dots C_p(-l^0_{r+p,j} + 1)\, C_p(-l^0_{r+p,j}) \Big)\, \zeta_\mu, \tag{4.18}$$

where the product is taken over the pairs (p, j) with $p = 1, \dots, n-1$ and $j = 1, \dots, r+p$ in the order

$$\begin{aligned}(n-1,1), \dots, (1,1),\ (n-1,2), \dots, (1,2),\ \dots,\ (n-1,r+1), \dots, (1,r+1),\\ (n-1,r+2), \dots, (2,r+2),\ (n-1,r+3), \dots, (3,r+3),\ \dots,\\ (n-1,r+n-2), (n-2,r+n-2),\ (n-1,r+n-1).\end{aligned}$$

Proposition 4.7. *All evaluations of* $C_p(u)$ *involved in the expression* (4.18) *are well-defined. The vectors* $\zeta_{\mathcal{T}}$ *parameterized by the row-strict tableaux* $\mathcal{T}$ *form a basis of* $L(\lambda)^+_\mu$*. Moreover, the action of the generators of the Lie subalgebra* $\mathfrak{gl}_n$ *in this basis is given by the formulas* (2.1), (2.2) *and* (2.3) *with* $s \geqslant m+1$.

Proof. The rational function $C_p(u)$ coincides with the image of the series

$$t^{\,1,\dots,p-1,p+1}_{\,1,\dots,p}(u)\, t^{\,1,\dots,p-1}_{\,1,\dots,p-1}(u)^{-1}\, (u - h_1 - p + 1) \tag{4.19}$$

under the homomorphism of Proposition 3.4. Note that h_1 acts in $L(\lambda)^+_\mu$ as multiplication by the scalar $\mu_1 = r$. Now we find the image of the series (4.19) (with h_1 replaced by r) in the skew representation $\overline{L}(\lambda')^+_{\mu'}$ of $\mathrm{Y}(\mathfrak{gl}_n)$. The formulas defining this representation can be written in an equivalent form with the use of quantum minors of the matrix $1 + \overline{E}u^{-1}$, where $\overline{E}$ denotes the $(r+n) \times (r+n)$ matrix whose (i,j) entry is $\overline{E}_{ij}$. Namely, the representation is defined by

$$t_{ij}(u) \mapsto \Big[\big(1 + \overline{E}\, u^{-1}\big)^{1,\dots,r}_{1,\dots,r}\Big]^{-1} \cdot \big(1 + \overline{E}\, u^{-1}\big)^{1,\dots,r,r+i}_{1,\dots,r,r+j}$$

for $i, j \in \{1, \dots, n\}$. Moreover, the images of the quantum minors occurring in (4.19) are then found by

$$t^{\,1,\dots,p-1,p+1}_{\,1,\dots,p}(u) \mapsto \Big[\big(1 + \overline{E}\, u^{-1}\big)^{1,\dots,r}_{1,\dots,r}\Big]^{-1} \cdot \big(1 + \overline{E}\, u^{-1}\big)^{1,\dots,r,r+1,\dots,r+p-1,r+p+1}_{1,\dots,r,r+1,\dots,r+p},$$

$$t^{\,1,\dots,p-1}_{\,1,\dots,p-1}(u) \mapsto \Big[\big(1 + \overline{E}\, u^{-1}\big)^{1,\dots,r}_{1,\dots,r}\Big]^{-1} \cdot \big(1 + \overline{E}\, u^{-1}\big)^{1,\dots,r,r+1,\dots,r+p-1}_{1,\dots,r,r+1,\dots,r+p-1};$$

see e.g. [15, Sec. 8.5] for proofs of these statements. Hence, calculating the image of the series (4.19) in the skew representation $\overline{L}(\lambda')^+_{\mu'}$ we conclude that the rational function $C_p(u)$, regarded as an operator in $\overline{L}(\lambda')^+_{\mu'}$ can be written as

$$C_p(u) = \big(u + \overline{E}\big)^{1,\dots,r+p-1,r+p+1}_{1,\dots,r+p} \cdot \Big[\big(u + \overline{E}\big)^{1,\dots,r+p-1}_{1,\dots,r+p-1}\Big]^{-1}.$$

However, as was observed in [16], for an appropriate value of u such an operator takes a vector of the Gelfand–Tsetlin basis of $\overline{L}(\lambda')^+_{\mu'}$ to another vector of this basis; see also [15, Sec. 5.4]. More precisely, if $\mathcal{V}$ and $\mathcal{V}^-$ are trapezium patterns of the form described in Sec. 2 such that $\mathcal{V}^-$ is obtained from $\mathcal{V}$ by replacing an entry $\lambda'_{r+p,j}$ by $\lambda'_{r+p,j} - 1$, then for the corresponding basis vectors $\zeta_{\mathcal{V}}$ and $\zeta_{\mathcal{V}^-}$ of $\overline{L}(\lambda')^+_{\mu'}$ we have

$$C_p(-l'_{r+p,j})\,\zeta_{\mathcal{V}} = \zeta_{\mathcal{V}^-}.$$

Applying the isomorphism of Theorem 4.5 we conclude that

$$C_p(-l'_{r+p,j})\,\zeta_{\mathcal{T}} = \zeta_{\mathcal{T}^-}$$

in the representation $L(\lambda)^+_\mu$, there $\mathcal{T}$ and $\mathcal{T}^-$ are the row-strict tableaux corresponding to the patterns $\mathcal{V}$ and $\mathcal{V}^-$, respectively, so that $\mathcal{T}^-$ is obtained from $\mathcal{T}$ by replacing an entry $m + p$ by $m + p + 1$ in column j. This shows that the vectors given by (4.18) are well-defined and they form a basis of $L(\lambda)^+_\mu$.

Comparing the actions of Yangian in $L(\lambda)^+_\mu$ and $\overline{L}(\lambda')^+_{\mu'}$ given by (3.16) and (4.9), we can conclude that the elements of the subalgebra $\mathfrak{gl}_n$ of $\mathfrak{gl}_{m|n}$ act on the basis vectors $\zeta_{\mathcal{T}}$ of $L(\lambda)^+_\mu$ by the same formulas as the elements of the subalgebra $\mathfrak{gl}_n$ of $\mathfrak{gl}_{r+n}$ act on the basis vectors $\zeta_{\mathcal{V}}$ of $\overline{L}(\lambda')^+_{\mu'}$, thus completing the proof. □

Corollary 4.8. *Let $\mathcal{T}$ be a row-strict tableaux of shape Γ_λ/μ. Then for $p = 1, \dots, n$*

$$A_p(u)\,\zeta_{\mathcal{T}} = \frac{(u + l'_{r+p,1}) \dots (u + l'_{r+p,r+p})}{(u + l'_{r+p-1,1}) \dots (u + l'_{r+p-1,r+p-1})}\,\zeta_{\mathcal{T}}. \tag{4.20}$$

Moreover, for $p \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, r+p\}$ we have

$$C_p(-l'_{r+p,j})\,\zeta_{\mathcal{T}} = \zeta_{\mathcal{T}^-}, \tag{4.21}$$

if $\mathcal{T}$ contains an entry $m+p$ in column j and the replacement of this entry by $m+p+1$ yields a row-strict tableau $\mathcal{T}^-$. Otherwise, $C_p(-l'_{r+p,j})\,\zeta_{\mathcal{T}} = 0$.

For the same values of the parameters p and j we have

$$B_p(-l'_{r+p,j})\,\zeta_{\mathcal{T}} = \zeta_{\mathcal{T}^+}, \tag{4.22}$$

if $\mathcal{T}$ contains an entry $m+p+1$ in column j and the replacement of this entry by $m+p$ yields a row-strict tableau $\mathcal{T}^+$. Otherwise, $B_p(-l'_{r+p,j})\,\zeta_{\mathcal{T}} = 0$.

Proof. We argue as in the proof of Proposition 4.7. The rational function $A_p(u)$ coincides with the image of the series

$$t^{\,1,\dots,p}_{\,1,\dots,p}(u)\, t^{\,1,\dots,p-1}_{\,1,\dots,p-1}(u)^{-1}\,(u - h_1 - p + 1)$$

under the homomorphism of Proposition 3.4. The image of this series (with h_1 replaced by r) in the skew representation $\overline{L}(\lambda')^+_{\mu'}$ is given by

$$A_p(u) = \big(u + \overline{E}\big)^{1,\dots,r+p}_{1,\dots,r+p} \cdot \Big[\big(u + \overline{E}\big)^{1,\dots,r+p-1}_{1,\dots,r+p-1}\Big]^{-1}.$$

Hence, using again the formulas for the Yangian action in the Gelfand–Tsetlin basis of $\overline{L}(\lambda')^+_{\mu'}$ we get the first relation.

