TODA FRAMES, HARMONIC MAPS AND EXTENDED DYNKIN DIAGRAMS

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ABSTRACT. This paper proves two main theorems. The first is that all cyclic primitive immersions of a genus one surface into G/T can be constructed by integrating a pair of commuting vector fields on a finite dimensional vector subspace of a Lie algebra. Here G is any simple real Lie group (not necessarily compact), T is a Cartan subgroup and G/T has a k-symmetric space structure induced from the Coxeter automorphism. If G is not compact, such a structure may not exist. We characterise the G/T to which the theory applies in terms of extended Dynkin diagrams, first observing that a Coxeter automorphism preserves the real Lie algebra $\mathfrak{g}$ if and only if any corresponding Cartan involution defines a permutation of the extended Dynkin diagram for $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. The second main result is that every involution of the extended Dynkin diagram for a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is induced by a Cartan involution of a real form of $\mathfrak{g}^{\mathbb{C}}$.

1. Introduction

The last few decades have seen significant progress in the understanding and classification of harmonic maps from surfaces into compact real Lie groups and symmetric spaces. An important class of harmonic maps are those of *finite type*, which are obtained as the solutions to a pair of ordinary differential equations on a finite dimensional vector subspace of a loop algebra. This is a far simpler process than attempting to solve the Laplace-Beltrami equation directly, and so motivates us to determine circumstances under which harmonic maps are of finite type. Similarly, when the target manifold is a k-symmetric space, $k > 2$, it is natural to restrict our attention to those harmonic maps which are cyclic primitive and ask when these maps are of finite type. Many papers (e.g. [10, 14, 2, 9, 4, 3, 6]) have addressed these questions when the target Lie group or (k)-symmetric space is *compact*. We remove the need for this compactness assumption and in Theorem 5.2 show that all maps from a genus one surface into a k-symmetric space G/T possessing a Toda frame are of finite type, where G is any simple real Lie group preserved by a Coxeter automorphism and T is the corresponding Cartan subgroup. A natural generalisation of the usual 2-dimensional affine Toda field equations provides the integrability condition for the existence of a Toda frame, and so we make contact with classical integrable systems theory. To determine the spaces G/T and the harmonic maps into them to which this theory applies we address the following two questions, each of independent interest:

(1) *When is G preserved by a Coxeter automorphism with respect to the Cartan subgroup T?* and
(2) *Assuming this, when does a map from a surface into G/T possess a Toda frame?*

The first of these does not arise in the compact situation, since a Coxeter automorphism for a complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ automatically preserves a compact real form $\mathfrak{g}$. The need to address this question, and the rather interesting answer which arises, is the most significant difference between the general situation studied here and the compact case. We characterise when a Coxeter automorphism preserves a real form of a complex simple Lie algebra, which is equivalent to the corresponding real Lie group G being preserved whenever G simply connected or adjoint. Given simple roots for $\mathfrak{g}^{\mathbb{C}}$ spanning a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$, let σ be the associated Coxeter automorphism and Θ a Cartan involution with respect to $\mathfrak{g}$ that preserves $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{t}^{\mathbb{C}}$. Then σ preserves $\mathfrak{g}$ if and only if Θ defines a permutation of the extended Dynkin diagram, so in particular whenever $\mathfrak{t}$ is a maximally compact Cartan subalgebra (Proposition 3.1). In Theorem 3.2 we prove that all involutions of the extended Dynkin diagram for a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ arise from a Cartan involution for some real form $\mathfrak{g}$. The second question is answered in Theorem 4.2, where it is proven that a map from a surface into G/T locally has a Toda frame precisely when it is cyclic primitive and a certain function is constant. Cyclic primitive maps are in particular harmonic and play an analogous role for k-symmetric spaces as harmonic maps do for symmetric spaces. This and our finite-type result are the natural extensions of results obtained in [3] in the case when G is compact.

Harmonic maps from surfaces into Lie groups and symmetric spaces arise naturally in many geometric and physical problems. On the geometric side, strong motivation comes from the study of surfaces with particular curvature properties. For example, minimal surfaces are described by conformal harmonic maps and both constant mean curvature and Willmore surfaces are characterised by having harmonic Gauss maps into particular symmetric spaces. From the physics viewpoint, these harmonic maps are interesting because of their relationship with the appropriate Yang-Mills equations and non-linear sigma-models. Indeed the harmonic map equations on a Riemann surface are precisely the reduction of the Yang-Mills equations on $\mathbb{R}^{2,2}$ obtained by considering solutions invariant under translation in the directions of negative signature. Classical solutions of sigma-models are given by harmonic maps into (non-compact) as pseudo-Riemannian manifolds. In [7] we study an explicit example, namely harmonic tori in de Sitter spaces S^m_1. In particular we apply the theory of this paper to the superconformal such maps with globally defined harmonic sequence to see that they may all be obtained by integrating a pair of commuting vector fields on a finite-dimensional vector space. It follows that all Willmore tori in S^3 without umbilic points may be obtained in this simple way.

The structure of this paper is as follows. In section 2 we give the general theory for harmonic maps of surfaces into symmetric spaces and for primitive maps into k-symmetric spaces when the relevant Lie group G is equipped with a bi-invariant pseudo-metric. The question of when a Coxeter automorphism preserves the real form of the complex simple Lie algebra is addressed in section 3 in terms of Cartan involutions and extended Dynkin diagrams. In section 4 we use the affine Toda equations to find the conditions under which a map $\psi : \mathbb{C} \to G/T$ processes a Toda frame. Section 5 contains the proof that if an immersion of a genus one surface into G/T is cyclic primitive then it is of finite type.

It is a pleasure to thank Anthony Henderson for helpful conversations regarding the Lie-theoretic results of section 3.

2. Finite type maps into symmetric spaces

The fact that a harmonic map from a surface to a Lie group corresponds to a loop of flat connections [15, 17] is the fundamental observation that enables one to apply integrable systems techniques to the study of these maps. The Cartan map $G/H \to G$ from a symmetric space to the relevant Lie group is well-known to be a totally geodesic immersion when G is compact and equipped with a bi-invariant Riemannian metric. The composition of a harmonic map with a totally geodesic one is again harmonic, so this enables harmonic maps into symmetric spaces to be studied using the same tools as those into Lie groups, and in particular in terms of a loop of flat connections. We show in Theorem 2.1 that when G has merely a bi-invariant pseudo-metric that the Cartan map is again a totally geodesic immersion. This is a reasonably straightforward extension of the arguments for the compact case. In particular all reductive Lie groups possess a bi-invariant pseudo-metric. We can hence study harmonic maps into G/H using integrable systems methods regardless of whether G is compact. The main purpose of this section is to provide necessary background information and to fix notation. In particular we hope that it will benefit those readers interested in our Lie-theoretic results, who may not be fully versed in the integrable systems literature. With the exception of Theorem 2.1, the material in this section is not new.

