Homogeneous Planar and Two-Dimensional Mean-Field Antidynamo Theorems with Zero Mean Flow

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Abstract

In an electrically conducting fluid two types of turbulence with a preferred direction are distinguished: planar turbulence, in which every velocity of the turbulent ensemble of flows has no component in the given direction; and two-dimensional turbulence, in which every velocity in the turbulent ensemble is invariant under translation in the preferred direction. Under the additional assumptions of two-scale and homogeneous turbulence with zero mean flow, the associated alpha- and beta-effects are derived in the second-order smoothing approximation when the electrically conducting fluid occupies all space. Two antidynamo theorems, which establish necessary conditions for dynamo action, are shown to follow from the special structures of these alpha and beta effects. The theorems are analogues of the laminar planar velocity and two-dimensional antidynamo theorems. The mean magnetic field is general in the planar theorem but only two-dimensional in the two-dimensional theorem. The laminar theorems imply decay of the total magnetic field for any velocity of the associated turbulent ensemble. However, the mean-field theorems are not fully consistent with this, because further conditions beyond those arising from the turbulence must be imposed on the beta-effect to establish decay of the mean magnetic field. The two mean-field theorems relax the previous restriction to turbulence which is both two-dimensional and planar.

KEYWORDS: magnetohydrodynamics, dynamo theory, mean-field electrodynamics, alpha-effect, beta-effect, antidynamo theorem

1 Introduction

In an electrically conducting fluid, which occupies all space E^3 and moves with a prescribed velocity $\mathbf{v}$, the magnetic induction field $\mathbf{B}$ is governed by the equations,

$$\partial_t \mathbf{B} = \eta \nabla^2 \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0, \quad \text{in } E^3. \tag{1.1}$$

We assume the magnetic diffusivity η is uniform. Further conditions must be imposed on $\mathbf{v}$ and $\mathbf{B}$ for the magnetic field $\mathbf{B}$ to be self-exciting. The velocity acts as a dynamo if there is a magnetic field satisfying (1.1), which does not decay to zero as $t \to \infty$. An antidynamo theorem (ADT) is a collection of results (Ivers 1984), which establishes conditions under which a magnetic field cannot be maintained by the inductive effect of the velocity, i.e. dynamo action.

1.1 The Laminar Planar Velocity and Two-Dimensional ADT's

In the laminar planar velocity ADT, which is the cartesian analogue of the toroidal velocity ADT (Elsasser 1946, Bullard & Gellman 1954), the flow is planar and the magnetic field is general. A velocity field $\mathbf{v}$ is *planar* if there is a cartesian coordinate system (x_1, x_2, x_3) with unit vectors $(\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3)$, such that $\mathbf{v}$ is of the form

$$\mathbf{v} = v_1(x_1, x_2, x_3, t)\mathbf{1}_1 + v_2(x_1, x_2, x_3, t)\mathbf{1}_2 \,. \tag{1.2}$$

Thus $\mathbf{v}$ is everywhere parallel to the x_1x_2-plane. Results have been established by Moffatt (1978), Zeldovich & Ruzmaikin (1980) and Ivers & James (1988). A second theorem of present interest is the laminar two-dimensional ADT, which is the cartesian analogue of Cowling's (1934) axisymmetric ADT. The velocity and magnetic fields are *two-dimensional*, i.e. there is a cartesian coordinate system (x_1, x_2, x_3), such that $\mathbf{v}$ and $\mathbf{B}$ are independent of x_3. Thus $\mathbf{v}$ is of the form

$$\mathbf{v} = v_1(x_1, x_2, t)\mathbf{1}_1 + v_2(x_1, x_2, t)\mathbf{1}_2 + v_3(x_1, x_2, t)\mathbf{1}_3 \,. \tag{1.3}$$

Results of varying generality have been established by Cowling (1957), Zeldovich (1957), Lortz (1968), Vainshtein & Zeldovich (1972), Lortz & Meyer-Spasche (1982a,b,c), Lortz, Meyer-Spasche & Stredulinsky (1984) and Stredulinsky, Meyer-Spasche & Lortz (1986).

1.2 Two-Scale Turbulence and ADT's

We derive mean-field electrodynamic analogues of the laminar planar velocity and two-dimensional ADT's. The velocity and magnetic field are decomposed into mean and fluctuating parts $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$ and $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$, where the overline denotes the ensemble average. We assume throughout that the mean velocity is zero, $\overline{\mathbf{v}} = \mathbf{0}$. We consider only two-scale turbulence (Krause & Rädler 1980, Moffatt 1978), which leads to the mean magnetic induction equation,

$$\partial_t \overline{\mathbf{B}} = \eta \nabla^2 \overline{\mathbf{B}} + \nabla\times(\boldsymbol{\alpha}\cdot\overline{\mathbf{B}} + \boldsymbol{\beta}\cdot\cdot\nabla\overline{\mathbf{B}})\,, \quad \text{in } E^3. \tag{1.4}$$

In cartesian component form $(\boldsymbol{\beta}\cdot\cdot\nabla\overline{\mathbf{B}})_i = \beta_{ijk}\partial_k \overline{B}_j$. The mean magnetic field is solenoidal everywhere,

$$\nabla\cdot\overline{\mathbf{B}} = 0\,, \quad \text{in } E^3. \tag{1.5}$$

The alpha-effect acts as a dynamo if there is a mean magnetic field satisfying (1.7), which does not decay to zero as $t \to \infty$.

To isolate the alpha-effect from the beta-effect and simplify the analysis the isotropic beta-effect,

$$\boldsymbol{\beta} := \beta\boldsymbol{\varepsilon}\,, \quad \text{in } E^3, \tag{1.6}$$

is often assumed, where $\boldsymbol{\varepsilon}$ is the unit rank-3 alternating tensor. Thus, if the scalar β is uniform, the mean induction equation reduces to

$$\partial_t \overline{\mathbf{B}} = \eta_T \nabla^2 \overline{\mathbf{B}} + \nabla\times(\boldsymbol{\alpha}\cdot\overline{\mathbf{B}})\,, \quad \text{in } E^3, \tag{1.7}$$

where $\eta_T := \eta + \beta$ is the turbulent magnetic diffusion. Also commonly considered is the general beta-effect with one invariant direction $\mathbf{e}$; in cartesian component form,

$$\beta^e_{ijk} := \beta\varepsilon_{ijk} + \beta_1 e_i\delta_{jk} + \beta_2 e_j\delta_{ki} + \beta_3 e_k\delta_{ij} + \beta_4 e_i e_m\varepsilon_{mjk} + \beta_5 e_j e_m\varepsilon_{imk} + \beta_6 e_k e_m\varepsilon_{ijm} + \beta_7 e_i e_j e_k \,. \tag{1.8}$$

The β_1 term does not contribute to (1.4) due to (1.5). Often only terms linear in $\mathbf{e}$ are included (Krause & Rädler 1980). Since $\boldsymbol{\beta}$ is a pseudo-tensor, the coefficients β^e and $\beta_{4:6}$ are (proper) scalars, and $\beta_{1:3,7}$ are pseudo-scalars, if $\mathbf{e}$ is a proper (polar) vector; β and $\beta_{1:7}$ are (proper) scalars, if $\mathbf{e}$ is a pseudo (axial) vector. Irrespective of the mirror-symmetry of $\mathbf{e}$, $\boldsymbol{\beta}^e = \mathbf{0}$ (indeed all $\boldsymbol{\beta} = \mathbf{0}$) if the turbulence is mirror-symmetric (see Moffatt 1978 p.155). In fact, if the turbulence is mirror-symmetric, then $\overline{\mathbf{B}} = \mathbf{0}$.