The claim involving the operators $C_p(-l'_{r+p,j})$ was established in the proof of Proposition 4.7. Similarly, the rational function $B_p(u)$ coincides with the image of the series

$$-\,t^{\,1,\dots,p}_{\,1,\dots,p-1,p+1}(u)\, t^{\,1,\dots,p+1}_{\,1,\dots,p+1}(u)^{-1}\,\frac{1}{u - h_1 - p}$$

under the homomorphism of Proposition 3.4. The image of this series in $\overline{L}(\lambda')^+_{\mu'}$ is given by

$$B_p(u) = -\big(u + \overline{E}\big)^{1,\dots,r+p}_{1,\dots,r+p-1,r+p+1} \cdot \Big[\big(u + \overline{E}\big)^{1,\dots,r+p+1}_{1,\dots,r+p+1}\Big]^{-1}.$$

Let $\mathcal{V}$ be the trapezium pattern corresponding to the supertableau $\mathcal{T}$. The formulas for the Yangian action in the Gelfand–Tsetlin basis of $\overline{L}(\lambda')^+_{\mu'}$ now give

$$B_p(-l'_{r+p,j})\,\zeta_{\mathcal{V}} = \zeta_{\mathcal{V}^+},$$

where $\mathcal{V}^+$ is the array obtained from $\mathcal{V}$ by replacing the entry $\lambda'_{r+p,j}$ by $\lambda'_{r+p,j} + 1$, and the vector $\zeta_{\mathcal{V}^+}$ is considered to be equal to zero if $\mathcal{V}^+$ is not a pattern; see [15, Sec. 5.4]. $\square$

Corollary 4.8 implies the following identities for the parameters of the initial tableau $\mathcal{T}^0$ of shape Γ_λ/μ.

Lemma 4.9. *For each* $p = 1, \dots, n$ *we have*

$$\frac{(u + l^0_{r+p,1} + p - 1) \dots (u + l^0_{r+p,r+p} + p - 1)}{(u + l^0_{r+p-1,1} + p - 1) \dots (u + l^0_{r+p-1,r+p-1} + p - 1)} = \frac{(u + \lambda_{m+p} + m)(u - r)}{u + m} \prod_{j=1,\ \lambda_j - \mu_j \geqslant p}^{m} \frac{u - \sigma_j + 1}{u - \sigma_j}.$$

Proof. Calculate $A_p(u+p-1)\,\zeta_{\mathcal{T}^0}$ in two different ways and compare the eigenvalues. First take $\mathcal{T} = \mathcal{T}^0$ in (4.20) and replace u by $u + p - 1$. This gives the left hand side of the equality. On the other hand, since $\zeta_{\mathcal{T}^0} = \zeta_\mu$ is the highest vector of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$, the eigenvalue of the operator $Z^{m+1,\dots,m+p}_{m+1,\dots,m+p}(u)$ on ζ_μ can be found from (4.3) and (4.11). Hence, the application of (4.15) yields the eigenvalue of $A_p(u + p - 1)$ in the form of the right hand side of the equality. □

Using the Gelfand–Tsetlin basis [7] of each representation $L'(\mu)$ of $\mathfrak{gl}_m$ with μ satisfying the assumptions of Corollary 4.3, and the basis $\zeta_{\mathcal{T}}$ of $L(\lambda)^+_\mu$ formed by the row-strict tableaux $\mathcal{T}$ of shape Γ_λ/μ, we will construct basis vectors of the vector spaces $L'(\mu) \otimes L(\lambda)^+_\mu$ occurring in (1.2). More precisely, consider the raising and lowering operators s_{ik} and s_{ki} for $k = 2, \dots, m$ and $i = 1, \dots, k - 1$ which are elements of $\mathrm{U}(\mathfrak{gl}_m)$ given by the explicit formulas analogous to (3.2) and (3.3),

$$s_{ik} = \sum_{i > i_1 > \dots > i_p \geqslant 1} E_{ii_1} E_{i_1 i_2} \dots E_{i_{p-1} i_p} E_{i_p k} (h_i - h_{j_1}) \dots (h_i - h_{j_r}),$$

$$s_{ki} = \sum_{i < i_1 < \dots < i_p < k} E_{i_1 i} E_{i_2 i_1} \dots E_{i_p i_{p-1}} E_{k i_p} (h_i - h_{j_1}) \dots (h_i - h_{j_r}),$$

where p runs over nonnegative integers, $h_i = E_{ii} - i + 1$ and $\{j_1, \dots, j_r\}$ is the complementary subset to $\{i_1, \dots, i_p\}$ in the set $\{1, \dots, i - 1\}$ or $\{i + 1, \dots, k - 1\}$, respectively. Note that the lowering operators s_{ki} and s_{kj} commute for any $i, j \in \{1, \dots, k - 1\}$.

Any supertableau Λ of shape Γ_λ uniquely determines a partition $\mu = (\mu_1, \dots, \mu_m)$ as the shape of the subtableau with entries in $\{1, \dots, m\}$, as well as the corresponding pattern $\mathcal{U}$ with entries λ_{ij}; see Sec. 2. Moreover, a row-strict tableau $\mathcal{T}$ is found as the subtableau of Λ with entries in $\{m + 1, \dots, m + n\}$. Define the vector of $L(\lambda)$ corresponding to Λ by

$$\zeta_\Lambda = \prod_{k=2,\dots,m}^{\longrightarrow} \Big(s_{k1}^{\lambda_{k1} - \lambda_{k-1,1}} \dots s_{k,k-1}^{\lambda_{k,k-1} - \lambda_{k-1,k-1}} \Big)\, \zeta_{\mathcal{T}}, \tag{4.23}$$

where $\zeta_{\mathcal{T}}$ is defined in (4.18). The following is implied by the properties of the Gelfand–Tsetlin bases; see e.g. [14] for a proof.

Proposition 4.10. *The action of the generators of the Lie subalgebra* $\mathfrak{gl}_m$ *of* $\mathfrak{gl}_{m|n}$ *on the vectors* ζ_Λ *of* $L(\lambda)$ *is given by formulas* (2.1) *with* $1 \leqslant s \leqslant m$ *and by formulas* (2.2), (2.3) *with* $1 \leqslant s \leqslant m-1$. □

Due to Propositions 4.7 and 4.10, in order to determine the action of all elements of $\mathfrak{gl}_{m|n}$ on the vectors ζ_Λ of $L(\lambda)$, it will be sufficient to find explicit expansions of $E_{m,m+1}\,\zeta_\Lambda$ and $E_{m+1,m}\,\zeta_\Lambda$ as linear combinations of these vectors. The following two lemmas are well-known in the classical case (i.e. for the Lie algebra $\mathfrak{gl}_{m+1}$) and their proofs are not essentially different in the super case; cf. [14, Sec. 2.3] and [28]. Here we regard the raising and lowering operators given by (3.2) and (3.3) as elements of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_{m|n})$.

Lemma 4.11. *The following relation holds in* $\mathrm{U}'(\mathfrak{gl}_{m|n})$,

$$E_{m,m+1} = \sum_{i=1}^{m} s_{mi}\, z_{i,m+1} \frac{1}{(h_i - h_1)\dots\wedge_i\dots(h_i - h_m)},$$

where $s_{mm} = 1$. □

Lemma 4.12. *For any nonnegative integers* $k_1, \dots, k_{m-1}$, *in* $\mathrm{U}'(\mathfrak{gl}_{m|n})$ *we have*

$$E_{m+1,m}\, s_{m1}^{k_1} \dots s_{m,m-1}^{k_{m-1}} = \sum_{i=1}^{m} s_{m1}^{k_1} \dots s_{mi}^{k_i - 1} \dots s_{m,m-1}^{k_{m-1}}\, z_{m+1,i} \frac{(h_i - h_1 + k_1)\dots(h_i - h_{m-1} + k_{m-1})}{(h_i - h_1)\dots\wedge_i\dots(h_i - h_m)}.$$

□

Observe that $E_{m,m+1}$ commutes with all lowering operators s_{ki}. Therefore, using (4.23) we get

$$E_{m,m+1}\,\zeta_\Lambda = \prod_{k=2,\dots,m}^{\longrightarrow} \Big(s_{k1}^{\lambda_{k1} - \lambda_{k-1,1}} \dots s_{k,k-1}^{\lambda_{k,k-1} - \lambda_{k-1,k-1}} \Big)\, E_{m,m+1}\,\zeta_{\mathcal{T}}. \tag{4.24}$$

By Lemma 4.11, to find the expansion of $E_{m,m+1}\,\zeta_\Lambda$ we need to calculate $z_{i,m+1}\,\zeta_{\mathcal{T}}$ in terms of the basis vectors of $L(\lambda)^+_{\mu+\delta_i}$. Similarly, $E_{m+1,m}$ commutes with the lowering operators s_{ki} for $k \leqslant m-1$. Hence, due to Lemma 4.12, to find the expansion of $E_{m+1,m}\,\zeta_\Lambda$ we need to calculate $z_{m+1,i}\,\zeta_{\mathcal{T}}$ in terms of the basis vectors of $L(\lambda)^+_{\mu-\delta_i}$.