Let G be a semisimple Lie group. Recall that a homogeneous space G/H is a k-*symmetric space* ($k > 1$) if there is an automorphism $\tau : G \to G$ of order k such that

$$(G^\tau)_0 \subset H \subset G^\tau$$

where G^τ denotes the fixed point set of τ, and $(G^\tau)_0$ the identity component of G^τ. When $k = 2$, we say that G/H is a *symmetric space*. We have the induced action

$$\begin{aligned} \tau : G/H &\to G/H \\ gH &\mapsto \tau(g)H. \end{aligned}$$

We write τ also for the induced automorphism of $\mathfrak{g}$ and note the $\mathbb{Z}_k$-grading

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{j=0}^{k-1} \mathfrak{g}_j^\tau, \ [\mathfrak{g}_j^\tau, \mathfrak{g}_l^\tau] \subset \mathfrak{g}_{j+l}^\tau,$$

where $\mathfrak{g}_j^\tau$ denotes the $e^{j\frac{2\pi i}{k}}$-eigenspace of τ.

We shall be interested in harmonic maps from a Riemann surface Σ into a symmetric space G/H. When G is compact, the Killing form on $\mathfrak{g}$ induces a bi-invariant metric on G/H and the harmonic map equations for $f : \Sigma \to G/H$ may either be calculated directly [20], using Noether's Theorem [16], or by composing f with the Cartan map $G/H \to G$, which is well-known in this case to be a totally geodesic immersion [8]. Recall here that the Cartan map of a symmetric space is given by

$$\begin{aligned} \iota : \ & G/H \to G \\ & gH \mapsto g\tau(g^{-1}). \end{aligned}$$

We suppose merely that G has a bi-invariant pseudo-metric. Then analogous computations hold; in particular we can reduce the problem to studying harmonic maps into the Lie group G, due to the following result.

Theorem 2.1. *Let G be a semisimple Lie group with bi-invariant pseudo-metric $\langle \cdot, \cdot \rangle$ and G/H a symmetric space with respect to the involution $\tau : G \to G$. Then $\iota : gH \mapsto \tau(g)g^{-1}$ is a totally geodesic immersion $G/H \to G$. If $H = G^\tau$, then ι is additionally an embedding.*

Let us call a Lie group G *reductive* if its Lie algebra $\mathfrak{g}$ is reductive, that is has radical equal to its centre. Then $\mathfrak{g}$ may be written as the direct sum of a semisimple Lie algebra and an abelian one. On the semisimple Lie algebra the Cartan-Killing form is non-degenerate, whilst on the abelian algebra any bilinear form is invariant under the adjoint action of the group. Combining these we obtain the existence of a bi-invariant pseudometric on any reductive Lie group, and hence the above theorem in particular applies when G is reductive.

Proof. ι is an immersion: Suppose $d\iota_{gH}(\gamma'(0)) = 0$ for some smooth path γ in G/H with $\gamma(0) = gH$. Take a lift $\tilde\gamma$ of γ to G with $\tilde\gamma(0) = g$ and write $\pi : G \to G/H$ for the projection. Then

$$0 = \frac{d}{dt}\bigg|_{t=0} \left(\tau\left(\tilde\gamma(t)\right)\left(\tilde\gamma(t)\right)^{-1}\right) = d\tau_g(\tilde\gamma'(0))g^{-1} - \tau(g)g^{-1}\tilde\gamma'(0)g^{-1},$$

so

$$d\tau_e(g^{-1}\tilde\gamma'(0)) = \tau(g^{-1})d\tau_g(\tilde\gamma'(0)) = g^{-1}\tilde\gamma'(0)$$

and $\gamma'(0)$ is zero in $T_{gH}(G/H)$ so $d\iota_{gH}$ is injective.

ι is totally geodesic: Let ∇^l denote the connection on G obtained by trivialising TG by left translation, and similarly ∇^r that induced from trivialising by right translation. A computation shows that $\nabla^r = \nabla^l + \mathrm{ad}_{g^{-1}dg}$ and hence

$$\nabla = \frac{1}{2}(\nabla^l + \nabla^r)$$

is the Levi-Civita connection of the pseudo-metric $\langle \cdot, \cdot \rangle$.

Denote by $\exp : \mathfrak{g} \to G$ the Lie-theoretic exponential map, and by e the differential-geometric exponential map associated to the Levi-Civita connection ∇. Note that as in the definite case, for each $X \in \mathfrak{g}$ the map

$$\begin{aligned} \gamma_X : \mathfrak{g} &\to G \\ t &\mapsto e^{tX} \end{aligned}$$

is a geodesic, i.e. $\nabla_{\gamma'_X}\gamma'_X = 0$, so exp and e agree on the domain of e. Since the pseudo-metric is bi-invariant, we conclude that the geodesics through $g \in G$ are locally of the form $\gamma(t) = ge^{tX}$. Denote by $\mathfrak{m}$ the (-1)-eigenspace of $\tau : \mathfrak{g} \to \mathfrak{g}$, and note that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H. The lift $\tilde{\gamma}(t) = ge^{tX}H$ is horizontal, in the sense that $\tilde{\gamma}'(t) \in ge^{tX}\mathfrak{m}$. Thus the geodesics in G/H through gH are locally of the form $\tilde{\gamma}(t) = ge^{tX}H$. Since

$$\iota(ge^{tX}H) = ge^{tX}\tau(e^{-tX})\tau(g^{-1}) = ge^{2tX}\tau(g^{-1}) = g\tau(g^{-1})e^{t\tau(g)X\tau(g^{-1})}$$

is again a geodesic, we conclude that ι is totally geodesic.