Krause & Rudiger (1974) defined turbulence to be 'two-dimensional' in E^3, if any velocity in the turbulent ensemble is of the form $\mathbf{v} = v_1(x_1, x_2, t)\mathbf{1}_1 + v_2(x_1, x_2, t)\mathbf{1}_2$. This more restricted form, which is the intersection of the forms (1.2) and (1.3), is a consequence of the Proudman-Taylor theorem (Proudman 1916) for rapidly rotating fluids. Krause & Rudiger (1974) also defined isotropic 'two-dimensional' turbulence. Krause (1976; see also Krause and Rädler 1980) showed that in two-scale isotropic 'two-dimensional' homogeneous turbulence in E^3, the alpha-effect must be of the form

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & 0 & \alpha_{13} \\ 0 & 0 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 0 \end{pmatrix}\,, \tag{1.9}$$

where the α_{ij} are constants. Krause proved that, for general $\boldsymbol{\beta}$, the alpha-effect (1.9) cannot produce dynamo action, even for general mean magnetic fields with spatial dependence $e^{i\mathbf{k}\cdot\mathbf{r}}$, if $0 < |\mathbf{k}| \ll 1$. However, the argument fails if $\boldsymbol{\alpha}$ does not contribute positively to $\mathrm{Re}\,\gamma$. The basis of the theorem is the following elegant result (Krause 1973,1976):

Theorem 1. The alpha-effect in isotropic homogeneous two-scale turbulence in E^3 is a dynamo for $0 < |\mathbf{k}| \ll 1$ only if the adjugate $\mathrm{adj}\,\boldsymbol{\alpha}^S$ of the symmetric part $\boldsymbol{\alpha}^S$ of $\boldsymbol{\alpha}$ is not negative semi-definite.

The superscript S denotes the symmetric part. This theorem is only valid for small wave-vectors $0 < |\mathbf{k}| \ll 1$. In cartesian component form, the element A_{ij} of the adjugate $\operatorname{adj}\boldsymbol{\alpha}^S$ is the cofactor of α^S_{ji} and $\boldsymbol{\alpha}^S \cdot \operatorname{adj}\boldsymbol{\alpha}^S = \mathbf{I}\det\boldsymbol{\alpha}^S$. The converse fails since $\boldsymbol{\alpha}$ need not contribute to $\operatorname{Re}\gamma$. For the alpha-effect (1.9)

$$\operatorname{adj}\boldsymbol{\alpha}^S = \frac{1}{4}\begin{pmatrix} -(\alpha_{23}+\alpha_{32})^2 & -(\alpha_{13}+\alpha_{31})(\alpha_{23}+\alpha_{32}) & 0 \\ -(\alpha_{13}+\alpha_{31})(\alpha_{23}+\alpha_{32}) & -(\alpha_{13}+\alpha_{31})^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.10}$$

which is negative semi-definite. Rüdiger (1978) invoked this result to explain the observation that magnetic (nonmagnetic) A stars are mostly slow (fast) rotators.

For turbulence under rapid rotation with Rossby number $|\mathbf{v}|/\Omega\ell \ll 1$ and magnetic Reynolds number $R'_{\mathrm{m}} := \Omega\ell^2/\eta \gg 1$, where ℓ is the fluctuating length-scale, and with viscous dissipation negligible compared to ohmic dissipation, Moffat (1970b) derived an alpha-effect of the form $\boldsymbol{\alpha} = \alpha_1(\mathbf{I} - \boldsymbol{\Omega\Omega}/\Omega^2)$, where $\mathbf{I} = \mathbf{1}_1\mathbf{1}_1 + \mathbf{1}_2\mathbf{1}_2 + \mathbf{1}_3\mathbf{1}_3$ is the identity tensor and $\boldsymbol{\Omega}$ is the rotation rate. This alpha-effect is inconsistent with (1.9). To explain this inconsistency we define two distinct forms of turbulence, planar turbulence and two-dimensional turbulence, analogously to the laminar dynamo case.

Definition 1 (Planar Turbulence)**.** The turbulence is planar if there is a direction $\mathbf{e}$ such that $\mathbf{e}\cdot\mathbf{v} = 0$ for all velocities $\mathbf{v}$ in the ensemble.

The direction $\mathbf{e}$ is the same for all velocities. If the x_3-axis is aligned with $\mathbf{e}$, then the turbulence is planar if any velocity $\mathbf{v}$ in the ensemble is of the form (1.2). We show in section 3.1 that, if the mean velocity is zero and the turbulence is homogeneous and planar, then the alpha-effect is of the cartesian form

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & 0 & \alpha_{13} \\ 0 & 0 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{1.11}$$

where the α_{ij} are constants. The matrix (1.11) may be non-symmetric.

If the turbulence is planar the beta-effect is also not isotropic and must be restricted. Thus it is shown in section 3.1 that in planar turbulence the beta-effect must satisfy the conditions,

$$\beta_{ijk} = 0\,, \quad i,j = 1,2\,. \tag{1.12}$$

The isotropic beta-effect (1.6) fails this condition. It can be minimally modified by zeroing the relevant components as follows,

$$\boldsymbol{\beta} = \beta(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\cdot\mathbf{e}\otimes\mathbf{e})\,, \quad \text{in } E^3, \tag{1.13}$$

where $\otimes$ denotes the tensor product. In cartesian component form, $\beta_{ijk} = \beta(\varepsilon_{ijk} - \varepsilon_{ij3}\delta_{k3})$. If the beta-effect shares the preferred direction $\mathbf{e}$ of the turbulence in definition 1 then it must also be of the form (1.8). Imposing (1.12) on $\boldsymbol{\beta}^e$ in (1.8) with $\beta_1 = 0$ gives $\beta_3 = 0$ and $\beta_6 = -\beta$, i.e.

$$\beta^{\mathrm{pl}}_{ijk} := \beta(\varepsilon_{ijk} - \varepsilon_{ij3}\delta_{k3}) + \beta_2\delta_{j3}\delta_{ki} + \beta_4\delta_{i3}\varepsilon_{3jk} + \beta_5\delta_{j3}\varepsilon_{i3k} + \beta_7\delta_{i3}\delta_{j3}\delta_{k3}\,. \tag{1.14}$$

The beta-effect (1.13) is a special case of $\boldsymbol{\beta}^{\mathrm{pl}}$.

Definition 2 (Two-Dimensional Turbulence)**.** The turbulence is two-dimensional if there is a direction $\mathbf{e}$ such that $\mathbf{e}\cdot\nabla\mathbf{v} = 0$ for all velocities $\mathbf{v}$ in the ensemble.

The direction $\mathbf{e}$ is the same for all velocities. If the x_3-axis is aligned with $\mathbf{e}$, then the turbulence is two-dimensional if any velocity $\mathbf{v}$ in the ensemble is of the form (1.3). We show in section 3.2 that, if the mean velocity is zero and the turbulence is homogeneous and two-dimensional, then the alpha-effect is of the form

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 0 \end{pmatrix}, \tag{1.15}$$

where the α_{ij} are constants. The matrix (1.15) may be non-symmetric. This is consistent with Moffat's (1970b) alpha-effect referred to above. The alpha-effects (1.11) and (1.15) constitute a splitting of Krause's alpha-effect (1.9). It is also shown in section 3.2 that the $\boldsymbol{\beta}$-effect must be restricted by the necessary conditions

$$\beta_{ij3} = 0\,, \quad \beta_{33k} = 0\,. \tag{1.16}$$

The isotropic beta-effect (1.6) fails these two-dimensional conditions and must be modified, as in (1.13). If $\boldsymbol{\beta}$ is invariant in the direction $\mathbf{e}$ of definition 2, then imposing conditions (1.16) on (1.8) yields $\beta_3 = 0$, $\beta_6 = -\beta$ and $\beta_7 = -(\beta_1 + \beta_2)$, i.e.

$$\beta^{\mathrm{td}}_{ijk} := \beta(\varepsilon_{ijk} - \varepsilon_{ij3}\delta_{k3}) + \beta_2\delta_{j3}(\delta_{ki} - \delta_{i3}\delta_{k3}) + \beta_4\delta_{i3}\varepsilon_{3jk} + \beta_5\delta_{j3}\varepsilon_{i3k}\,. \tag{1.17}$$

The coefficients are independent of x_3. The beta-effect (1.13) is also a special case of $\boldsymbol{\beta}^{\mathrm{td}}$.