Lemma 4.13. *For any* $p \in \{1, \dots, n-1\}$ *we have in* $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$:

$$B_p(u)\, z_{m+1,i} = z_{m+1,i}\, B_p(u), \qquad \text{for} \quad i \geqslant 2,$$

and

$$B_p(u)\, z_{m+1,1} = z_{m+1,1}\, B_p(u)\, \frac{u - h_1 - p}{u - h_1 - p + 1}.$$

Proof. It suffices to apply (4.13) to permute $z_{m+1,i}$ with $B_p(u)$, and use the relation $z_{m+1,i}\,(u-h_i-p+1)=(u-h_i-p)\,z_{m+1,i}$ for $i\geqslant 2$. □

Lemma 4.14. *We have the relations in* $\mathrm{Z}(\mathfrak{gl}_{m|n},\mathfrak{gl}_m)$*:*

$$\begin{aligned} z_{1,m+1}\,C_1(u)&=C_1(u)\,z_{1,m+1}\,\frac{u-h_1+1}{u-h_1-1},\\ z_{i,m+1}\,C_1(u)&=C_1(u)\,z_{i,m+1}\,\frac{u-h_i+1}{u-h_i},\qquad \text{if}\quad i\geqslant 2,\end{aligned}$$

and

$$\begin{aligned} z_{i,m+1}\,C_p(u)&=C_p(u)\,z_{i,m+1}, && \text{if}\quad i,p\geqslant 2,\\ z_{1,m+1}\,C_p(u)&=C_p(u)\,z_{1,m+1}\,\frac{u-h_1-p+1}{u-h_1-p}, && \text{if}\quad p\geqslant 2.\end{aligned}$$

Proof. All relations follow by the application of (4.14). □

Proposition 4.15. *For any* $i\in\{1,\dots,m\}$ *we have*

$$z_{i,m+1}\,\zeta_{\mathcal{T}}=b_{i,\mathcal{T}}\prod_{(p,j)}\Big(C_p(-l'_{r+p,j}-1)\dots C_p(-l^0_{r+p,j}+1)\,C_p(-l^0_{r+p,j})\Big)\,z_{i,m+1}\,\zeta_{\mu},\qquad (4.25)$$

where the product over the pairs (p,j) *is taken in the same ordering as in* (4.18) *and* $b_{i,\mathcal{T}}$ *is a constant given by*

$$b_{i,\mathcal{T}}=\prod_{p=1}^{n-1}\prod_{j=1}^{r+p}\frac{l'_{r+p,j}+\sigma_1+p}{l^0_{r+p,j}+\sigma_1+p}\cdot\prod_{j=1}^{r+1}\frac{l'_{r+1,j}+\sigma_1}{l^0_{r+1,j}+\sigma_1}\qquad \text{if}\quad i=1,\qquad (4.26)$$

and

$$b_{i,\mathcal{T}}=\prod_{j=1}^{r+1}\frac{l'_{r+1,j}+\sigma_i}{l^0_{r+1,j}+\sigma_i}\qquad \text{if}\quad i\geqslant 2.\qquad (4.27)$$

Proof. To verify (4.27), note that if $i\geqslant 2$, then by Lemma 4.14, $z_{i,m+1}$ is permutable with the operators of the form $C_k(u)$ with $k\geqslant 2$. Furthermore, the lemma implies that for $p\geqslant 0$

$$\begin{aligned} &z_{i,m+1}\,C_1(u+p-1)\dots C_1(u+1)\,C_1(u)\\ &\qquad=C_1(u+p-1)\dots C_1(u+1)\,C_1(u)\,z_{i,m+1}\,\frac{u-h_i+p}{u-h_i}\end{aligned}$$

which yields (4.27). Relation (4.26) is verified by a similar calculation with the use of Lemma 4.14. □

Due to Proposition 4.15, the calculation of $z_{i,m+1}\,\zeta_{\mathcal{T}}$ is reduced to expanding of $z_{i,m+1}\,\zeta_\mu$. If $\lambda_i-\mu_i=0$ for some $i\in\{1,\dots,m\}$, then $z_{i,m+1}\,\zeta_\mu=0$ by Corollary 4.2. On the other hand, if $k=\lambda_i-\mu_i\geqslant 1$, then applying (4.7) with μ replaced by $\mu+\delta_i$ we obtain

$$\zeta_\mu=c_{i,\mu}\,z_{m+k,i}\,\zeta_{\mu+\delta_i},\tag{4.28}$$

where

$$c_{i,\mu}=\prod_{j=1}^{i-1}(-1)^{\lambda_j-\mu_j}\,\frac{\sigma_i-l_j}{\sigma_i-\sigma_j}\prod_{j=1,\ \lambda_j-\mu_j\geqslant k}^{i-1}\frac{\sigma_i-\sigma_j}{\sigma_i-\sigma_j+1}.$$

Furthermore, using (3.10), we derive

$$z_{i,m+1}\,z_{m+k,i}\,\zeta_{\mu+\delta_i}=-Z_{m+k,m+1}(h_i-1)\,\zeta_{\mu+\delta_i}=-Z_{m+k,m+1}(\sigma_i)\,\zeta_{\mu+\delta_i}.$$

If $k=1$, then by (4.3), this vector equals $\zeta_{\mu+\delta_i}$ multiplied by a scalar. Now suppose that $k\geqslant 2$. In this case relations (3.4) imply that

$$Z_{m+k,m+1}(\sigma_i)=[E_{m+k,m+1},Z_{m+1,m+1}(\sigma_i)],$$

and so

$$z_{i,m+1}\,z_{m+k,i}\,\zeta_{\mu+\delta_i}=Z_{m+1,m+1}(\sigma_i)\,E_{m+k,m+1}\,\zeta_{\mu+\delta_i}\tag{4.29}$$

since $Z_{m+1,m+1}(\sigma_i)\,\zeta_{\mu+\delta_i}=0$ by (4.3). We will identify $\zeta_{\mu+\delta_i}$ with the basis vector $\zeta_{\mathcal{T}^0}$ of $L(\lambda)^+_{\mu+\delta_i}$ corresponding to the initial tableau $\mathcal{T}^0$ of shape $\Gamma_\lambda/(\mu+\delta_i)$, obtained by filling in the boxes of each row by the consecutive numbers $m+1,m+2,\dots$ from left to right. We suppose first that $i\geqslant 2$ and use the corresponding parameters $\lambda^0_{r+p,j}$ of $\mathcal{T}^0$ defined in (4.16) together with $l^0_{r+p,j}$ defined in (4.17).

As we showed in Proposition 4.7, the expansion of $E_{m+k,m+1}\,\zeta_{\mathcal{T}^0}$ in terms of the basis vectors $\zeta_{\mathcal{T}}$ can be found by the respective case of the formula (2.3). Writing

$$E_{m+k,m+1}=[E_{m+k,m+k-1},\dots,[E_{m+3,m+2},E_{m+2,m+1}]\dots]\tag{4.30}$$

we can see that for $i\geqslant 2$ the expansion of $E_{m+k,m+1}\,\zeta_{\mathcal{T}^0}$ will contain only those vectors $\zeta_{\mathcal{T}}$ for which exactly one of the parameters $\lambda^0_{r+1,1},\dots,\lambda^0_{r+1,r+1}$ is decreased by 1. Due to the subsequent application of the operator $Z_{m+1,m+1}(\sigma_i)$, there will be only one of such vectors occurring in the expansion of (4.29) with a nonzero coefficient. Indeed, using (4.20) with $p=1$ we derive that for any skew tableau $\mathcal{T}$ of shape $\Gamma_\lambda/(\mu+\delta_i)$,

$$Z_{m+1,m+1}(u)\,\zeta_{\mathcal{T}}=\frac{(u+l'_{r+1,1})\dots(u+l'_{r+1,r+1})}{(u+l'_{r,1})\dots(u+l'_{r,r})}\tag{4.31}$$
$$\times(u-\sigma_2)\dots(u-\sigma_i-1)\dots(u-\sigma_{m+1})\,\zeta_{\mathcal{T}}.$$

Set $s = \mu_i$ so that $\mu'_{s+1} = i-1$. If $\mathcal{T}$ occurs in the expansion of $E_{m+k,m+1}\,\zeta_{\mathcal{T}^0}$ and $\lambda'_{r+1,s+2} = \mu'_{s+1}+1$, then

$$l'_{r+1,s+2} = \lambda'_{r+1,s+2} - s - 1 = \mu'_{s+1} - s = i - 1 - \mu_i = -\sigma_i$$

and $Z_{m+1,m+1}(\sigma_i)\,\zeta_{\mathcal{T}} = 0$. Hence, if $Z_{m+1,m+1}(\sigma_i)\,\zeta_{\mathcal{T}} \neq 0$ then $\lambda'_{r+1,s+2} = \mu'_{s+1}$. The betweenness conditions then give

$$\lambda'_{r+1,s+2} = \dots = \lambda'_{r+k-1,s+k} = \mu'_{s+1},$$

while for the remaining parameters we have $\lambda'_{r+p,j} = \lambda^0_{r+p,j}$. This determines a unique tableau which we denote by $\mathcal{T}^+$. Thus,

$$z_{i,m+1}\, z_{m+k,i}\, \zeta_{\mu+\delta_i} = d_{i,\mu}\, \zeta_{\mathcal{T}^+}, \tag{4.32}$$

for a nonzero constant $d_{i,\mu}$ and any $k \geqslant 1$. To calculate its value, note that by (2.3) and (4.30) the coefficient of the basis vector $\zeta_{\mathcal{T}^+}$ in the expansion of $E_{m+k,m+1}\,\zeta_{\mu+\delta_i}$ coincides with the coefficient of this vector in the expansion of

$$(-1)^k\, E_{m+2,m+1}\, E_{m+3,m+2} \dots E_{m+k,m+k-1}\, \zeta_{\mu+\delta_i}$$

and it is found by the formula

$$(-1)^k \prod_{p=1}^{k-1} \frac{(\sigma_i + l^0_{r+p-1,1} + p - 1) \dots (\sigma_i + l^0_{r+p-1,r+p-1} + p - 1)}{(\sigma_i + l^0_{r+p,1} + p - 1) \dots \wedge_{s+p+1} \dots (\sigma_i + l^0_{r+p,r+p} + p - 1)}. \tag{4.33}$$