If $H = G^\tau$, then ι is an embedding: In this case if $\iota(g_1H) = \iota(g_2H)$, then $g_1^{-1}g_2 = \tau(g_1^{-1}g_2)$, and so $g_1^{-1}g_2 \in H$, and thus ι is injective. □

Let $F : U \to G$ be a smooth lift of $f : U \to G/H$ on some simply connected $U \subset \Sigma$, where we assume henceforth that G is semisimple and has a bi-invariant pseudo-metric (we will later restrict our attention to simple such G.). By the above theorem, f is harmonic if and only if $\iota \circ f$ is. The Maurer-Cartan form on G is the unique left-invariant $\mathfrak{g}$-valued 1-form which acts as the identity on $\mathfrak{g}$. We denote it by ω, and note that if G is a linear group, then $\omega = g^{-1}dg$. We will use this notation throughout even in the non-linear case. Write $\tilde{f} = \iota \circ f$ and $\Phi = \tilde{f}^*(\omega) = \tilde{f}^{-1}d\tilde{f}$. For any smooth $\tilde{f}$, the form Φ satisfies the zero-curvature condition

$$d\Phi + \frac{1}{2}[\Phi \wedge \Phi] = 0, \tag{1}$$

known as the Maurer-Cartan equation. Recall that for vector fields X, Y,

$$[\Phi \wedge \Phi](X, Y) = 2[\Phi, \Phi](X, Y) = [\Phi(X), \Phi(Y)].$$

The condition that the map $\tilde{f} : \Sigma \to G$ is harmonic can be written as

$$d * \Phi = 0. \tag{2}$$

Noting that $\tilde{f} = \tau(F)F^{-1}$, we have

$$\Phi = F\left(\tau(F)^{-1}d(\tau(F)) - F^{-1}dF\right)F^{-1} = -2\mathrm{Ad}_F(\varphi_\mathfrak{m}), \tag{3}$$

where $\varphi = \varphi_\mathfrak{h} + \varphi_\mathfrak{m}$ is the decomposition of $\varphi := F^{-1}dF$ into the eigenspaces of τ. Then (2) becomes

$$0 = d(\mathrm{Ad}_F(*\varphi_\mathfrak{m})) = \mathrm{Ad}_F(d * \varphi_\mathfrak{m} + [\varphi \wedge *\varphi_\mathfrak{m}]) \tag{4}$$

or equivalently,

$$d * \varphi_\mathfrak{m} + [\varphi \wedge *\varphi_\mathfrak{m}] = 0. \tag{5}$$

One can also compute the harmonic map equations directly for f. Writing $[\mathfrak{m}]$ for the subbundle of $G/H \times \mathfrak{g}$ whose fibre at $g \cdot x$ is $\mathrm{Ad}_g(\mathfrak{m})$, we have an isomorphism $[\mathfrak{m}] \cong T(G/H)]$ given by

$$\begin{aligned} [\mathfrak{m}]_y &\to T_y G/H \\ Y &\mapsto \frac{d}{dt}\bigg|_{t=0} e^{tY} \cdot y. \end{aligned}$$

The inverse of this isomorphism defines a $\mathfrak{g}$-valued 1-form θ on the symmetric space G/H, which we term its Maurer-Cartan form. Then [16] f is harmonic if and only if

$$d * (f^*\theta) = 0$$

and using that

$$f^*\theta = \mathrm{Ad}_F(\varphi_{\mathfrak{m}})$$

we recover (4). Write $\varphi'_{\mathfrak{m}} + \varphi''_{\mathfrak{m}}$ for the decomposition of $\varphi_{\mathfrak{m}}$ into dz and $d\bar{z}$ parts. Since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, a straightforward computation shows (1) and (5) are equivalent to the requirement that for each $\lambda \in S^1$, the form

$$\varphi_\lambda = \lambda \varphi'_{\mathfrak{m}} + \varphi_{\mathfrak{h}} + \lambda^{-1} \varphi''_{\mathfrak{m}} \tag{6}$$

satisfies the Maurer-Cartan equation

$$d\varphi_\lambda + \frac{1}{2}[\varphi_\lambda \wedge \varphi_\lambda] = 0. \tag{7}$$

Some solutions to (7) can be obtained simply by solving a pair of commuting ordinary differential equations on a finite-dimensional loop algebra. These unusually simple solution is said to be of and finite type.

Let G/K be a k-symmetric space for $k > 2$ and τ the corresponding kth order involution. As we shall now explain when mapping into a k-symmetric space for $k > 2$ it is natural to restrict our attention to a subclass of harmonic maps consisting of those which are primitive, a notion that we now define. Again we have the reductive splitting

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with

$$\mathfrak{p}^{\mathbb{C}} = \bigoplus_{j=1}^{k-1} \mathfrak{g}_j^\tau, \qquad \mathfrak{k}^{\mathbb{C}} = \mathfrak{g}_0^\tau.$$

Similarly to before we may define the Maurer-Cartan form θ of the k-symmetric space G/K when $k > 2$. For any smooth lift $F : U \to G$ of $\psi : U \to G/K$, writing $\varphi = F^*\omega$ we have

$$\psi^*\theta = \mathrm{Ad}_F \varphi_{\mathfrak{p}}.$$

We say that a smooth map ψ of a surface Σ into G/K is *primitive* if the image of $\psi^*\theta'$ is contained in $[\mathfrak{g}_1]$. Equivalently, it is primitive precisely when $\varphi' = F^{-1}\partial F$

takes values in $\mathfrak{g}_0^\tau \oplus \mathfrak{g}_1^\tau$. Using that $[\mathfrak{g}_1^\tau, \mathfrak{g}_{-1}^\tau] \subset \mathfrak{g}_0^\tau$, the Maurer-Cartan equation for φ decomposes into $\mathfrak{g}_1^\tau$, $\mathfrak{g}_0^\tau$ and $\mathfrak{g}_{-1}^\tau$ components as

$$\begin{aligned} d\varphi'_{\mathfrak{p}} + [\varphi_{\mathfrak{k}} \wedge \varphi'_{\mathfrak{p}}] &= 0 \\ d\varphi_{\mathfrak{k}} + \frac{1}{2}[\varphi_{\mathfrak{k}} \wedge \varphi_{\mathfrak{k}}] + [\varphi'_{\mathfrak{p}} \wedge \varphi''_{\mathfrak{p}}] &= 0 \\ d\varphi''_{\mathfrak{p}} + [\varphi_{\mathfrak{k}} \wedge \varphi''_{\mathfrak{p}}] &= 0. \end{aligned} \tag{8}$$

From these equations one easily verifies that primitive maps are in particular harmonic. Moreover [5] if G/H is a symmetric space with $K \subset H$ and the corresponding reductive splitting preserved under τ, then the projection of $\psi : \Sigma \to G/K$ into G/H is harmonic. An analogous calculation to that above shows that on simply connected subsets $U \subset \Sigma$, a primitive map $\psi : U \to G/K$ is equivalent to a loop

$$\varphi_\lambda = \lambda\varphi'_{\mathfrak{p}} + \varphi_{\mathfrak{k}} + \lambda^{-1}\varphi''_{\mathfrak{p}}, \quad \lambda \in S^1 \tag{9}$$

of $\mathfrak{g}$-valued 1-forms each satisfying the Maurer-Cartan equation. We see then that both harmonic maps into symmetric spaces and primitive maps into k-symmetric spaces are governed by the same equation (7) so we turn now to the question of constructing solutions to this equation.