The turbulence definitions 1 and 2 are based on $\mathbf{v}$ not $\boldsymbol{\alpha}$. We distinguish planar and two-dimensional alpha-effects as follows:

Definition 3 (Planar Alpha-Effect)**.** The alpha-effect is two-dimensional if there is a direction $\mathbf{e}$ such that $\mathbf{e}\cdot\boldsymbol{\alpha} = \mathbf{0}$ and $\boldsymbol{\alpha}\cdot\mathbf{e} = \mathbf{0}$.

Definition 4 (Two-Dimensional Alpha-Effect)**.** The alpha-effect is two-dimensional if there is a direction $\mathbf{e}$ such that $\mathbf{e}\cdot\nabla\boldsymbol{\alpha} = \mathbf{0}$.

A planar alpha-effect should not be confused with a planar-turbulence alpha-effect of definition 1. Similarly for a two-dimensional alpha-effect. The alpha-effect (1.11) is not planar nor is (1.15) two-dimensional. The ADT's proved below are based on planar turbulence and two-dimensional turbulence.

In section 3 we derive mean field electrodynamic analogues of the two-dimensional and planar velocity ADT's. In section 2 we outline the derivation of formulae for the alpha- and beta-effects in two-scale homogeneous turbulence in E^3. In section 3 we derive the alpha-effects (1.9) and (1.11) and associated restrictions on the beta-effect for planar and two-dimensional turbulence. We also prove the main results, Theorems 3.1 and 3.2, for beta-effects of the restricted form (1.13). Extensions of the theorems to general beta-effects are also established under necessary conditions on $\boldsymbol{\beta}$. We also examine the consistency of the mean-field ADT's with the corresponding laminar ADT: since each velocity in the planar or two-dimensional ensembles satisfies the related laminar ADT, the mean magnetic field should decay according to that laminar theorem.

2 Solution of the Fluctuating Magnetic Induction Equation

2.1 The Second-Order Correlation Approximation and Green's Tensor Solution

The mean of equation (1.1) yields the mean magnetic induction equation,

$$(\partial_t - \eta\nabla^2)\overline{\mathbf{B}} = \nabla\times\boldsymbol{\mathcal{E}}\,, \tag{2.1}$$

where $\boldsymbol{\mathcal{E}}(\mathbf{r},t) := \overline{\mathbf{v}'(\mathbf{r},t)\times\mathbf{B}'(\mathbf{r},t)}$. Subtracting (2.1) from (1.1) leaves the fluctuating magnetic induction equation,

$$(\partial_t - \eta\nabla^2)\mathbf{B}' = \nabla\times(\mathbf{v}'\times\overline{\mathbf{B}}) + \nabla\times(\mathbf{v}'\times\mathbf{B}' - \overline{\mathbf{v}'\times\mathbf{B}'})\,. \tag{2.2}$$

We make the second-order correlation approximation, in which $\nabla\times(\mathbf{v}'\times\mathbf{B}' - \overline{\mathbf{v}'\times\mathbf{B}'})$ is neglected compared to $\partial_t\mathbf{B}'$ or $\eta\nabla^2\mathbf{B}'$ (see Krause and Rädler 1980). Thus (2.2) reduces to

$$(\partial_t - \eta\nabla^2)\mathbf{B}' = \nabla\times(\mathbf{v}'\times\overline{\mathbf{B}})\,. \tag{2.3}$$

The solution of (2.3) can be given in terms of the Green's tensor $\mathbf{G}(\mathbf{r},t,\boldsymbol{\xi},\tau)$, which is the solution of the differential equation, $(\partial_t - \eta\nabla^2)\mathbf{G} = \delta^3(\boldsymbol{\xi}-\mathbf{r})\delta(\tau - t)\mathbf{I}$, where δ^3 and δ are Dirac delta distributions, subject to $\mathbf{G}(\mathbf{r},t,\boldsymbol{\xi},\tau) = \mathbf{0}$ if $t < \tau$, $\mathbf{G}(\mathbf{r},t,\boldsymbol{\xi},\tau) \to \mathbf{0}$ as $|\boldsymbol{\xi}-\mathbf{r}| \to \infty$ (Bräuer 1973). The solution is isotropic and given by $\mathbf{G}(\mathbf{r},t,\boldsymbol{\xi},\tau) = G(\mathbf{r}-\boldsymbol{\xi}, t-\tau)\mathbf{I}$, where the function

$$G(\mathbf{r},t) = \begin{cases} 0, & t \le 0; \\ \dfrac{\exp(-|\mathbf{r}|^2/4\eta t)}{(4\pi\eta t)^{3/2}}, & t > 0. \end{cases}$$

Thus the solution of (2.3) is

$$\mathbf{B}'(\mathbf{r},t) = \iint_{\mathbb{R}^3\times\mathbb{R}} G(\mathbf{r}-\boldsymbol{\xi}, t-\tau)\nabla_{\boldsymbol{\xi}}\times[\mathbf{v}'(\boldsymbol{\xi},\tau)\times\overline{\mathbf{B}}(\boldsymbol{\xi},\tau)]\,d^3\boldsymbol{\xi}\,d\tau\,. \tag{2.4}$$

A useful alternative form of (2.4) may be obtained using integration by parts and the identity tensor,

$$\mathbf{B}'(\mathbf{r},t) = -\iint_{\mathbb{R}^3\times\mathbb{R}} [\mathbf{I}\times\nabla_{\boldsymbol{\xi}} G(\mathbf{r}-\boldsymbol{\xi},t-\tau)]\times\mathbf{v}'(\boldsymbol{\xi},\tau)\cdot\overline{\mathbf{B}}(\boldsymbol{\xi},\tau)\,d^3\boldsymbol{\xi}\,d\tau\,.$$

The vector product with $\mathbf{v}'(\mathbf{r},t)$ and the ensemble mean yield

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = -\iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\mathbf{v}'(\mathbf{r},t)\times[\mathbf{I}\times\nabla_{\boldsymbol{\xi}} G(\mathbf{r}-\boldsymbol{\xi},t-\tau)]\times\mathbf{v}'(\boldsymbol{\xi},\tau)}\cdot\overline{\mathbf{B}}(\boldsymbol{\xi},\tau)\,d^3\boldsymbol{\xi}\,d\tau\,.$$

Change the variables of integration to $\boldsymbol{\xi}' = \mathbf{r}-\boldsymbol{\xi}$, $\tau' = t-\tau$ and then drop the primes,

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\mathbf{v}'(\mathbf{r},t)\times[\mathbf{I}\times\nabla_{\boldsymbol{\xi}} G(\boldsymbol{\xi},\tau)]\times\mathbf{v}'(\mathbf{r}-\boldsymbol{\xi},t-\tau)}\cdot\overline{\mathbf{B}}(\mathbf{r}-\boldsymbol{\xi},t-\tau)\,d^3\boldsymbol{\xi}\,d\tau\,. \tag{2.5}$$