We now use Lemma 4.9 (applied to $\Gamma_\lambda/(\mu+\delta_i)$ instead of Γ_λ/μ) to write this coefficient in a different form. Divide both sides of the identity of the lemma by $u - \sigma_i$ and set $u = \sigma_i$. We get the identity

$$\frac{(\sigma_i + l^0_{r+p-1,1} + p - 1) \dots (\sigma_i + l^0_{r+p-1,r+p-1} + p - 1)}{(\sigma_i + l^0_{r+p,1} + p - 1) \dots \wedge_{s+p+1} \dots (\sigma_i + l^0_{r+p,r+p} + p - 1)}$$
$$= -\frac{\sigma_i + m}{(\sigma_i - r)(\sigma_i + \lambda_{m+p} + m)} \prod_{j=1,\ j\neq i,\ \lambda_j - \mu_j \geqslant p}^{m} \frac{\sigma_i - \sigma_j}{\sigma_i - \sigma_j + 1}.$$

Taking the product over p, we find that the coefficient (4.33) equals

$$-\Big(\frac{\sigma_i + m}{\sigma_i - r}\Big)^{k-1} \prod_{p=1}^{k-1} \frac{1}{\sigma_i + \lambda_{m+p} + m} \prod_{j=1,\ j\neq i}^{m} \Big(\frac{\sigma_i - \sigma_j}{\sigma_i - \sigma_j + 1}\Big)^{\min\{\lambda_j - \mu_j,\, k-1\}}.$$

Hence, applying (4.31) for $u = \sigma_i$ and $\mathcal{T} = \mathcal{T}^+$, and combining this with (4.28), we conclude that in the case under consideration,

$$z_{i,m+1}\, \zeta_\mu = g_{i,\mu}\, \zeta_{\mathcal{T}^+}, \tag{4.34}$$

with

$$g_{i,\mu}=\prod_{j=1}^{i-1}(-1)^{\lambda_j-\mu_j}\,\frac{\sigma_i-l_j}{\sigma_i-\sigma_j}\prod_{j=1,\ \lambda_j-\mu_j\geqslant k}^{i-1}\frac{\sigma_i-\sigma_j}{\sigma_i-\sigma_j+1}$$

$$\times\Big(\frac{\sigma_i+m}{\sigma_i-r}\Big)^{k-1}\prod_{p=1}^{k-1}\frac{1}{\sigma_i+\lambda_{m+p}+m}\prod_{j=1,\ j\neq i}^{m}\Big(\frac{\sigma_i-\sigma_j}{\sigma_i-\sigma_j+1}\Big)^{\min\{\lambda_j-\mu_j,\,k-1\}}$$

$$\times\frac{(\sigma_i+l^0_{r+1,1})\dots(\sigma_i+l^0_{r+1,r+1})}{(\sigma_i+l^0_{r,1})\dots\wedge_{s+1}\dots(\sigma_i+l^0_{r,r})}\,(\sigma_i-\sigma_2)\dots\wedge_i\dots(\sigma_i-\sigma_{m+1}),$$

where the parameters $l^0_{r+p,j}$ are now associated with the initial tableau $\mathcal{T}^0$ of shape Γ_λ/μ via (4.16) and (4.17). The same formula for $g_{i,\mu}$ clearly remains valid in the case $k=1$ as well.

To extend the above calculation to the case $i=1$ we need to take into account the fact that the parameter $r=\mu_1$ changes to $r+1$ for the partition $\mu+\delta_1$. This time (4.31) is replaced by the relation

$$Z_{m+1,m+1}(u)\,\zeta_{\mathcal{T}}=\frac{(u+l'_{r+2,1})\dots(u+l'_{r+2,r+2})}{(u+l'_{r+1,1})\dots(u+l'_{r+1,r+1})}\,(u-\sigma_2)\dots(u-\sigma_{m+1})\,\zeta_{\mathcal{T}},\qquad(4.35)$$

where $\mathcal{T}$ is any skew tableau of shape $\Gamma_\lambda/(\mu+\delta_1)$. We will use the parameters $\lambda^0_{r+p,j}$ and $l^0_{r+p,j}$ associated with the initial tableau of shape Γ_λ/μ by (4.16) and (4.17). Relation (4.32) holds for $i=1$ as well, where the parameters of the tableau $\mathcal{T}^+$ of shape $\Gamma_\lambda/(\mu+\delta_1)$ are given by

$$\lambda'_{r+p+1,j}=\lambda^0_{r+p,j}\qquad\text{for}\quad p=0,\dots,n\quad\text{and}\quad j=1,\dots,r+p,\qquad(4.36)$$

while $\lambda'_{r+1,r+1}=1$ and $\lambda'_{r+p,r+p}=0$ for $p\geqslant 2$. The coefficient of the basis vector $\zeta_{\mathcal{T}^+}$ in the expansion of $E_{m+k,m+1}\,\zeta_{\mu+\delta_1}$ coincides with the coefficient of this vector in the expansion of

$$(-1)^k\,E_{m+2,m+1}\,E_{m+3,m+2}\dots E_{m+k,m+k-1}\,\zeta_{\mu+\delta_1}$$

and it is found by the formula

$$(-1)^k\prod_{p=1}^{k-1}\frac{(r+l^0_{r+p-1,1}+p-1)\dots(r+l^0_{r+p-1,r+p-1}+p-1)}{(r+l^0_{r+p,1}+p-1)\dots(r+l^0_{r+p,r+p}+p-1)}.\qquad(4.37)$$

Lemma 4.9 now gives

$$\frac{(r+l^0_{r+p-1,1}+p-1)\dots(r+l^0_{r+p-1,r+p-1}+p-1)}{(r+l^0_{r+p,1}+p-1)\dots(r+l^0_{r+p,r+p}+p-1)}$$

$$=\frac{r+m}{r+\lambda_{m+p}+m}\prod_{j=2,\ \lambda_j-\mu_j\geqslant p}^{m}\frac{r-\sigma_j}{r-\sigma_j+1},$$

so that taking the product over p, we find that the coefficient (4.37) equals

$$(-1)^k\,(r+m)^{k-1}\prod_{p=1}^{k-1}\frac{1}{r+\lambda_{m+p}+m}\,\prod_{j=2}^{m}\Big(\frac{r-\sigma_j}{r-\sigma_j+1}\Big)^{\min\{\lambda_j-\mu_j,\,k-1\}}.$$

Therefore, applying (4.35) for $u=r$ and $\mathcal{T}=\mathcal{T}^+$, we conclude that (4.34) holds for $i=1$ with

$$g_{1,\mu}=(-1)^{k-1}\,(r+m)^{k-1}\prod_{p=1}^{k-1}\frac{1}{r+\lambda_{m+p}+m}\,\prod_{j=2}^{m}\Big(\frac{r-\sigma_j}{r-\sigma_j+1}\Big)^{\min\{\lambda_j-\mu_j,\,k-1\}}$$

$$\times\,\frac{(r+l^0_{r+1,1})\dots(r+l^0_{r+1,r+1})}{(r+l^0_{r,1})\dots(r+l^0_{r,r})}\,(r-\sigma_2)\dots(r-\sigma_{m+1}),$$

which is valid for $k\geqslant 1$.

Combining the above calculation with Proposition 4.15 we come to the following.

Proposition 4.16. *Suppose that for some* $i\in\{1,\dots,m\}$ *the following condition holds:* $\mu+\delta_i$ *is a partition, and a row-strict tableau* $\mathcal{T}$ *of shape* Γ_λ/μ *contains the entry* $m+1$ *in the box* (i,μ_i+1)*. Then*

$$z_{i,m+1}\,\zeta_{\mathcal{T}}=b_{i,\mathcal{T}}\,g_{i,\mu}\,\zeta_{\mathcal{T}_i^+},\tag{4.38}$$

where $\mathcal{T}_i^+$ *is the tableau obtained from* $\mathcal{T}$ *by removing the entry* $m+1$ *from the box* (i,μ_i+1)*, and the coefficients* $b_{i,\mathcal{T}}$ *and* $g_{i,\mu}$ *are nonzero. Moreover, if the condition does not hold, then* $z_{i,m+1}\,\zeta_{\mathcal{T}}=0$.

Proof. If the condition holds, then the claim follows from the formulas (4.21) and the definition (4.18) of the vectors $\zeta_{\mathcal{T}}$ together with (4.25) and (4.34). Now suppose that $\mu+\delta_i$ is not a partition. If the vector $\xi=z_{i,m+1}\,\zeta_{\mathcal{T}}$ were nonzero, its cyclic span $\mathrm{U}(\mathfrak{gl}_m)\,\xi$ would be a highest weight $\mathfrak{gl}_m$-module with the highest weight $\mu+\delta_i$. Since $\mu+\delta_i$ is not a partition, this module must be infinite-dimensional, a contradiction.