Let ΩG be the loop group $\Omega G = \{\gamma : S^1 \to G\}$ with corresponding loop algebra $\Omega\mathfrak{g} := \{\xi : S^1 \to \mathfrak{g}\}$, where the loops are assumed real analytic without further comment. We use $\Omega\mathfrak{g}^{\mathbb{C}}$ to denote loops in the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. For studying maps into k-symmetric spaces it is helpful to consider the twisted loop group

$$\Omega^\tau G = \{\gamma : S^1 \to G : \gamma(e^{\frac{2\pi i}{k}}\lambda) = \tau(\gamma(\lambda))\}$$

and corresponding twisted loop algebra $\Omega^\tau\mathfrak{g}$ along with its complexification $\Omega^\tau\mathfrak{g}^{\mathbb{C}}$. The (possibly doubly infinite) Laurent expansion

$$\xi(\lambda) = \sum_j \xi_j\lambda^j, \quad \xi_j \in \mathfrak{g}_j^\tau \subset \mathfrak{g}^{\mathbb{C}}, \quad \Phi_{-j} = \bar{\Phi}_j$$

allows us to filter $\Omega^\tau\mathfrak{g}^{\mathbb{C}}$ by finite-dimensional subspaces

$$\Omega_d^\tau = \{\xi \in \Omega\mathfrak{g} \mid \xi_j = 0 \text{ whenever } |j| > d\}.$$

Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{k}$ and recall that a non-zero $\alpha \in (\mathfrak{t}^{\mathbb{C}})^*$ is a *root* with corresponding *root space* $\mathcal{G}^\alpha \subset \mathfrak{g}^{\mathbb{C}}$ if $[X_1, X_2] = \alpha(X_1)X_2$ for all $X_1 \in \mathfrak{t}$ and $X_2 \in \mathcal{G}^\alpha$. Note that the root spaces are necessarily 1-dimensional. We denote the set of roots by Δ and employ the same notation for the root system formed by considering Δ as a subset of $(\mathfrak{t}^{\mathbb{C}})^*$. Choose a set of *simple roots*, that is a subset $\{\alpha_1, \ldots, \alpha_N\}$ of Δ such that every root $\alpha \in \Delta$ can be written uniquely as

$$\alpha = \sum_{j=1}^N n_j\alpha_j,$$

where the n_j are either all positive integers or all negative integers. The *height* of α is $h(\alpha) = \sum_{j=1}^N n_j$ and the root(s) of maximal height are called *highest root(s)* whilst those of minimal height are termed *lowest root(s)*.

We similarly define the root spaces of $\mathfrak{k}^{\mathbb{C}}$. Let $\mathfrak{n}$ be the nilpotent algebra consisting of the positive root spaces of $\mathfrak{k}^{\mathbb{C}}$ with respect to a choice of simple roots and consider the resulting decomposition

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{n} \oplus \mathfrak{t}^{\mathbb{C}} \oplus \bar{\mathfrak{n}} \tag{10}$$

of $\mathfrak{k}^{\mathbb{C}}$. Then for $\eta \in \mathfrak{k}^{\mathbb{C}}$ and a local coordinate z on Σ, decomposing according to (10) we have

$$(\eta dz)_{\mathfrak{h}} = r(\eta)dz + \overline{r(\eta)}d\bar{z}$$

where $r : \mathfrak{k}^{\mathbb{C}} \to \mathfrak{k}^{\mathbb{C}}$ is defined by

$$r(\eta) = \eta_{\bar{\mathfrak{n}}} + \frac{1}{2}\eta_{\mathfrak{k}}. \tag{11}$$

The key observation here is that if $\xi : \Sigma \to \Omega_d^\tau$ satisfies

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda\xi_d + r(\xi_{d-1})] \tag{12}$$

then

$$\varphi_\lambda = (\lambda\xi_d + r(\xi_{d-1}))dz + (\lambda^{-1}\xi_{-d} + \overline{r(\xi_{d-1})})d\bar{z}$$

satisfies the Maurer-Cartan equation (7) (c.f. [5], Theorem 2.5). The equation

$$\frac{1}{2}(X(\xi) - iY(\xi)) = (\lambda\xi_d + r(\xi_{d-1}))$$

defines vector fields X, Y on Ω_d. A straightforward computation shows that these vector fields commute and so finding solutions to (12) is merely a matter of solving a pair of commuting ordinary differential equations. This yields a rather special class of solutions to the Maurer-Cartan equations (7) and hence of harmonic maps to symmetric spaces and primitive maps to k-symmetric spaces, $k > 2$. The flows of X, Y are easily seen to evolve on spheres in Ω_d. When G is compact, so are these spheres and hence X, Y are complete and for any initial condition the differential equation (12) has a unique solution. However when G is non-compact the global existence of solutions is not guaranteed.

Definition 2.2. A harmonic map $f : \Sigma \to G/H$ to a symmetric space or a primitive map $\psi : \Sigma \to G/K$ to a k-symmetric space, $k > 2$ is said to be of *finite type* if it has a lift $F : \Sigma \to G$ for which there exists a smooth map $\xi : \mathbb{R} \to \Omega_d^\tau\mathfrak{g}$ satisfying

$$d\xi = [\xi, \varphi_\lambda] \tag{13}$$

and

$$\varphi_\lambda = (\lambda\xi_d + r(\xi_{d-1}))dz + (\lambda^{-1}\xi_{-d} + \overline{r(\xi_{d-1})})d\bar{z}. \tag{14}$$

Here φ_λ and r are defined in (9) and (11) for the primitive case and in (6) and the obvious analogue to (11) for the harmonic case.

We introduce some terminology for later use. A *formal Killing field* for f or ψ is a smooth map $\xi : \Sigma \to \Omega^\tau\mathfrak{g}$ satisfying the Lax equation (13). When ξ takes values in some Ω_d it is termed a *polynomial Killing field* of degree d and when it additionally satisfies (14) it is an *adapted polynomial Killing field.*

When the automorphism $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is of the form $\tau = \mathrm{Ad}_{\exp M}$ for some $M \in \mathfrak{t}^{\mathbb{C}}$ where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, then we can express the eigenspaces $\mathfrak{g}_j^\tau$ of τ in terms of root spaces.