2.2 Two-Scale Turbulence and the Alpha-Effect

We assume that the turbulence is two-scale with a clear separation between the mean and fluctuating length and time scales. Thus we expand $\overline{\mathbf{B}}(\mathbf{r}-\boldsymbol{\xi},t-\tau)$ in (2.5) in a Taylor series about $\mathbf{r}$,

$$\overline{\mathbf{B}}(\mathbf{r}-\boldsymbol{\xi},t-\tau) = \overline{\mathbf{B}}(\mathbf{r},t) - \boldsymbol{\xi}\cdot\nabla\overline{\mathbf{B}}(\mathbf{r},t) + \boldsymbol{\mathcal{O}}(|\boldsymbol{\xi}|^2)\,.$$

Note that the divergence with respect to $\boldsymbol{\xi}$ of each term on the right side vanishes. Then $\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \boldsymbol{\alpha}\cdot\overline{\mathbf{B}}(\mathbf{r},t) + \boldsymbol{\beta}\cdot\cdot\nabla\overline{\mathbf{B}}(\mathbf{r},t)$, which together with (2.1) yields (1.4). In coordinate-free form,

$$\boldsymbol{\alpha} = \iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\mathbf{v}'(\mathbf{r},t)\times[\mathbf{I}\times\nabla_{\boldsymbol{\xi}} G(\boldsymbol{\xi},\tau)]\times\mathbf{v}'(\mathbf{r}-\boldsymbol{\xi},t-\tau)}\,d^3\boldsymbol{\xi}\,d\tau\,, \tag{2.6}$$

$$\boldsymbol{\beta} = -\iint_{\mathbb{R}^3\times\mathbb{R}} \overline{\mathbf{v}'(\mathbf{r},t)\times[\mathbf{I}\times\nabla_{\boldsymbol{\xi}} G(\boldsymbol{\xi},\tau)]\times\mathbf{v}'(\mathbf{r}-\boldsymbol{\xi},t-\tau)}\otimes\boldsymbol{\xi}\,d^3\boldsymbol{\xi}\,d\tau\,. \tag{2.7}$$

In terms of the two-point velocity correlation tensor, $\mathbf{Q}(\mathbf{r},t,\boldsymbol{\xi},\tau) := \overline{\mathbf{v}'(\mathbf{r},t)\otimes\mathbf{v}'(\mathbf{r}+\boldsymbol{\xi},t+\tau)}$, the cartesian tensor components of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are

$$\alpha_{ij} = \varepsilon_{ilm}\varepsilon_{mnp}\varepsilon_{pqj}\iint_{\mathbb{R}^3\times\mathbb{R}} Q_{lq}(\mathbf{r},t,-\boldsymbol{\xi},-\tau)\frac{\partial G(\boldsymbol{\xi},\tau)}{\partial\xi_n}\,d^3\boldsymbol{\xi}\,d\tau\,, \tag{2.8}$$

$$\beta_{ijk} = -\varepsilon_{ilm}\varepsilon_{mnp}\varepsilon_{pqj}\iint_{\mathbb{R}^3\times\mathbb{R}} \xi_k Q_{lq}(\mathbf{r},t,-\boldsymbol{\xi},-\tau)\frac{\partial G(\boldsymbol{\xi},\tau)}{\partial\xi_n}\,d^3\boldsymbol{\xi}\,d\tau\,. \tag{2.9}$$

The velocity correlation tensor satisfies the symmetry property,

$$\mathbf{Q}(\mathbf{r}-\boldsymbol{\xi},t-\tau,\boldsymbol{\xi},\tau) = \overline{\mathbf{v}'(\mathbf{r}-\boldsymbol{\xi},t-\tau)\otimes\mathbf{v}'(\mathbf{r},t)} = \{\overline{\mathbf{v}'(\mathbf{r},t)\otimes\mathbf{v}'(\mathbf{r}-\boldsymbol{\xi},t-\tau)}\}^T = \mathbf{Q}^T(\mathbf{r},t,-\boldsymbol{\xi},-\tau)\,.$$

Moreover, if $\mathbf{v}'$ is incompressible, then

$$\nabla_{\boldsymbol{\xi}}\cdot\mathbf{Q}^T(\mathbf{r},t,\boldsymbol{\xi},\tau) = \overline{\{\nabla_{\boldsymbol{\xi}}\cdot\mathbf{v}'(\mathbf{r}+\boldsymbol{\xi},t+\tau)\}\mathbf{v}'(\mathbf{r},t)} = \mathbf{0}\,. \tag{2.10}$$

2.3 Homogeneous Turbulence and Fourier Representation of the Alpha-Effect

Henceforth we assume that turbulence is homogeneous, in the sense that $\mathbf{Q}(\mathbf{r},t,\boldsymbol{\xi},\tau)$ is independent of $\mathbf{r}$ and t. We can then write $\mathbf{Q} = \mathbf{Q}(\boldsymbol{\xi},\tau)$. In particular, $\mathbf{Q}^T(\boldsymbol{\xi},\tau) = \mathbf{Q}(-\boldsymbol{\xi},-\tau)$. Fourier techniques are particularly effective, if turbulence is homogeneous and the conducting region is E^3. The full Fourier transform of a scalar function $f(\boldsymbol{\xi},\tau):\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ is defined by

$$\widehat{f}(\mathbf{k},\omega) = \mathcal{F}\{f\} := \frac{1}{(2\pi)^4}\iint_{\mathbb{R}^3\times\mathbb{R}} f(\boldsymbol{\xi},\tau)e^{-i(\mathbf{k}\cdot\boldsymbol{\xi}-\omega\tau)}\,d^3\boldsymbol{\xi}\,d\tau\,,$$

and the inverse transform by

$$f(\boldsymbol{\xi},\tau) = \mathcal{F}^{-1}\{f\} := \iint_{\mathbb{R}^3\times\mathbb{R}} \widehat{f}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\boldsymbol{\xi}-\omega\tau)}\,d^3\mathbf{k}\,d\omega\,.$$

Since $\mathbf{v}'$ is real, the Fourier transform of the velocity correlation tensor satisfies the symmetry, $\widehat{\mathbf{Q}}^*(\mathbf{k},\omega) = \widehat{\mathbf{Q}}(-\mathbf{k},-\omega)$, where the asterisk denotes complex conjugation. Thus $\widehat{\mathbf{Q}}^* = \widehat{\mathbf{Q}}^T$, i.e. $\widehat{\mathbf{Q}}$ is hermitian. Moreover, $\mathcal{F}\{\nabla_{\boldsymbol{\xi}} \cdot \mathbf{Q}^T(\boldsymbol{\xi},\tau)\} = i\widehat{\mathbf{Q}}(\mathbf{k},\omega)\cdot\mathbf{k}$. If all flows in the ensemble are incompressible, then $\widehat{\mathbf{Q}}(\mathbf{k},\omega)\cdot\mathbf{k} = \mathbf{0}$ by (2.10) and $\mathbf{k}\cdot\widehat{\mathbf{Q}}(\mathbf{k},\omega) = \mathbf{k}\cdot\widehat{\mathbf{Q}}^T(-\mathbf{k},-\omega) = \mathbf{0}$.

Using Parseval's identity

$$\int_{\mathbb{R}^4} f(\boldsymbol{\xi},\tau)g^*(\boldsymbol{\xi},\tau)\,d^3\xi\,d\tau = (2\pi)^4\int_{\mathbb{R}^4}\widehat{f}(\mathbf{k},\omega)\widehat{g}^*(\mathbf{k},\omega)\,d^3\mathbf{k}\,d\omega$$

and

$$\widehat{G}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\frac{1}{\eta k^2 - i\omega}\,,$$

the Fourier transforms of (2.8) and (2.9) in the homogeneous case yield

$$\alpha_{ij} = i(\epsilon_{ilq}\delta_{nj} - \epsilon_{ilj}\delta_{nq})\int_{\mathbb{R}^4}\frac{k_n\widehat{Q}_{lq}(\mathbf{k},\omega)}{\eta|\mathbf{k}|^2 - i\omega}\,d^3\mathbf{k}\,d\omega \tag{2.11}$$

and

$$\beta_{ijk} = -(\epsilon_{ilq}\delta_{nj} - \epsilon_{ilj}\delta_{nq})\int_{\mathbb{R}^4}\frac{k_n}{\eta|\mathbf{k}|^2 - i\omega}\frac{\partial\widehat{Q}_{lq}(\mathbf{k},\omega)}{\partial k_k}\,d^3\mathbf{k}\,d\omega\,, \tag{2.12}$$

where $\widehat{Q}_{lq}$ are the cartesian tensor components of the Fourier transform of the correlation tensor. The hermitian property of $\widehat{\mathbf{Q}}$ ensures that α_{ij} is real.