Furthermore, if $\mu+\delta_i$ is a partition, then the condition will be violated if the box (i,μ_i+1) is outside the diagram Γ_λ/μ. In this case $\lambda_i=\mu_i$ and the claim follows from Corollary 4.2, as $z_{i,m+1}\,\zeta_\mu=0$. Finally, let the diagram Γ_λ/μ contain the box (i,μ_i+1) with the entry of $\mathcal{T}$ in this box greater than $m+1$. Set $j=\mu_i+1$. Then all entries in column j of the tableau $\mathcal{T}^+$ occurring in (4.34) should also exceed $m+1$. On the other hand, the product in (4.18) defining the vector $\zeta_{\mathcal{T}}$ contains the factor $C_1(-l'_{r+1,j}-1)\dots C_1(-l^0_{r+1,j})$. Hence, the operator $C_1(-l'_{r+1,j}-1)$ in (4.25) will have to be applied to a vector of the form $\zeta_{\mathcal{T}'}$, where $\mathcal{T}'$ is a tableau of shape $\Gamma_\lambda/(\mu+\delta_i)$ whose entries in column j all exceed $m+1$. The result of this application is zero by Corollary 4.8. □

Proposition 4.16 together with Lemma 4.11 and relation (4.24) provide explicit formulas for the coefficients in the expansion of $E_{m,m+1}\,\zeta_\Lambda$ as a linear combination of the basis vectors. We will express these coefficients in terms of the parameters of Λ in Theorem 4.18 below.

We now turn to the calculation of $z_{m+1,i}\,\zeta_{\mathcal{T}}$ for an arbitrary row-strict tableau $\mathcal{T}$ of shape Γ_λ/μ. Suppose that the condition of Proposition 4.16 holds. Apply $z_{m+1,i}$ to both sides of (4.38). Using (3.9), we get

$$b_{i,\mathcal{T}}\, g_{i,\mu}\, z_{m+1,i}\, \zeta_{\mathcal{T}_i^+} = z_{m+1,i}\, z_{i,m+1}\, \zeta_{\mathcal{T}} = Z_{m+1,m+1}(h_i)\, \zeta_{\mathcal{T}} = Z_{m+1,m+1}(\sigma_i)\, \zeta_{\mathcal{T}} \tag{4.39}$$

which equals $\zeta_{\mathcal{T}}$ multiplied by a scalar found for $i = 1$ and $i \geqslant 2$ from (4.35) and (4.31), respectively. This allows us to calculate $z_{m+1,i}\,\zeta_{\mathcal{T}_i^+}$.

Proposition 4.17. *Suppose that for some $i \in \{1, \dots, m\}$ the following condition holds: $\mu - \delta_i$ is a partition and a row-strict tableau $\mathcal{T}$ of shape Γ_λ/μ does not contain the entry $m+1$ in the box $(i, \mu_i + 1)$. Then*

$$z_{m+1,i}\, \zeta_{\mathcal{T}} = f_{i,\mathcal{T}}\, \zeta_{\mathcal{T}_i^-}, \tag{4.40}$$

where $\mathcal{T}_i^-$ is the tableau obtained from $\mathcal{T}$ by adding the entry $m+1$ in the box (i, μ_i), and $f_{i,\mathcal{T}}$ is a constant. If the condition does not hold, then $z_{m+1,i}\,\zeta_{\mathcal{T}} = 0$.

Proof. The first part of the proposition follows from (4.39). Now suppose that the condition is violated and $\mu - \delta_i$ is not a partition. If $i < m$ then the claim follows by the same argument as in the proof of Proposition 4.16 by considering the $\mathrm{U}(\mathfrak{gl}_m)$-cyclic span of the vector $z_{m+1,i}\,\zeta_{\mathcal{T}}$. If $i = m$ then we must have $\mu_m = 0$. We will show that

$$z_{m+1,m}\, \zeta_{\mathcal{T}} = 0 \tag{4.41}$$

for all row-strict tableaux $\mathcal{T}$ of skew shapes Γ_λ/μ with μ running over the partitions with $\mu_m = 0$. Using (4.20) with $p = 1$ we derive that

$$Z_{m+1,m+1}(u)\, \zeta_{\mathcal{T}} = \frac{(u + l'_{r+1,1}) \dots (u + l'_{r+1,r+1})}{(u + l'_{r,1}) \dots (u + l'_{r,r})} \cdot (u - \sigma_2) \dots (u - \sigma_{m+1})\, \zeta_{\mathcal{T}}. \tag{4.42}$$

Therefore, by (3.10) we have

$$z_{m,m+1}\, z_{m+1,m}\, \zeta_{\mathcal{T}} = -Z_{m+1,m+1}(h_m - 1)\, \zeta_{\mathcal{T}} = -Z_{m+1,m+1}(-m)\, \zeta_{\mathcal{T}} = 0, \tag{4.43}$$

where we used the assumption $\mu_m = 0$ to conclude that all parameters l'_{rj} in (4.42) do not exceed $m-1$, while $\sigma_{m+1} = -m$. We proceed by induction on the weights $\omega(\mathcal{T})$ of the vectors $\zeta_{\mathcal{T}}$. The base of induction is the case $\zeta_{\mathcal{T}} = \zeta_\mu$ with $\mu = (\lambda_1, \dots, \lambda_{m-1}, 0)$ so that

$$\omega(\mathcal{T}) = (\lambda_1, \dots, \lambda_{m-1}, 0 \,|\, \lambda_{m+1} + 1, \dots, \lambda_{m+k} + 1, \lambda_{m+k+1}, \dots, \lambda_{m+n}),$$

where $k = \lambda_m$. In this case $z_{m+1,m}\,\zeta_\mu = z_{m+1,m}\,z_{m+k,m}\dots z_{m+1,m}\,\zeta = 0$ since $z^2_{m+1,m} = 0$ and $z_{m+1,m}\,z_{m+p,m} = -z_{m+1,m}\,z_{m+p,m}$ by (3.5). Now, given an arbitrary $\mathcal{T}$ we will show that the vector $z_{m+1,m}\,\zeta_{\mathcal{T}}$ is annihilated by all operators $E_{a,a+1}$ with $a = 1,\dots,m+n-1$. Indeed, this is clear for $a = 1,\dots,m-1$. Furthermore, each operator $E_{a,a+1}$ with $a > m$ commutes with $z_{m+1,m}$ so that

$$E_{a,a+1}\,z_{m+1,m}\,\zeta_{\mathcal{T}} = z_{m+1,m}\,E_{a,a+1}\,\zeta_{\mathcal{T}}$$

which is zero by the induction hypothesis, since $E_{a,a+1}\,\zeta_{\mathcal{T}}$ is a linear combination of the vectors $\zeta_{\mathcal{T}'}$ with the weights $\omega(\mathcal{T}')$ exceeding $\omega(\mathcal{T})$. To calculate $E_{m,m+1}\,z_{m+1,m}\,\zeta_{\mathcal{T}}$ use Lemma 4.11. Using (3.6) we find that for $i < m$

$$z_{i,m+1}\,z_{m+1,m}\,\zeta_{\mathcal{T}} = -z_{m+1,m}\,z_{i,m+1}\,\zeta_{\mathcal{T}}$$

which is zero by Proposition 4.16 and the induction hypothesis. Finally, the case $i = m$ is taken care of by (4.43). Thus, if a vector $z_{m+1,m}\,\zeta_{\mathcal{T}}$ were nonzero, it would generate a proper nonzero submodule of $L(\lambda)$. This is a contradiction since $L(\lambda)$ is irreducible. This proves (4.41).

It remains to show that $z_{m+1,i}\,\zeta_{\mathcal{T}} = 0$ in the case where $\mu - \delta_i$ is a partition and a row-strict tableau $\mathcal{T}$ of shape Γ_λ/μ contains the entry $m+1$ in the box (i,μ_i+1). By the first part of the proposition, the vector $\zeta_{\mathcal{T}}$ can be written as $\zeta_{\mathcal{T}} = c\,z_{m+1,i}\,\zeta_{\overline{\mathcal{T}}}$, where c is a constant and the tableau $\overline{\mathcal{T}}$ is obtained from $\mathcal{T}$ by removing the entry $m+1$ in the box (i,μ_1+1). Then $z_{m+1,i}\,\zeta_{\mathcal{T}} = 0$ since $z^2_{m+1,i} = 0$. □

We will now use Propositions 4.16 and 4.17 to produce a basis of $L(\lambda)$ and to prove the main theorem. Recall the vectors ζ_Λ of $L(\lambda)$ constructed in (4.23). As before, to each supertableau Λ of shape Γ_λ we associate a partition $\mu = (\mu_1,\dots,\mu_m)$ as the shape of the subtableau with entries in $\{1,\dots,m\}$. The parameters $l^0_{r+p,j}$ of the initial tableau $\mathcal{T}^0$ of shape Γ_λ/μ are defined in (4.17), where $r = r(\Lambda)$ is defined by $r = \lambda_{m1} = \mu_1$.