Given our chosen set of simple roots α_j, denote by η_j the corresponding dual basis of $\mathfrak{t}^{\mathbb{C}}$. For any root $\alpha = m_1\alpha_1 + \dots m_N\alpha_N$, smooth map $s_j : \Sigma \to \mathbb{C}$ and root vector $R_\alpha \in \mathcal{G}^\alpha$, a straightforward computation shows that

$$\mathrm{Ad}_{\exp(s_1\eta_1 + \dots s_N\eta_N)} R_\alpha = \exp(m_1 s_1 + \dots m_N s_N) R_\alpha. \tag{15}$$

Note that $\exp(m_1 s_1 + \dots m_N s_N)$ is a scalar function. Given $\tau = \mathrm{Ad}_{\exp(\frac{2\pi i}{k}(\sum s_j\eta_j))}$ we have

$$\mathfrak{g}_l^\tau = \mathrm{span}\{R_\alpha | \alpha = \sum_{j=1}^N m_j\alpha_j, \sum_{j=1}^N s_j m_j = l \bmod (k)\}.$$

In particular if we let $k-1$ denote the maximal height of a root of $\mathfrak{g}^{\mathbb{C}}$ and suppose

$$\sigma := \mathrm{Ad}_{\exp(\frac{2\pi i}{k}\sum_{j=1}^N \eta_j)}, \tag{16}$$

then σ is of order k and from (15) it acts on the root spaces by

$$\sigma(R_\alpha) = \exp\left(\frac{2\pi i h(\alpha)}{k}\right) R_\alpha. \tag{17}$$

We recognise the inner automorphism σ as the Coxeter automorphism associated to the identity transformation of the simple roots [1]. It plays an important role here because when it preserves the real Lie group G, it allows us to view G/T as a k-symmetric space for which $\mathfrak{g}_1^\sigma$ is the sum of the simple and lowest root spaces. Here T is a Cartan subgroup with Lie algebra $\mathfrak{t}$. Furthermore since $K = T$ in this case, the map r described in (11) is simply multiplication by $\frac{1}{2}$ and so the adapted polynomial Killing field condition (14) simplifies. Taking this N-symmetric space structure on G/T, we say that a smooth map $\psi : \Sigma \to G/T$ is *cyclic primitive* if it is primitive and satisfies the condition that the image of $\psi^*\theta'$ contains a cyclic element. Writing α_0 for the lowest root, an element in $\left(\bigoplus_{j=0}^N \mathcal{G}^{\alpha_j}\right)$ is *cyclic* if its projection to each of the root spaces $\mathcal{G}^{\alpha_0}, \mathcal{G}^{\alpha_1}, \dots, \mathcal{G}^{\alpha_N}$ is non-zero. We henceforth assume that G is simple in order to guarantee the uniqueness of the lowest root (that is, we assume that G is connected and $\mathfrak{g}$ is simple). We shall write the lowest root as

$$\alpha_0 = \sum j = 1^N m_j\alpha_j. \tag{18}$$

3. Extended Dynkin diagrams and Cartan involutions

To ascertain the k-symmetric spaces to which our theory will apply we now give conditions under which a choice of real form $\mathfrak{g}$ of a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and simple roots α_j yield a Coxeter automorphism σ which preserves the real Lie algebra $\mathfrak{g}$. When $G^{\mathbb{C}}$ is a simply connected or adjoint simple Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, this ensures that the Coxeter automorphism preserves the real group G. Let $^{-}$ denote the complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ corresponding to the real form $\mathfrak{g}$. Define the conjugate of a root α by

$$\bar{\alpha}(X) = \overline{\alpha(\bar{X})}.$$

Then from (17) we see that the condition for the Coxeter automorphism σ to preserve $\mathfrak{g}$ is that for all roots α, the height $h(\alpha)$ satisfies

$$h(\bar{\alpha}) = -h(\alpha) \bmod k,$$

or equivalently that for $j = 1, \ldots, N$ we have

$$\bar{\alpha}_j \in \{-\alpha_0, \ldots, -\alpha_N\}.$$

We will now use a Cartan involution to express this reality condition in terms of the extended Dynkin diagram for $\alpha_0, \ldots, \alpha_N$. A Cartan involution is an involution Θ of $\mathfrak{g}$ such that

$$\langle X, Y\rangle_\Theta = -\langle X, \Theta(Y)\rangle$$

is positive definite, where $\langle\cdot,\cdot\rangle$ denotes the Killing form. Using complex-linearity, Θ extends to an involution of $\mathfrak{g}^{\mathbb{C}}$. We may [12, Prop. 6.59] choose a Cartan involution Θ which preserves the Cartan subalgebra $\mathfrak{t}$.

Proposition 3.1. Let $\mathfrak{g}$ be a real simple Lie algebra, $\mathfrak{t}$ a Cartan subalgebra and Θ be a Cartan involution preserving $\mathfrak{t}$. Choose simple roots $\alpha_1, \ldots, \alpha_N$ for the root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and let σ be the corresponding Coxeter automorphism of $\mathfrak{g}^{\mathbb{C}}$ defined in (16). Then the following are equivalent:

(1) σ preserves the real form $\mathfrak{g}$,
(2) σ commutes with Θ,
(3) Θ defines a permutation of the extended Dynkin diagram for $\mathfrak{g}^{\mathbb{C}}$ consisting of the usual Dynkin diagram augmented with the lowest root α_0.

Proof. Write $\mathfrak{t} = \mathfrak{l} \oplus \mathfrak{p}$, where $\mathfrak{l}$, $\mathfrak{p}$ are respectively the $(+1)$-eigenspace and (-1)-eigenspace of the action of Θ on $\mathfrak{t}$. Then [12, Cor. 6.49] all roots α are real on $\mathfrak{p}$ and imaginary on $\mathfrak{l}$, and defining the action of Θ on roots by $\Theta(\alpha)(X) = \alpha(\Theta(X))$ we have that

$$\Theta(\alpha) = -\bar{\alpha} \quad \text{for all roots } \alpha.$$

If R_α is a root vector for α, then $\bar{R}_\alpha$ is a root vector for $\bar{\alpha}$ and $\Theta(R_\alpha)$ is a root vector for $\Theta(\alpha)$. We assume that our root vectors are chosen so that

$$R_{\bar{\alpha}} = \bar{R}_\alpha$$

and write $R_{\Theta(\alpha)} = c_\alpha \Theta(R_\alpha)$. Then using (17), a straightforward computation shows that $\sigma \circ \Theta(R_\alpha) = \Theta \circ \sigma(R_\alpha)$ if and only if $\sigma(\bar{R}_{-\alpha}) = \overline{\sigma(R_{-\alpha})}$, proving the equivalence of conditions (1) and (2) above.