3 Homogeneous Alpha-Effect ADT's in E^3

The homogeneous property of the turbulence implies the mean induction equation (1.4) possesses solutions of the form

$$\overline{\mathbf{B}} = \widehat{\mathbf{B}}e^{i\mathbf{k}\cdot\mathbf{r}+\gamma t}\,,\qquad \mathbf{k}\neq\mathbf{0}\,, \tag{3.1}$$

where $\widehat{\mathbf{B}}$ is a constant vector. Substitution into (1.4) and (1.5) gives

$$\mathbf{M}\cdot\widehat{\mathbf{B}} = \mathbf{0}\,,\quad \mathbf{M} := (\gamma+\eta k^2)\mathbf{I} - i\mathbf{k}\times(\boldsymbol{\alpha} + i\boldsymbol{\beta}\cdot\mathbf{k})\,;\qquad \mathbf{k}\cdot\widehat{\mathbf{B}} = 0 \tag{3.2}$$

The non-solenoidal mode with $\mathbf{k}\cdot\widehat{\mathbf{B}}\neq 0$ always decays, irrespective of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$: the scalar product of (3.2)(a) with $\mathbf{k}$ implies $(\gamma+\eta k^2)\mathbf{k}\cdot\widehat{\mathbf{B}} = 0$, i.e. $\gamma = -\eta k^2$.

More general magnetic field solutions can be constructed by linearity,

$$\overline{\mathbf{B}}(\mathbf{r},t) = \int_{\mathbb{R}^3}\widehat{\mathbf{B}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}+\gamma(\mathbf{k})t}\,d^3\mathbf{k}\quad\text{or}\quad \overline{\mathbf{B}}(\mathbf{r},t) = \sum_{\mathbf{k}}\widehat{\mathbf{B}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}+\gamma(\mathbf{k})t}\,,$$

and solutions may be periodic or non-periodic. A unified treatment can be given in terms of Stieltjes integrals (see Moffatt 1970a). Since the mean magnetic field may not necessarily vanish at infinity care is need with the self-excitation condition. Magnetic energy self-excitation criteria for a region D with boundary ∂D are based on the magnetic energy equation for D,

$$\frac{d}{dt}\int_D \tfrac{1}{2}\overline{\mathbf{B}}^2\,d^3\mathbf{r} = -\int_{\partial D}\overline{\mathbf{E}}\times\overline{\mathbf{B}}\cdot d\mathbf{S} - \int_D \eta(\nabla\times\overline{\mathbf{B}})^2\,d^3\mathbf{r} + \int_D \nabla\times\overline{\mathbf{B}}\cdot(\boldsymbol{\alpha}\cdot\overline{\mathbf{B}} + \boldsymbol{\beta}\cdot\cdot\nabla\overline{\mathbf{B}})\,d^3\mathbf{r}\,,$$

where $\overline{\mathbf{E}}$ is the mean electric field. This equation can be derived using the mean forms of Faraday's Law, $\nabla\times\overline{\mathbf{E}} = -\partial_t\overline{\mathbf{B}}$, Ohm's Law and Ampere's Law, which give $\overline{\mathbf{E}} = \eta\nabla\times\overline{\mathbf{B}} - \boldsymbol{\alpha}\cdot\overline{\mathbf{B}} - \boldsymbol{\beta}\cdot\cdot\nabla\overline{\mathbf{B}}$. Self-excitation of the mean magnetic field depends only on the Poynting flux term on the right side; the second and third terms physically represent the ohmic dissipation and magnetic energy transfer between the mean and fluctuating magnetic fields. Periodic magnetic fields are clearly self-excited: if D is a periodic cell there is no net flow of magnetic energy into D and dynamo action corresponds to non-decay of the magnetic energy in D. In the non-periodic Fourier transform case, we assume

$$\int_{\mathbb{R}^3}(1+|\mathbf{k}|^2)|\widehat{\mathbf{B}}(\mathbf{k},\omega)|^2\,d^3\mathbf{k} < \infty\,,$$

and dynamo action corresponds to non-decay of the energy of the mean magnetic field. In the non-periodic random wave case $D = [-L, L]^3$, the volume-averaged Poynting flux vanishes,

$$\lim_{L\to\infty} \frac{1}{8L^3} \int_{\partial D} \overline{\mathbf{E}} \times \overline{\mathbf{B}} \cdot d\mathbf{S} = 0$$

and dynamo action corresponds to non-decay of the spatially-averaged energy of the mean magnetic field,

$$\lim_{L\to\infty} \frac{1}{8L^3} \int_{D} \overline{\mathbf{B}}^2 \, d^3\mathbf{r} = \sum_{\mathbf{k}} |\widehat{\mathbf{B}}(\mathbf{k})|^2 e^{2\gamma(\mathbf{k})t} \, .$$

Since the main interest of the present work is the alpha-effect, we assume in the two main results, theorem 2 and theorem 3, that the beta-effect has the simpler restricted form (1.13) of $\boldsymbol{\beta}$, where the scalar $\beta = \frac{1}{4}\varepsilon_{ijk}\beta_{ijk}$, i.e.

$$\beta = -\int_{\mathbb{R}^4} \frac{2k_l}{\eta|\mathbf{k}|^2 - i\omega} \frac{\partial \widehat{Q}^S_{kl}(\mathbf{k},\omega)}{\partial k_k} \, d^3\mathbf{k}\, d\omega \, .$$

The superscript S denotes the symmetric part. As indicated above this anisotropic beta-effect differs from the commonly used isotropic (1.6). Since $\mathbf{k} \times \boldsymbol{\beta} \cdot \mathbf{k} = \beta(k_\perp^2 \mathbf{I} - \mathbf{k}_\perp \otimes \mathbf{k})$ equation (3.2) can be replaced by $[(\gamma + \eta_T k_\perp^2 + \eta k_3^2)\mathbf{I} - i\mathbf{k} \times \boldsymbol{\alpha} - \beta \mathbf{k}_\perp \otimes \mathbf{k}] \cdot \widehat{\mathbf{B}} = \mathbf{0}$. Further reduction is possible by imposing the solenoidal condition $\mathbf{k} \cdot \widehat{\mathbf{B}} = 0$,

$$\mathbf{M} \cdot \widehat{\mathbf{B}} = \mathbf{0} \, , \quad \mathbf{M} := (\gamma + \eta_T k_\perp^2 + \eta k_3^2)\mathbf{I} - i\mathbf{k} \times \boldsymbol{\alpha} \, . \tag{3.3}$$

This modifies but does not eliminate the non-solenoidal mode in (3.2): in particular, the scalar product of (3.3) with $\mathbf{k}$ gives the modified growth rate $\gamma = -\eta_T k_\perp^2 - \eta k_3^2$ for the mode.