Theorem 4.18. *The vectors ζ_Λ parameterized by all supertableaux Λ of shape Γ_λ form a basis of the representation $L(\lambda)$ of $\mathfrak{gl}_{m|n}$. Moreover, the action of the generators of $\mathfrak{gl}_{m|n}$ is given by* (2.1)–(2.3) *together with the formulas*

$$E_{m,m+1}\,\zeta_\Lambda = \sum_{\Lambda'} c_{\Lambda\Lambda'}\,\zeta_{\Lambda'}, \tag{4.44}$$

$$E_{m+1,m}\,\zeta_\Lambda = \sum_{\Lambda'} d_{\Lambda\Lambda'}\,\zeta_{\Lambda'}, \tag{4.45}$$

where the sums are taken over supertableaux Λ' obtained from Λ respectively by replacing an entry $m+1$ by m and by replacing an entry m by $m+1$. The coefficients are found by *the formulas*

$$c_{\Lambda\Lambda'} = \frac{(l_{mi}+l'_{r+1,1})\dots(l_{mi}+l'_{r+1,r+1})}{(l_{mi}+l^0_{r,1})\dots\wedge_{\lambda_{mi}+1}\dots(l_{mi}+l^0_{r,r})}\left(\frac{l_{mi}+m}{l_{mi}-l_{m1}}\right)^k$$

$$\times\prod_{p=1}^{k-1}\frac{1}{l_{mi}+\lambda_{m+p}+m}\prod_{j=1}^{i-1}(-1)^{\lambda_j-\lambda_{mj}}\frac{l_{mi}-l_j}{l_{mi}-l_{mj}}$$

$$\times\prod_{j=1,\ \lambda_j-\lambda_{mj}\geqslant k}^{i-1}\frac{l_{mi}-l_{mj}}{l_{mi}-l_{mj}+1}\prod_{j=1,\ j\neq i}^{m}\left(\frac{l_{mi}-l_{mj}}{l_{mi}-l_{mj}+1}\right)^{\min\{\lambda_j-\lambda_{mj},\,k-1\}},$$

and

$$d_{\Lambda\Lambda'} = \frac{(l_{mi}-l_{m-1,1})\dots(l_{mi}-l_{m-1,m-1})}{(l_{mi}-l_{m1})\dots\wedge_i\dots(l_{mi}-l_{mm})}\left(\frac{l_{mi}-l_{m1}-1}{l_{mi}+m-1}\right)^k$$

$$\times\prod_{p=1}^{k}(l_{mi}+\lambda_{m+p}+m-1)\prod_{j=1}^{i-1}(-1)^{\lambda_j-\lambda_{mj}}\frac{l_{mi}-l_{mj}-1}{l_{mi}-l_j-1}$$

$$\times\prod_{j=1,\ \lambda_j-\lambda_{mj}>k}^{i-1}\frac{l_{mi}-l_{mj}}{l_{mi}-l_{mj}-1}\prod_{j=1,\ j\neq i}^{m}\left(\frac{l_{mi}-l_{mj}}{l_{mi}-l_{mj}-1}\right)^{\min\{\lambda_j-\lambda_{mj},\,k\}},$$

where the replacement occurs in row $i\geqslant 2$ and $k=\lambda_i-\lambda_{mi}$;

$$c_{\Lambda\Lambda'} = (-1)^{k-1}(l_{m1}+m)^k\,\frac{(l_{m1}+l'_{r+1,1})\dots(l_{m1}+l'_{r+1,r+1})}{(l_{m1}+l^0_{r,1})\dots(l_{m1}+l^0_{r,r})}\prod_{p=1}^{k-1}\frac{1}{l_{m1}+\lambda_{m+p}+m}$$

$$\times\prod_{p=1}^{n-1}\prod_{j=1}^{r+p}\frac{l_{m1}+l'_{r+p,j}+p}{l_{m1}+l^0_{r+p,j}+p}\prod_{j=2}^{m}\left(\frac{l_{m1}-l_{mj}}{l_{m1}-l_{mj}+1}\right)^{\min\{\lambda_j-\lambda_{mj},\,k-1\}},$$

and

$$d_{\Lambda\Lambda'} = \frac{(-1)^k}{(l_{m1}+m-1)^k}\,\frac{(l_{m1}-l_{m-1,1})\dots(l_{m1}-l_{m-1,m-1})}{(l_{m1}-l_{m2})\dots(l_{m1}-l_{mm})}$$

$$\times\prod_{p=1}^{k}(l_{m1}+\lambda_{m+p}+m-1)\prod_{j=2}^{m}\left(\frac{l_{m1}-l_{mj}}{l_{m1}-l_{mj}-1}\right)^{\min\{\lambda_j-\lambda_{mj},\,k\}}$$

$$\times\prod_{p=1}^{n-1}\prod_{j=1}^{r+p-1}\frac{l_{m1}+l^0_{r+p,j}+p-1}{l_{m1}+l'_{r+p,j}+p-1},$$

where the replacement occurs in row 1 *and* $k = \lambda_1 - \lambda_{m1}$.

Proof. Consider the subspace K of $L(\lambda)$, spanned by all vectors ζ_Λ. By Propositions 4.7 and 4.10 the subspace K is invariant with respect to the action of the subalgebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ of $\mathfrak{gl}_{m|n}$. Moreover, Lemmas 4.11 and 4.12 together with Propositions 4.16 and 4.17 imply that K is also invariant with respect to the action of the elements $E_{m,m+1}$ and $E_{m+1,m}$. Hence, K is a $\mathfrak{gl}_{m|n}$-submodule of $L(\lambda)$. Since K contains the highest vector ζ, and $L(\lambda)$ is irreducible we can conclude that $K = L(\lambda)$.

Furthermore, for each partition $\mu = (\mu_1, \dots, \mu_m)$ such that $0 \leqslant \lambda_i - \mu_i \leqslant n$ for all i, the vectors $\zeta_{\mathcal{T}}$ parameterized by skew tableaux $\mathcal{T}$ of shape Γ_λ/μ form a basis of the vector space $L(\lambda)^+_\mu$. For any fixed $\mathcal{T}$ the vectors of the form (4.23) parameterized by the column-strict μ-tableaux with entries in $\{1, \dots, m\}$ form a basis of the $\mathfrak{gl}_m$-module $L'(\mu)$. This implies that the vectors ζ_Λ are linearly independent and hence form a basis of $L(\lambda)$.

We will now calculate the expansions $E_{m,m+1}\,\zeta_\Lambda$ and $E_{m+1,m}\,\zeta_\Lambda$ as linear combinations of the basis vectors. Taking into account the denominator in the relation of Lemma 4.11 and using (4.34) for $i \geqslant 2$ we get

$$\frac{g_{i,\mu}}{(\sigma_i - \sigma_1) \dots \wedge_i \dots (\sigma_i - \sigma_m)} = \prod_{j=1}^{i-1} (-1)^{\lambda_j - \mu_j} \frac{\sigma_i - l_j}{\sigma_i - \sigma_j} \prod_{j=1,\ \lambda_j - \mu_j \geqslant k}^{i-1} \frac{\sigma_i - \sigma_j}{\sigma_i - \sigma_j + 1}$$

$$\times \Big(\frac{\sigma_i + m}{\sigma_i - r}\Big)^k \prod_{p=1}^{k-1} \frac{1}{\sigma_i + \lambda_{m+p} + m} \prod_{j=1,\ j\neq i}^{m} \Big(\frac{\sigma_i - \sigma_j}{\sigma_i - \sigma_j + 1}\Big)^{\min\{\lambda_j - \mu_j,\, k-1\}}$$

$$\times \frac{(\sigma_i + l^0_{r+1,1}) \dots (\sigma_i + l^0_{r+1,r+1})}{(\sigma_i + l^0_{r,1}) \dots \wedge_{s+1} \dots (\sigma_i + l^0_{r,r})},$$

where $s = \mu_i$. The calculation is completed by applying Proposition 4.15 with formula (4.27). Similarly, using the case $i = 1$ of (4.34), we find that

$$\frac{g_{1,\mu}}{(r - \sigma_2) \dots (r - \sigma_m)} = (-1)^{k-1}\, (r + m)^k \prod_{p=1}^{k-1} \frac{1}{r + \lambda_{m+p} + m}$$

$$\times \frac{(r + l^0_{r+1,1}) \dots (r + l^0_{r+1,r+1})}{(r + l^0_{r,1}) \dots (r + l^0_{r,r})} \prod_{j=2}^{m} \Big(\frac{r - \sigma_j}{r - \sigma_j + 1}\Big)^{\min\{\lambda_j - \mu_j,\, k-1\}}.$$

The calculation is completed by applying Proposition 4.15 with (4.26) thus proving the formulas for $c_{\Lambda\Lambda'}$ in (4.44).