The Cartan involution Θ commutes with σ if and only if for all roots α, the height function h satisfies

$$h(\Theta(\alpha)) \equiv h(\alpha) \bmod k,$$

or equivalently when Θ defines a permutation of $\alpha_0, \alpha_1, \ldots, \alpha_N$. All automorphisms of a Lie algebra preserve the Killing form and hence a Cartan involution Θ as above defines a permutation of the extended Dynkin diagram and we see the equivalence of conditions (2) and (3). □

We next show that every involution of the extended Dynkin diagram for $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ does indeed arise from a Cartan involution for some real form $\mathfrak{g}$ with Θ-stable Cartan subalgebra $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{t}^{\mathbb{C}}$. A Θ-stable Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is maximally compact if and only if Θ preserves the set of simple roots for the root system $\Delta(\mathfrak{g}, \mathfrak{t})$ [12, p 387] and so when $\mathfrak{t}$ is maximally compact, a Coxeter automorphism σ must preserve the real form $\mathfrak{g}$. (In particular, all Cartan subalgebras of a compact real form $\mathfrak{g}$ are maximally compact.) Hence the Cartan subalgebra $\mathfrak{t}$ not being maximally compact corresponds to the involution of the extended Dynkin diagram acting nontrivially on the lowest root α_0.

Theorem 3.2. *Every involution of the extended Dynkin diagram for a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is induced by a Cartan involution of a real form of $\mathfrak{g}^{\mathbb{C}}$.*

More precisely, let $\mathfrak{g}^{\mathbb{C}}$ be a simple complex Lie algebra with Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and choose simple roots $\alpha_1, \ldots, \alpha_N$ for the root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Given an involution π of the extended Dynkin diagram for Δ, there exists a real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ and a Cartan involution Θ of $\mathfrak{g}$ preserving $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{t}^{\mathbb{C}}$ such that Θ induces π and $\mathfrak{t}$ is a real form of $\mathfrak{t}^{\mathbb{C}}$. The Coxeter automorphism σ determined by $\alpha_1, \ldots, \alpha_N$ preserves the real form $\mathfrak{g}$.

Proof. Given an involution π of the extended Dynkin diagram, let us denote also by π the corresponding involution of the set $\{0, 1, \ldots, N\}$. Then π determines an involution $\hat{\pi}$ of $(\mathfrak{t}^{\mathbb{C}})^*$ preserving the root system Δ and such that $\hat{\pi}(\alpha_j) = \alpha_{\pi(j)}$.

Let $\{H_\alpha, R_\alpha \mid \alpha \in \Delta\}$ be a Chevalley basis. That is, writing $\alpha^\#$ for the dual of the root α with respect to the Killing form κ we set $H_\alpha = (2/\kappa(\alpha^\#, \alpha^\#))\alpha^\#$ and we may choose the root vectors R_α so that

$$[R_\alpha, R_{-\alpha}] = H_\alpha.$$

Given any $b_j \in \mathbb{C}$ for $j = 1, \ldots, N$, we obtain an automorphism Θ of $\mathfrak{g}^{\mathbb{C}}$ compatible with π by requiring that $\Theta(R_{\alpha_j}) = b_j R_{\hat{\pi}(\alpha_j)}$ for $j = 1, \ldots, N$ and that $\{\hat{\pi}(H_\alpha), \Theta(R_\alpha) \mid \alpha \in \Delta\}$ is a Chevalley basis. Given π and $b_1, \ldots, b_N$ we define $b_0 \in \mathbb{C}$ by the equation $\Theta(R_{\alpha_0}) = b_0 R_{\hat{\pi}(\alpha_0)}$. Our first task is to verify that for an appropriate choice of b_j, the resulting Θ is an involution.

The automorphism Θ will be an involution precisely when $b_j b_{\pi(j)} = 1$ for $j = 1, \ldots, N$. For the j such that $\pi(j) \neq 0$, this is guaranteed by taking $b_{\pi(j)} = b_j^{-1}$. This is achieved by choosing $b_j = \pm 1$ when $\pi(j) = j$ and $b_j = 1$ when $\pi(j) \neq j$). We need to show that some such choice of b_j gives $b_0 b_{\pi(0)} = 1$.

We can write R_0 as $C[R_{-\beta_1}, [R_{\beta_2}, \ldots, [R_{-\beta_{K-1}}, R_{-\beta_K}]] \ldots]$ for some non-zero constant C and β_i simple roots such that $\sum_{i=1}^K \beta_i = -\alpha_0$. Now

$$\Theta(R_0) = C \prod_{j=1}^N b_{-j}^{m_j} [R_{-\pi(\beta_1)}, [R_{-\pi(\beta_2)}, \ldots, [R_{\pi(\beta_{K-1})}, R_{-\pi(\beta_K)}]] \ldots] \tag{19}$$

and $\Theta^2(R_0) = \prod_{j=1}^N (b_{-j} b_{-\pi(j)})^{m_j} R_0$, implying

$$b_0 b_{\pi(0)} = \Pi_{j=1}^N (b_{-j} b_{-\pi(j)})^{m_j}. \tag{20}$$

Recall that $[R_{\alpha_j}, R_{-\alpha_j}] = (2/\kappa(\alpha_j^\#, \alpha_j^\#))\alpha_j^\#$. Applying Θ we know $[b_j R_{\alpha_j}, b_{-j} R_{-\alpha_j}] = (2/\kappa(\alpha_j^\#, \alpha_j^\#))\pi(\alpha_j)^\#$. Both sides are multiples of $H_{\pi(\alpha_j)}$, in particular,

$$b_j b_{-j} = \frac{\kappa(\pi(\alpha_j)^\#, \pi(\alpha_j)^\#)}{\kappa(\alpha_j^\#, \alpha_j^\#)}.$$

This means that $b_j b_{-j} b_{\pi(j)} b_{-\pi(j)} = 1$ and $b_{-j} b_{-\pi(j)} = (b_j b_{\pi(j)})^{-1}$ which is 1 for $j \neq \pi(0)$.