The rank-2 tensor $\boldsymbol{\alpha}$ in (3.3) can be replaced by its symmetric part $\boldsymbol{\alpha}^S$. Decompose $\boldsymbol{\alpha} = \boldsymbol{\alpha}^S + \mathbf{I} \times \mathbf{a}$, where the vector $\mathbf{a} := \frac{1}{2}[(\alpha_{23} - \alpha_{32})\mathbf{1}_x + (\alpha_{31} - \alpha_{13})\mathbf{1}_y + (\alpha_{12} - \alpha_{21})\mathbf{1}_z]$ is constructed from the antisymmetric part of $\boldsymbol{\alpha}$. Since $\mathbf{k} \times (\mathbf{I} \times \mathbf{a}) \cdot \overline{\mathbf{B}} = (\mathbf{k} \cdot \overline{\mathbf{B}})\mathbf{a} - (\mathbf{k} \cdot \mathbf{a})\overline{\mathbf{B}}$, $\mathbf{M}$ in (3.3) can be reduced to

$$\mathbf{M} := (\gamma + \eta_T k_\perp^2 + \eta k_3^2 - i\mathbf{k} \cdot \mathbf{a})\mathbf{I} - i\mathbf{k} \times \boldsymbol{\alpha}^S \, , \tag{3.4}$$

noting the solenoidal condition on $\overline{\mathbf{B}}$. The antisymmetric part of $\boldsymbol{\alpha}$ contributes only to the frequency of the magnetic field, not its growth rate $\operatorname{Re}\gamma$.

3.1 Homogeneous Planar Alpha-Effect ADT in E^3

We now assume that the turbulence is planar in the sense of definition 1 and that the magnetic field has the general form (3.1). Since any velocity in the ensemble is planar, $v_3 = 0$, and hence the 2-point velocity correlation tensor for planar turbulence has zero components, $Q_{i3} = 0$, $Q_{3j} = 0$. The corresponding components of its Fourier transform are also zero, $\widehat{Q}_{i3} = 0$, $\widehat{Q}_{3j} = 0$. Hence $\alpha_{ij} = 0$ if $i, j = 1, 2$, and hence the alpha-effect is of the form (1.11). Similarly, the beta-effect must satisfy the necessary conditions (1.12). The isotropic beta-effect (1.6) does not satisfy these conditions. However, the minimal change to (1.6), in the sense that only the elements which violate (1.12) are zeroed, i.e. $\beta_{ijk} = \beta^\varepsilon_{ijk} - \beta^\varepsilon_{ij3}\delta_{k3}$, yields the beta-effect (1.13). Similarly, imposing (1.12) on (1.8) yields (1.14)

We prove theorem 2 from first principles without using theorem 1, since the corollary 1 then follows directly.

Theorem 2. (Homogeneous Planar Alpha-Effect ADT). Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy (1.11) and (1.13), respectively, with constant α_{ij} and β_{ijk}. If the magnetic field is of the form (3.1), then $\operatorname{Re}\gamma = -\eta_T k_\perp^2 - \eta k_3^2$. In general, if $\eta_T > 0$, then the magnetic modes (3.1) all decay to zero.

Proof. For non-trivial solutions to (3.3) $\det \mathbf{M} = 0$. Thus, since $\boldsymbol{\alpha}$ is of the form (1.11),

$$\begin{vmatrix} \lambda - ik_2\alpha_{31} & -ik_2\alpha_{32} & -i(k_2\alpha_{33} - k_3\alpha_{23}) \\ ik_1\alpha_{31} & \lambda + ik_1\alpha_{32} & -i(k_3\alpha_{13} - k_1\alpha_{33}) \\ 0 & 0 & \lambda - i(k_1\alpha_{23} - k_2\alpha_{13}) \end{vmatrix} = 0 \, ,$$

where $\lambda := \gamma + \eta_T k_\perp^2 + \eta k_3^2$. The two zeros greatly simplify the evaluation of the determinant. Solving for λ yields the growth rates,

$$\gamma = -\eta_T k_\perp^2 - \eta k_3^2 + i(k_1\alpha_{23} - k_2\alpha_{13}) \, , -\eta_T k_\perp^2 - \eta k_3^2 - i(k_1\alpha_{32} - k_2\alpha_{31}) \, . \tag{3.5}$$

The solenoidal condition $\mathbf{k} \cdot \widehat{\mathbf{B}} = 0$ has removed the mode with $\gamma = -\eta_T k_\perp^2 - \eta k_3^2$. Since α_{ij}, k_1, k_2 are real, $\operatorname{Re}\gamma = -\eta_T k_\perp^2 - \eta k_3^2$ for all modes. ☐

We now consider beta-effects more general than (1.13), but restricted by the necessary conditions (1.12). The coefficient matrix $\mathbf{M}$ is now given by (3.2). For these general beta-effects it is useful to define the matrices

$$\mathbf{M}_1 := \begin{pmatrix} \beta_{231} & \beta_{232} & \beta_{233} \\ -\beta_{131} & -\beta_{132} & -\beta_{133} \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{M}_2 := \begin{pmatrix} -\beta_{321} & -\beta_{322} & -\beta_{323} \\ \beta_{311} & \beta_{312} & \beta_{313} \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

The following corollary establishes sufficient conditions on the beta-effect for decay of the magnetic modes (3.1).

Corollary 1. (General Planar-Turbulence Beta-Effect). Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy (1.11) and (1.12), respectively, with constant α_{ij} and β_{ijk}, and let the magnetic field be of the form (3.1). If the matrices $\eta\mathbf{I} + \mathbf{M}_i^S$, $i = 1, 2$, where $\mathbf{M}_i$ is given by (3.6) and the superscript S denotes the symmetric part, are positive definite, then $\operatorname{Re}\gamma < 0$ and the magnetic modes (3.1) all decay to zero. In terms of the minimum eigenvalues, if $\lambda_{\min}(\mathbf{M}_i^S) > -\eta$, $i = 1, 2$, then $\operatorname{Re}\gamma < 0$.

Proof. By the conditions (1.11) and (1.12) on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the matrix $\mathbf{M}$ in (3.2) has elements $m_{31} = 0$ and $m_{32} = 0$. Thus replacing $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha} + i\boldsymbol{\beta}\cdot\mathbf{k}$ and η_T by η in (3.5) yields $\operatorname{Re}\gamma = -\eta|\mathbf{k}|^2 - \mathbf{k}^T\mathbf{M}_1\mathbf{k}, -\eta|\mathbf{k}|^2 - \mathbf{k}^T\mathbf{M}_2\mathbf{k}$. The result follows. ☐

Of course, the single mode (3.1) decays if $\boldsymbol{\beta}$ and the wave-vector $\mathbf{k}$ satisfy $\mathbf{k}^T\mathbf{M}_1\mathbf{k} > -\eta|\mathbf{k}|^2$ and $\mathbf{k}^T\mathbf{M}_2\mathbf{k} > -\eta|\mathbf{k}|^2$.