To prove the formula for the coefficients $d_{\Lambda\Lambda'}$ in (4.45), apply $z_{m+1,i}$ with $i \geqslant 2$ to both sides of (4.34) and use (3.9) to get

$$g_{i,\mu}\, z_{m+1,i}\, \zeta_{\mathcal{T}^+} = z_{m+1,i}\, z_{i,m+1}\, \zeta_\mu = Z_{m+1,m+1}(h_i)\, \zeta_\mu = Z_{m+1,m+1}(\sigma_i)\, \zeta_\mu$$

so that

$$z_{m+1,i}\,\zeta_{\mathcal{T}^+}=g_{i,\mu}^{-1}\,Z_{m+1,m+1}(\sigma_i)\,\zeta_\mu.$$

Then (4.42) implies

$$Z_{m+1,m+1}(\sigma_i)\,\zeta_\mu=\frac{(\sigma_i+l^0_{r+1,1})\dots(\sigma_i+l^0_{r+1,r+1})}{(\sigma_i+l^0_{r,1})\dots\wedge_{s+1}\dots(\sigma_i+l^0_{r,r})}\cdot(\sigma_i-\sigma_2)\dots\wedge_i\dots(\sigma_i-\sigma_{m+1})\,\zeta_\mu,$$

where we identify ζ_μ with the vector $\zeta_{\mathcal{T}^0}$ associated with the initial tableau $\mathcal{T}^0$ of the skew shape Γ_λ/μ. Using the formula for $g_{i,\mu}$ we get

$$\begin{aligned}z_{m+1,i}\,\zeta_{\mathcal{T}^+}=&\Big(\frac{\sigma_i-r}{\sigma_i+m}\Big)^{k-1}\prod_{p=1}^{k-1}(\sigma_i+\lambda_{m+p}+m)\prod_{j=1,\ j\neq i}^{m}\Big(\frac{\sigma_i-\sigma_j+1}{\sigma_i-\sigma_j}\Big)^{\min\{\lambda_j-\mu_j,\,k-1\}}\\&\times\prod_{j=1}^{i-1}(-1)^{\lambda_j-\mu_j}\,\frac{\sigma_i-\sigma_j}{\sigma_i-l_j}\prod_{j=1,\ \lambda_j-\mu_j\geqslant k}^{i-1}\frac{\sigma_i-\sigma_j+1}{\sigma_i-\sigma_j}\,\zeta_\mu.\end{aligned}$$

Replace μ by $\mu-\delta_i$ so that k will be replaced by $k+1$ and $\mathcal{T}^+$ will be associated with μ. Thus, for $k\geqslant 0$ we have

$$\begin{aligned}z_{m+1,i}\,\zeta_{\mathcal{T}^+}=&\Big(\frac{\sigma_i-r-1}{\sigma_i+m-1}\Big)^{k}\prod_{p=1}^{k}(\sigma_i+\lambda_{m+p}+m-1)\prod_{j=1,\ j\neq i}^{m}\Big(\frac{\sigma_i-\sigma_j}{\sigma_i-\sigma_j-1}\Big)^{\min\{\lambda_j-\mu_j,\,k\}}\\&\times\prod_{j=1}^{i-1}(-1)^{\lambda_j-\mu_j}\,\frac{\sigma_i-\sigma_j-1}{\sigma_i-l_j-1}\prod_{j=1,\ \lambda_j-\mu_j>k}^{i-1}\frac{\sigma_i-\sigma_j}{\sigma_i-\sigma_j-1}\,\zeta_{\mu-\delta_i}.\end{aligned}$$

Now we use (4.22) and the first relation of Lemma 4.13 to calculate $z_{m+1,i}\,\zeta_{\mathcal{T}}$ for an arbitrary skew tableau of shape Γ_λ/μ. Applying appropriate operators of the form $B_p(-l'_{r+p,j})$ to both sides of (4.40), we conclude that the coefficient $f_{i,\mathcal{T}}$ coincides with $f_{i,\mathcal{T}^+}$, where the tableau $\mathcal{T}^+$ is associated with μ as above. Together with Lemma 4.12 this proves the formula for the coefficient $d_{\Lambda\Lambda'}$ in (4.45), for $i\geqslant 2$.

Finally, consider the case $i=1$ of the formula (4.45). Apply $z_{m+1,1}$ to both sides of (4.34) with $i=1$ and use (3.9) to get

$$g_{1,\mu}\,z_{m+1,1}\,\zeta_{\mathcal{T}^+}=z_{m+1,1}\,z_{1,m+1}\,\zeta_\mu=Z_{m+1,m+1}(h_1)\,\zeta_\mu=Z_{m+1,m+1}(r)\,\zeta_\mu$$

so that

$$z_{m+1,1}\,\zeta_{\mathcal{T}^+}=g_{1,\mu}^{-1}\,Z_{m+1,m+1}(r)\,\zeta_\mu.$$

Now (4.42) gives

$$Z_{m+1,m+1}(r)\,\zeta_\mu=\frac{(r+l^0_{r+1,1})\dots(r+l^0_{r+1,r+1})}{(r+l^0_{r,1})\dots(r+l^0_{r,r})}\cdot(r-\sigma_2)\dots(r-\sigma_{m+1})\,\zeta_\mu,$$

where we identify ζ_μ with the vector $\zeta_{\mathcal{T}^0}$ associated with the initial tableau $\mathcal{T}^0$ of the skew shape Γ_λ/μ. Hence, using the above formula for $g_{1,\mu}$ we find that

$$z_{m+1,1}\,\zeta_{\mathcal{T}^+} = \frac{(-1)^{k-1}}{(r+m)^{k-1}} \prod_{p=1}^{k-1}(r+\lambda_{m+p}+m) \prod_{j=2}^{m} \Big(\frac{r-\sigma_j+1}{r-\sigma_j}\Big)^{\min\{\lambda_j-\mu_j,\,k-1\}} \zeta_\mu,$$

where $\mathcal{T}^+$ is the tableau defined in (4.36). We need to replace μ by $\mu-\delta_1$ so that r is replaced by $r-1$ and $k=\lambda_1-r$ is replaced by $k+1$. Assuming that $\mathcal{T}^+$ is now associated with $\mu-\delta_1$, the above relation takes the form

$$z_{m+1,1}\,\zeta_{\mathcal{T}^+} = \frac{(-1)^{k}}{(r+m-1)^{k}} \prod_{p=1}^{k}(r+\lambda_{m+p}+m-1) \prod_{j=2}^{m} \Big(\frac{r-\sigma_j}{r-\sigma_j-1}\Big)^{\min\{\lambda_j-\mu_j,\,k\}} \zeta_{\mu-\delta_1}.$$

To calculate $z_{m+1,1}\,\zeta_{\mathcal{T}}$ for an arbitrary skew tableau of shape Γ_λ/μ, we use (4.22) and the second relation of Lemma 4.13. Applying appropriate operators of the form $B_p(-l'_{r+p,j})$ to both sides of (4.40), we find that

$$f_{1,\mathcal{T}} = f_{1,\mathcal{T}^+} \prod_{p=1}^{n-1} \prod_{j=1}^{r+p-1} \frac{r+l^0_{r+p,j}+p-1}{r+l'_{r+p,j}+p-1},$$

where the parameters $l^0_{r+p,j}$ and $l'_{r+p,j}$ are associated with the tableaux $\mathcal{T}^0$ and $\mathcal{T}$, respectively. This completes the proof of (4.45) and the theorem. □

Remark 4.19. Note that any normalization of the vectors ζ_μ with normalization constants depending only on μ (and λ) will only affect the formulas for the action of $E_{m,m+1}$ and $E_{m+1,m}$ and leave the formulas for the action of the generators of the subalgebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ of $\mathfrak{gl}_{m|n}$ in the new basis unchanged. This can be used to construct basis vectors $\xi_\Lambda = N(\Lambda)\,\zeta_\Lambda$ with simpler expansions of $E_{m,m+1}\,\xi_\Lambda$ and $E_{m+1,m}\,\xi_\Lambda$. □

Theorem 4.18 implies the formula for the character of $L(\lambda)$ defined in (2.4), which was originally found by Berele and Regev [3] and Sergeev [23]. Recall that given a Young diagram ρ, the corresponding supersymmetric Schur polynomial $s_\rho(x)$ in the variables $x=(x_1,\dots,x_m/x_{m+1},\dots,x_{m+n})$ is defined by the formula

$$s_\rho(x) = \sum_{\Lambda} \prod_{\alpha\in\rho} x_{T(\alpha)},$$

summed over the supertableaux Λ of shape ρ, where $T(\alpha)$ denotes the entry in the box α of the diagram ρ. An alternative expression for $s_\rho(x)$ is provided by the Sergeev–Pragacz formula; see e.g. [12, p. 61].