If $\pi(0) = 0$ we use $b_0 b_{\pi(0)} = \Pi_{j=1}^N (b_{-j} b_{-\pi(j)})^{m_j}$ to see $b_0^2 = 1$ as $b_{-j} b_{-\pi(j)} = 1$ for all j. Thus we will assume that $\pi(0) \neq 0$.

The lowest root is the sum of simple roots $\alpha_0 = -\sum m_j \alpha_j$. This implies that $\pi(\alpha_0) = -\sum m_j \pi(\alpha_j)$ which can be rewritten as $m_{\pi(0)} \alpha_0 = -\alpha_{\pi(0)} - \sum_{j \neq \pi(0)} m_j \pi(\alpha_j)$. Substituting our formula for α_0 we obtain

$$m_{\pi(0)} \sum_{j=1}^N m_j \alpha_j = \alpha_{\pi(0)} + \sum_{j \neq \pi(0)} m_j \alpha_{\pi(j)}.$$

The simple roots $\alpha_1, \alpha_2, \ldots, \alpha_N$ are linearly independent so $m_{\pi(0)}^2 = 1$ and since it is a positive integer $m_{\pi(0)} = 1$. We can also conclude $m_j = m_{\pi(j)}$.

Recall that $b_0 b_{\pi(0)} = \Pi_{j=1}^N (b_{-j} b_{-\pi(j)})^{m_j}$ (20). Using $b_{-j} b_{-\pi(j)} = 1$ for $j \neq 0, \pi(0)$ and $m_{\pi(0)} = 1$ we know $b_0 b_{\pi(0)} = b_{-0} b_{-\pi(0)} = (b_0 b_{\pi(0)})^{-1}$ and hence $b_0 b_{\pi(0)} = \pm 1$.

Suppose that there exists some j such that $\pi(j) = j$ and m_j is odd. Considering (19) shows that by switching the sign of b_j we also switch the sign of b_0 (and obviously this does not affect $b_{\pi(0)}$). This means we switch the sign of $b_0 b_{\pi(0)}$. Since $b_0 b_{\pi(0)} = \pm 1$ this method allows us to choose the b_j appropriately so that $b_0 b_{\pi(0)} = 1$.

We now consider a method of proof for when there is no fixed α_j with m_j odd. Suppose δ is a positive root such that

- the expression of δ as a sum of simple roots does not contain $\alpha_{\pi(0)}$,

- $\pi(\delta) + \alpha_{\pi(0)}$ is also a root and
- $\delta + \alpha_0 = -\pi(\delta) - \alpha_{\pi(0)}$.

Now $[R_\delta, R_0] = C_1 R_{\delta+\alpha_0}$ and $[R_{\pi(\delta)}, R_{\pi(0)}] = C_2 R_{\pi(\delta)+\alpha_{\pi(0)}}$ for some non-zero constants C_1, C_2. Since $\pi(\delta + \alpha_0) = -(\delta + \alpha_0)$ we know that

$$[[R_\delta, R_0], [R_{\pi(\delta)}, R_{\pi(0)}]] = C_1 C_2 H_{\delta+\alpha_0}.$$

Applying Θ to each side gives

$$[[b_\delta R_{\pi(\delta)}, b_0 R_{\pi(0)}], [b_{\pi(\delta)} R_\delta, b_{\pi(0)} R_0]] = -C_1 C_2 H_{\delta+\alpha_0}$$

implying

$$b_\delta b_{\pi(\delta)} b_0 b_{\pi(0)} = 1. \tag{21}$$

We can write R_δ as $C'[R_{\beta_1}, [R_{\beta_2} \ldots]]$ with C' a non-zero constant and β_i simple roots not including $\alpha_{\pi(0)}$ and $\sum_i \beta_i = \delta$. This means that $\Theta^2(R_\delta) = \prod_i b_{\beta_i} b_{\beta_{\pi(i)}} R_\delta$. However we choose the b_j so that $b_{\beta_i} b_{\beta_{\pi(i)}} = 1$ for all i and hence $b_\delta b_{\pi(\delta)} = 1$. Substituting this into (21) implies that $b_0 b_{\pi(0)} = 1$.

A similar argument may be applied if there exist positive roots γ, β such that

- the expressions of γ, β as sums of simple roots do not contain $\alpha_{\pi(0)}$,
- $\pi(\gamma) + \alpha_{\pi(0)}$ and $\beta + \pi(\beta)$ are also roots and
- $-\beta - \pi(\beta) = \gamma + \pi(\gamma) + \alpha_0 + \alpha_{\pi(0)}$.

Here we know there is some non-zero constant C such that

$$[[R_\gamma, R_0], [R_{\pi(\gamma)}, R_{\pi(0)}]] = C[R_{-\beta}, R_{-\pi(\beta)}]$$

and applying Θ gives

$$-b_\gamma b_{\pi(\gamma)} b_0 b_{\pi(0)} [[R_\gamma, R_0], [R_{\pi(\gamma)}, R_{\pi(0)}]] = -b_{-\beta} b_{-\pi(\beta)} C[R_{-\beta}, R_{-\pi(\beta)}]$$

and hence $b_\gamma b_{\pi(\gamma)} b_0 b_{\pi(0)} = b_{-\beta} b_{-\pi(\beta)}$. Since $\alpha_{\pi(0)}$ is not contained in the sum of simple roots of either γ or β we know $b_\gamma b_{\pi(\gamma)} = 1$ and similarly $b_{-\beta} b_{-\pi(\beta)} = 1$. We conclude that $b_{\pi(0)} b_0 = 1$.

To show that every involution of the extended Dynkin diagram extends to an involution of the Lie algebra we now only need to consider the involutions of each of the diagrams and for those that don't fix some α_j with odd m_j, identify a suitable root γ or pair of roots γ, β.

For a root system of type A_N, $m_j = 1$ for all j. Thus any diagram involution fixing a simple root is induced by an involution of the Lie algebra. When N is even the only involutions of the extended Dynkin diagram must fix some root. For N odd, by inspection of the extended Dynkin diagram shown in Figure 1, we can see that we additionally have the rotation $\pi(j) = j + \frac{1}{2}(N+1) \bmod (N+1)$ and reflections.

For the involution $\pi(j) = j + \frac{1}{2}(N+1) \bmod (N+1)$ we may take $\delta = \alpha_1 + \alpha_2 + \ldots + \alpha_{\frac{1}{2}(N-1)}$.