As an application of the corollary we first consider a beta-effect $\boldsymbol{\beta}^\Omega$ which is affine in one invariant direction $\boldsymbol{\Omega}$, i.e. $\boldsymbol{\beta}^\Omega$ is of the form (1.8) with $\mathbf{e}$ replaced by $\boldsymbol{\Omega}$ and without the quadratic or cubic terms in $\boldsymbol{\Omega}$. In cartesian component form,

$$\beta^\Omega_{ijk} := \beta\varepsilon_{ijk} + \beta_2\Omega_j\delta_{ki} + \beta_3\Omega_k\delta_{ij}\,. \tag{3.7}$$

The beta-effect β^Ω does not satisfy the necessary conditions (1.12). One way to modify (1.8) to enforce the conditions with minimal changes is to zero the relevant components,

$$\beta_{ijk} = \beta^\Omega_{ijk} - \beta^\Omega_{11k}\delta_{i1}\delta_{j1} - \beta^\Omega_{12k}\delta_{i1}\delta_{j2} - \beta^\Omega_{21k}\delta_{i2}\delta_{j1} - \beta^\Omega_{22k}\delta_{i2}\delta_{j2}\,. \tag{3.8}$$

Hence $\beta_{231} = -\beta_{132} = \beta$, $\beta_{232} = \beta_{131} = \beta_2\Omega_3$ and $-\beta_{321} = \beta_{312} = \beta$, $\beta_{323} = \beta_2\Omega_2$, $\beta_{313} = \beta_2\Omega_1$. The remaining elements of $\mathbf{M}_1$ and $\mathbf{M}_2$ are zero. Thus the minimum eigenvalues $\lambda_{\min}(\mathbf{M}_1^S) = 0$ and $\lambda_{\min}(\mathbf{M}_2^S) = \frac{1}{2}(\beta - \sqrt{\beta^2 + \beta_2^2|\boldsymbol{\Omega}_\perp|^2})$, where $\boldsymbol{\Omega}_\perp := (\Omega_1, \Omega_2, 0)$. Hence $\eta\mathbf{I} + \mathbf{M}_1^S$ is positive definite, if $\eta + \beta > 0$; and $\eta\mathbf{I} + \mathbf{M}_2^S$ is positive definite, if $|\beta_2\boldsymbol{\Omega}_\perp| < 2\sqrt{\eta(\eta+\beta)}$, i.e. if the invariant direction $\boldsymbol{\Omega}$ only differs weakly from that of the planar turbulence.

We now consider the beta-effect $\boldsymbol{\beta}^{\mathrm{pl}}$, which is more physically realistic since it has the same invariant direction as the planar turbulence. Now $\beta_{231} = -\beta_{132} = \beta + \beta_5$, $\beta_{232} = \beta_{131} = \beta_2$, $-\beta_{321} = \beta_{312} = \beta + \beta_4$ and the remaining elements of $\mathbf{M}_1$ and $\mathbf{M}_2$ are zero. Thus $\lambda_{\min}(\mathbf{M}_1^S) = \min(\beta + \beta_5, 0)$ and $\lambda_{\min}(\mathbf{M}_2^S) = \min(\beta + \beta_4, 0)$. Hence $\eta\mathbf{I} + \mathbf{M}_i^S$ is positive definite, if $\beta_5 > -\eta - \beta$ and $\beta_4 > -\eta - \beta$ for $i = 1, 2$ respectively. These conditions are not obviously implied by the formula (2.12). Thus, the decay of the magnetic field appears to be conditional, which is not consistent with the laminar planar flow ADT.

A restricted form of theorem 2, in which $0 < |\mathbf{k}| \ll 1$, is a corollary of theorem 1, since (1.10) is true for the $\boldsymbol{\alpha}$ in (1.11); whereas theorem 2 holds for all $|\mathbf{k}| > 0$. However, the restriction $|\mathbf{k}| \ll 1$ is consistent with the two-scale assumption underlying (2.6) and (2.7), and hence the theorem.

3.2 Homogeneous Two-Dimensional Alpha-Effect ADT in E^3

We assume in this subsection that the turbulence is two-dimensional in the sense of Definition 2 in the introduction and that the magnetic field is also two-dimensional, i.e. $\mathbf{B}$ is independent of x_3. Thus (3.1) must be restricted to

$$\overline{\mathbf{B}} = \widehat{\mathbf{B}}e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp + \gamma t}\,, \qquad \mathbf{k}_\perp \neq \mathbf{0}\,, \tag{3.9}$$

where $\mathbf{r}_\perp := (x_1, x_2, 0)$ and $\mathbf{k}_\perp = (k_1, k_2, 0)$. The superposition of magnetic modes must be modified accordingly. Thus, for example,

$$\mathbf{B}(\mathbf{r}_\perp, t) = \int_{\mathbb{R}^2} \widehat{\mathbf{B}}(\mathbf{k})e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp + \gamma(\mathbf{k}_\perp)t}\,d^2\mathbf{k}_\perp\,, \quad \text{where} \quad \int_{\mathbb{R}^2}(1 + |\mathbf{k}_\perp|^2)|\widehat{\mathbf{B}}(\mathbf{k})|^2\,d^2\mathbf{k}_\perp < \infty\,.$$

The symmetry restrictions on the alpha and beta-effects differ from the planar case. The Fourier transforms must be modified for two-dimensional functions: if a function $f : \mathbb{R}^3 \to \mathbb{R}$ is independent of x_3, i.e. $f(\mathbf{r} + c\mathbf{1}_3) = f(\mathbf{r})$ for all $\mathbf{r} \in \mathbb{R}^3$ and $c \in \mathbb{R}$, then $\mathcal{F}\{f\} = \widehat{f}(\mathbf{k}, \omega) = \delta(k_3)\widehat{f}_\perp(\mathbf{k}_\perp, \omega)$, where $f_\perp(\mathbf{r}_\perp) := f(\mathbf{r})$.

$$\widehat{f}_\perp(\mathbf{k}_\perp, \omega) := \frac{1}{(2\pi)^3} \iint_{\mathbb{R}^3} f(\mathbf{r}, t) e^{-i(\mathbf{k}_\perp \cdot \mathbf{r}_\perp - \omega t)}\, d^2\mathbf{r}_\perp\, dt$$

using the Fourier representation of the Dirac delta distribution, $\delta(k) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-ikz}\, dz$. Integrating over k_3 yields

$$\alpha_{ij} = i(\epsilon_{ilq}\delta_{nj} - \epsilon_{ilj}\delta_{nq}) \int_{\mathbb{R}^3} \frac{k_n(1 - \delta_{3n})\widehat{Q}_{ql}(\mathbf{k}_\perp, \omega)}{\eta k_\perp^2 - i\omega}\, d^2\mathbf{k}_\perp\, d\omega \tag{3.10}$$

and

$$\beta_{ijk} = -(\epsilon_{ilq}\delta_{nj} - \epsilon_{ilj}\delta_{nq}) \int_{\mathbb{R}^3} \frac{k_n(1 - \delta_{3n})}{\eta k_\perp^2 - i\omega} \frac{\partial \widehat{Q}_{ql}(\mathbf{k}_\perp, \omega)}{\partial k_k}\, d^2\mathbf{k}_\perp\, d\omega\,,$$

In particular, $\alpha_{33} = 0$ since $i = j = 3$ forces $n = 3$, and hence the alpha-effect is of the form (1.15) with constant matrix elements. We do not require or assume that the alpha-effect is symmetric. By the same reasoning and the fact that $\widehat{Q}_{ql}$ is independent of k_3, the cartesian components of $\boldsymbol{\beta}$ are restricted by the conditions (1.16). The restricted anisotropic beta-effect (1.13) satisfies these conditions whereas the isotropic (1.6) does not. The coefficient matrix $\mathbf{M}$ in (3.3) reduces to

$$\mathbf{M} = (\gamma + \eta_T k_\perp^2)\mathbf{I} - i\mathbf{k}_\perp \times \boldsymbol{\alpha}\,. \tag{3.11}$$

Theorem 3. (Homogeneous Two-Dimensional Alpha-Effect ADT). Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy (1.15) and (1.13), respectively, with constant α_{ij} and β_{ijk}. If the magnetic field is of the two-dimensional form (3.9), then $\operatorname{Re}\gamma = -\eta_T k_\perp^2$. All magnetic modes (3.9) decay to zero.