Corollary 4.20. *The character* $\operatorname{ch} L(\lambda)$ *coincides with the supersymmetric Schur polynomial* $s_{\Gamma_\lambda}(x_1,\dots,x_m/x_{m+1},\dots,x_{m+n})$ *associated with the Young diagram* Γ_λ. □

Example 4.21. In the case $n = 1$, the basis ζ_Λ of the representation $L(\lambda_1, \dots, \lambda_m \,|\, \lambda_{m+1})$ of the Lie superalgebra $\mathfrak{gl}_{m|1}$ can be parameterized by the patterns

$$\mathcal{U} = \begin{array}{ccccccccc} \lambda_{m1} & & \lambda_{m2} & & & \cdots & & & \lambda_{mm} \\ & \lambda_{m-1,1} & & & \cdots & & & \lambda_{m-1,m-1} & \\ & & \cdots & & \cdots & & \cdots & & \\ & & & \lambda_{21} & & \lambda_{22} & & & \\ & & & & \lambda_{11} & & & & \end{array}$$

see Sec. 2. The top row runs over partitions $(\lambda_{m1}, \dots, \lambda_{mm})$ such that either $\lambda_{mj} = \lambda_j$ or $\lambda_{mj} = \lambda_j - 1$ for each $j = 1, \dots, m$. The action of the generators E_{ss} with $s = 1, \dots, m$ and the generators $E_{s,s+1}$, $E_{s+1,s}$ with $s = 1, \dots, m-1$ on the basis vectors $\zeta_{\mathcal{U}}$ is given by the Gelfand–Tsetlin formulas

$$\begin{aligned} E_{ss}\, \zeta_{\mathcal{U}} &= \Big(\sum_{i=1}^{s} \lambda_{si} - \sum_{i=1}^{s-1} \lambda_{s-1,i}\Big)\, \zeta_{\mathcal{U}}, \\ E_{s,s+1}\, \zeta_{\mathcal{U}} &= -\sum_{i=1}^{s} \frac{(l_{si} - l_{s+1,1}) \dots (l_{si} - l_{s+1,s+1})}{(l_{si} - l_{s1}) \dots \wedge_i \dots (l_{si} - l_{ss})}\, \zeta_{\mathcal{U}+\delta_{si}}, \\ E_{s+1,s}\, \zeta_{\mathcal{U}} &= \sum_{i=1}^{s} \frac{(l_{si} - l_{s-1,1}) \dots (l_{si} - l_{s-1,s-1})}{(l_{si} - l_{s1}) \dots \wedge_i \dots (l_{si} - l_{ss})}\, \zeta_{\mathcal{U}-\delta_{si}}, \end{aligned}$$

where the array $\mathcal{U} \pm \delta_{si}$ is obtained from $\mathcal{U}$ by replacing λ_{si} with $\lambda_{si} \pm 1$, and $\zeta_{\mathcal{U}}$ is considered to be equal to zero if the array $\mathcal{U}$ is not a pattern. For the action of the generators $E_{m+1,m+1}$, $E_{m,m+1}$ and $E_{m+1,m}$ we have

$$E_{m+1,m+1}\, \zeta_{\mathcal{U}} = \Big(\sum_{i=1}^{m+1} \lambda_i - \sum_{i=1}^{m} \lambda_{mi}\Big)\, \zeta_{\mathcal{U}},$$

$$\begin{aligned} E_{m,m+1}\, \zeta_{\mathcal{U}} &= \sum_{i=1}^{m} (l_{mi} + \lambda_{m+1} + m) \\ &\times \prod_{j=1}^{i-1} (-1)^{\lambda_j - \lambda_{mj}}\, \frac{l_{mi} - l_j}{l_{mi} - l_{mj}} \prod_{j=i+1,\ \lambda_j - \lambda_{mj} = 1}^{m} \frac{l_{mi} - l_{mj} + 1}{l_{mi} - l_j + 1}\, \zeta_{\mathcal{U}+\delta_{mi}}, \end{aligned}$$

$$\begin{aligned} E_{m+1,m}\, \zeta_{\mathcal{U}} &= \sum_{i=1}^{m} \frac{(l_{mi} - l_{m-1,1}) \dots (l_{mi} - l_{m-1,m-1})}{(l_{mi} - l_{m1}) \dots \wedge_i \dots (l_{mi} - l_{mm})} \\ &\times \prod_{j=1}^{i-1} (-1)^{\lambda_j - \lambda_{mj}}\, \frac{l_{mi} - l_{mj} - 1}{l_{mi} - l_j - 1} \prod_{j=1,\ \lambda_j - \lambda_{mj} = 1}^{i-1} \frac{l_{mi} - l_{mj}}{l_{mi} - l_j}\, \zeta_{\mathcal{U}-\delta_{mi}}, \end{aligned}$$

where the coefficients in the expansion of $E_{m,m+1}\,\zeta_{\mathcal{U}}$ were simplified with the use of Lemma 4.9. The basis $\zeta_{\mathcal{U}}$ of $L(\lambda_1,\dots,\lambda_m\,|\,\lambda_{m+1})$ coincides with that of [20] up to a normalization. □

Example 4.22. In the case $m=1$, the basis ζ_Λ of the representation $L(\lambda_1\,|\,\lambda_2,\dots,\lambda_{n+1})$ of the Lie superalgebra $\mathfrak{gl}_{1|n}$ can be parameterized by the trapezium patterns

$$\mathcal{V}=\begin{array}{cccccccc} \lambda'_{r+n,1} & \lambda'_{r+n,2} & \cdots & & \cdots & & & \lambda'_{r+n,r+n}\\ & \ddots & \ddots & \cdots & \cdots & & \cdot^{\cdot^{\cdot}} & \\ & & \lambda'_{r+1,1} & \lambda'_{r+1,2} & \cdots & \lambda'_{r+1,r+1} & & \\ & & & 1 & 1 & \cdots & 1 & \end{array}$$

where the number r of 1's in the bottom row is nonnegative and varies between λ_1-n and λ_1. The top row coincides with $(\lambda'_1,\dots,\lambda'_q,0,\dots,0)$, where $q=\lambda_1$. The action of the generators E_{ss} with $s=1,\dots,n+1$ and the generators $E_{s,s+1}$, $E_{s+1,s}$ with $s=2,\dots,n$ on the basis vectors $\zeta_{\mathcal{V}}$ is given by the Gelfand–Tsetlin formulas (2.1)–(2.3). The formulas for the action of E_{12} and E_{21} are given in (4.44) and (4.45), respectively.

Specializing further and taking $n=2$ with $\lambda_1\geqslant 2$ we can parameterize the basis vectors $\zeta_{\mathcal{V}}$ of the $\mathfrak{gl}_{1|2}$-module $L(\lambda_1\,|\,\lambda_2,\lambda_3)$ by trapezium patterns $\mathcal{V}$ of four types:

$$\mathcal{V}^{(1)}_a \quad = \quad \begin{array}{cccccccc} \lambda_2+1 & \lambda_3+1 & 1 & \cdots & 1 & 0 & 0 \\ & a+1 & 1 & \cdots & 1 & 1 & 0 \\ & & 1 & \cdots & 1 & 1 & 1 \end{array}$$

the number of 1's in the bottom row is λ_1;

$$\mathcal{V}^{(2)}_a \quad = \quad \begin{array}{cccccc} \lambda_2+1 & \lambda_3+1 & 1 & \cdots & 1 & 0 \\ & a+1 & 1 & \cdots & 1 & 1 \\ & & 1 & \cdots & 1 & 1 \end{array}$$

the number of 1's in the bottom row is λ_1-1;

$$\mathcal{V}^{(3)}_a \quad = \quad \begin{array}{cccccc} \lambda_2+1 & \lambda_3+1 & 1 & \cdots & 1 & 0 \\ & a+1 & 1 & \cdots & 1 & 0 \\ & & 1 & \cdots & 1 & 1 \end{array}$$

the number of 1's in the bottom row is λ_1-1;

$$\mathcal{V}^{(4)}_a \quad = \quad \begin{array}{ccccc} \lambda_2+1 & \lambda_3+1 & 1 & \cdots & 1 \\ & a+1 & 1 & \cdots & 1 \\ & & 1 & \cdots & 1 \end{array}$$

the number of 1's in the bottom row is $\lambda_1 - 2$, where the parameter a in all cases runs over the integers such that $\lambda_3 \leqslant a \leqslant \lambda_2$. Thus, $\dim L(\lambda_1 \,|\, \lambda_2, \lambda_3) = 4(\lambda_2 - \lambda_3 + 1)$. The formulas for the action of the odd generators of the Lie superalgebra $\mathfrak{gl}_{1|2}$ provided by Theorem 4.18 have the form

$$E_{12}\,\zeta_{\mathcal{V}_a^{(1)}} = 0, \qquad E_{12}\,\zeta_{\mathcal{V}_a^{(2)}} = \frac{(a+\lambda_1)(a+\lambda_1+1)}{\lambda_1+\lambda_2+1}\,\zeta_{\mathcal{V}_a^{(1)}}$$

$$E_{12}\,\zeta_{\mathcal{V}_a^{(3)}} = 0, \qquad E_{12}\,\zeta_{\mathcal{V}_a^{(4)}} = -\frac{(\lambda_1-1)(a+\lambda_1)(a+\lambda_1-1)}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_2-1)}\,\zeta_{\mathcal{V}_a^{(3)}}$$

and

$$E_{21}\,\zeta_{\mathcal{V}_a^{(2)}} = 0, \qquad E_{21}\,\zeta_{\mathcal{V}_a^{(1)}} = \frac{\lambda_1+\lambda_2+1}{a+\lambda_1+1}\,\zeta_{\mathcal{V}_a^{(2)}}$$

$$E_{21}\,\zeta_{\mathcal{V}_a^{(4)}} = 0, \qquad E_{21}\,\zeta_{\mathcal{V}_a^{(3)}} = -\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_2-1)}{(\lambda_1-1)(a+\lambda_1)}\,\zeta_{\mathcal{V}_a^{(4)}}.$$

□

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