Proof. For non-trivial solutions in (3.3) $\det \mathbf{M} = 0$, where now $\mathbf{M}$ is given by (3.11). Since $k_3 = 0$ and $\boldsymbol{\alpha}$ is of the form (1.15),

$$\begin{vmatrix} \lambda - ik_2\alpha_{31} & -ik_2\alpha_{32} & 0 \\ ik_1\alpha_{31} & \lambda + ik_1\alpha_{32} & 0 \\ -i(k_1\alpha_{21} - k_2\alpha_{11}) & -i(k_1\alpha_{22} - k_2\alpha_{12}) & \lambda - i(k_1\alpha_{23} - k_2\alpha_{13}) \end{vmatrix} = 0\,,$$

where $\lambda := \gamma + \eta_T k_\perp^2$. The significant feature is the two zero elements, similar to the planar turbulence case. Solving for λ gives

$$\gamma = -\eta_T k_\perp^2 + i(k_1\alpha_{23} - k_2\alpha_{13})\,, \quad -\eta_T k_\perp^2 - i(k_1\alpha_{32} - k_2\alpha_{31})\,. \tag{3.12}$$

The solenoidal condition $\mathbf{k} \cdot \widehat{\mathbf{B}} = 0$ removes the non-solenoidal mode with $\gamma = -\eta_T k_\perp^2$. Since α_{ij}, k_1, k_2 are real, the result follows. □

The result is essentially unchanged if all the flows in the ensemble are incompressible, $\nabla \cdot \mathbf{v} = 0$. In this case $\mathbf{k} \cdot \widehat{\mathbf{Q}} = \mathbf{0}$ and $\widehat{\mathbf{Q}} \cdot \mathbf{k} = \mathbf{0}$. The last term in (3.10) contains $\delta_{nq}k_n\widehat{Q}_{ql} = k_n\widehat{Q}_{nl} = 0$. Thus

$$\alpha_{ij} = i\epsilon_{ilq} \int_{\mathbb{R}^3} \frac{k_j\widehat{Q}_{ql}(\mathbf{k}_\perp, \omega)}{\eta k_\perp^2 - i\omega}\, d^2\mathbf{k}_\perp\, d\omega$$

and hence $\alpha_{13} = 0 = \alpha_{23}$. Thus $\gamma = -\eta k_\perp^2\,, -i(k_1\alpha_{32} - k_2\alpha_{31}) - \eta k_\perp^2$. One mode is still oscillatory.

The theorem fails if the magnetic field depends on x_3. Thus if $k_3 \neq 0$ and $k_1, k_2 \ll k_3$, equation (3.11) implies $\mathbf{k} \times \boldsymbol{\alpha} \cdot \widehat{\mathbf{B}} = \lambda\widehat{\mathbf{B}}$, where *lambda* is defined by $\gamma = -\eta k^2 + i\lambda$ and

$$\mathbf{k} \times \boldsymbol{\alpha} = k_3 \begin{pmatrix} -\alpha_{21} & -\alpha_{22} & -\alpha_{23} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & 0 & 0 \end{pmatrix} + \boldsymbol{\mathcal{O}}(k_1, k_2)\,.$$

If $\boldsymbol{\alpha}$ is symmetric, then $\lambda = \pm ik_3\sqrt{A_{33}} + \mathcal{O}(k_1, k_2)$ where A_{33} is the cofactor of α_{33} and $\gamma = -\eta k_3^2 \pm k_3\sqrt{A_{33}} + \mathcal{O}(k_1, k_2)$. If $A_{33} > 0$ and the alpha-effect magnetic Reynolds number $\sqrt{A_{33}}/\eta|k_3| > 1$, then growing modes exist. The fastest growing modes occur for wavenumber $k_3 \approx \sqrt{A_{33}}/2\eta$ with growth rate $\operatorname{Re}\gamma \approx A_{33}/4\eta$. For example, Moffatt's (1970b) $\boldsymbol{\alpha}$-effect has $A_{33} = \alpha_{11}^2 > 0$, which gives growing modes.

If general beta-effects are considered, restricted only by (1.16), we get (3.3) with $\mathbf{M}$ reduced to

$$\mathbf{M} = (\gamma + \eta k_\perp^2)\mathbf{I} - i\mathbf{k}_\perp \times (\boldsymbol{\alpha} + i\boldsymbol{\beta} \cdot \mathbf{k}_\perp)\,. \tag{3.13}$$

Corollary 2. (General Two-Dimensional Turbulence Beta-Effect). Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy (1.15) and (1.13), respectively, with constant α_{ij} and β_{ijk}, and let the magnetic field be of the two-dimensional form (3.9). If the matrices $\eta_T\mathbf{I} + \mathbf{M}_i^S$, $i = 1, 2$, are positive definite when restricted to the subspace $\{\mathbf{k}_\perp|\mathbf{k} \in \mathbb{R}^3\}$, where $\mathbf{M}_i$ is given by (3.6), then $\mathrm{Re}\,\gamma < 0$ and all magnetic modes (3.9) decay to zero.

Proof. From (1.15) and (1.16) the matrix $\mathbf{M}$ in (3.13) has elements $m_{13} = 0$ and $m_{23} = 0$. Thus replacing $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha} + i\boldsymbol{\beta} \cdot \mathbf{k}$ in (3.12) yields $\mathrm{Re}\,\gamma = -\eta|\mathbf{k}_\perp|^2 - \mathbf{k}_\perp^T\mathbf{M}_1\mathbf{k}_\perp, -\eta|\mathbf{k}_\perp|^2 - \mathbf{k}_\perp^T\mathbf{M}_2\mathbf{k}_\perp$. The result follows. □

As an application of this corollary we consider the beta-effect $\boldsymbol{\beta}^{\mathrm{td}}$ with the preferred direction of the two-dimensional turbulence. Then $\beta_{231} = -\beta_{132} = \beta + \beta_5$, $\beta_{232} = \beta_{131} = \beta_2$, $-\beta_{321} = \beta_{312} = \beta + \beta_4$ and the remaining elements of $\mathbf{M}_1$ and $\mathbf{M}_2$ are zero. Thus $\eta\mathbf{I} + \mathbf{M}_i^S$ are positive definite when restricted to the two-dimensional subspace, if $\beta_5 > -\eta - \beta$ and $\beta_4 > -\eta - \beta$ for $i = 1, 2$ respectively, as in the planar case. Thus the magnetic modes (3.9) decay conditionally to zero.

4 Concluding Remarks

In the homogeneous two-dimensional and planar mean-field antidynamo theorems proven herein the alpha-effect is consistent with the corresponding laminar theorem, in the sense that no restrictions are imposed on the alpha-effect, apart from (1.11) and (1.15) which arise purely from the turbulence. This is false for the beta-effect. In addition to conditions (1.12) and (1.16), which arise from the turbulence, the beta-effect must either be restricted to (1.13) or satisfy the conditions of corollaries 1 and 2, Thus there remains the possibility that there may exist beta-effects in planar or two-dimensional turbulence which allow energy to flow from the fluctuating field $\mathbf{B}'$ to $\overline{\mathbf{B}}$ and $\overline{\mathbf{B}}$ to grow exponentially, even though $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$ must decay to zero to satisfy the associated laminar ADT. The difficulty is that (1.12) and (1.16) are only necessary conditions. Even imposing the additional condition of a preferred direction consistent with the turbulence does not remove the inconsistency.

The mean-field planar ADT with zero mean velocity fails if $\mathbf{e}$ in the definition of planar turbulence is different for each velocity, since neither the mean flow nor the fluctuating components must be planar. If the mean velocity is non-zero the ADT should generalise, even if $\mathbf{e}$ depends on the velocity. Similarly for the two-dimensional ADT. There is also the added complication of the mean magnetic field not being two-dimensional.

Extensions of the theorems, if the turbulence is inhomogeneous, or the mean velocity is non-zero or the conducting region is bounded in certain directions, are the subject of future work. It is known that the laminar planar velocity theorem may fail in a bounded conductor, e.g. a sphere (Bachtiar et al 2006). Fourier techniques are still useful for inhomogeneous turbulence but not so for non-zero mean velocity or finite conducting regions. Weaker forms of planar and two-dimensional turbulence are also possible, based on the statistical properties of the turbulence, such as the alpha-effect.

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