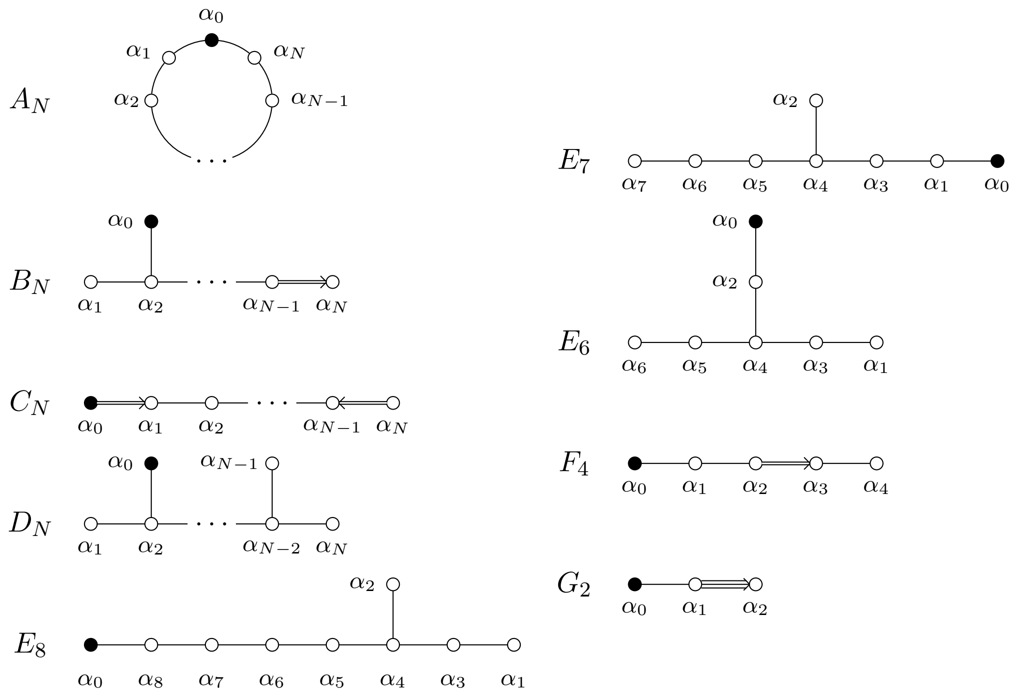

FIGURE 1. Extended Dynkin diagrams, with the lowest root α_0 coloured.

Consider now some involution that comes from a reflection. Since we have automatically covered the cases when there is a fixed root we can assume that there is an even number of roots between α_0 and $\pi(\alpha_0)$ going in either direction around the circle. Indeed the axis of reflection is between the roots $(\pi(0)-1)/2$ and $(\pi(0)+1)/2$ and between $(N+\pi(0))/2$ and $(N+\pi(0))/2+1$. We can set

$$\gamma = \alpha_1 + \alpha_2 + \ldots + \alpha_{(\pi(0)-1)/2} \quad \text{and} \quad \beta = \alpha_{\pi(0)+1} + \alpha_{\pi(0)+1} \cdots + \alpha_{(\pi(0)+N)/2}.$$

There is only one involution of the root system of type B_N which sends α_0 to α_1 and fixes everything else. We can choose $\delta = \alpha_2 + \ldots + \alpha_N$.

For roots systems of type C_N there is again only one involution; $\pi(\alpha_i) = \alpha_{N-i}$. Here choose $\delta = \alpha_1 + \ldots + \alpha_{N-1}$.

For D_N, $m_1 = m_{N-1} = m_N = 1$, and so we need only consider involutions which do not fix any of these vertices, of which there are three. These are involutions with $\pi(0) = 1, N-1$ or N. If $\pi(0) = 1$ and let γ be $\alpha_2 + \ldots + \alpha_{N-1}$, and if $\pi(0) = N-1$ or N let $\gamma = \alpha_1 + \alpha_2 + \ldots + \alpha_{N-2}$.

For the root system E_6, all involutions of the diagram fix the vertex α_4 and $m_4 = 3$ is odd.

A choice of all positive roots of E_7 is detailed for example in [18, p 1524-1530]. Let $\delta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$, so $\pi(\delta) = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ and

$\pi(\delta) + \alpha_{\pi(0)} = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$. Observe that $\delta + \alpha_{\pi(0)} + \pi(\delta) = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ which is the highest root.

The extended Dynkin diagrams of type E_8, F_4, G_2 do not possess any involutions.

We have then shown that given any involution π of an extended Dynkin diagram for $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, there exists an involution Θ of $\mathfrak{g}^{\mathbb{C}}$ preserving $\mathfrak{t}^{\mathbb{C}}$ and inducing π. It remains to show that there is a real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ for which Θ is a Cartan involution and such that $\mathfrak{g} \cap \mathfrak{t}^{\mathbb{C}}$ has full rank. For any choice of simple roots we may consider the corresponding *Borel subalgebra* $\mathfrak{b}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} \mathcal{G}^{\alpha}$ and it is easy to see that Θ preserves the set of simple roots if and only if it preserves the corresponding Cartan and Borel subalgebras. Now by [11, Theorem 8.6] there exists an automorphism Ψ of $\mathfrak{g}^{\mathbb{C}}$ such that $\Psi\Theta\Psi^{-1}$ acts on the corresponding simple and lowest root vectors R_{α_j} in the Chevalley basis simply by scaling them by ± 1, and hence preserves the Cartan and Borel subalgebras $\mathfrak{t}^{\mathbb{C}}$ and $\mathfrak{b}^{\mathbb{C}}$. Then Θ preserves the Cartan subalgebra $\Psi^{-1}(\mathfrak{t}^{\mathbb{C}})$ and the Borel subalgebra $\Psi^{-1}(\mathfrak{b}^{\mathbb{C}})$ and hence it preserves the set of simple roots $\Psi^{-1}\{\alpha_1, \ldots, \alpha_N\}$. Then there exists a real form $\mathfrak{g}'$ of $\mathfrak{g}^{\mathbb{C}}$ with respect to which Θ is a Cartan involution and such that $\mathfrak{g}'\Psi^{-1}(\mathfrak{t}^{\mathbb{C}})$ is a Cartan subalgebra of $\mathfrak{g}'$ [12, proof of Theorem 6.88].

Let $\mathfrak{l}^{\mathbb{C}}$ denote the $(+1)$-eigenspace of Θ and $L^{\mathbb{C}}$ a complex Lie group with Lie algebra $\mathfrak{l}^{\mathbb{C}}$. In [13, Theorem 1] (c.f. [19, Proposition 2.1]) it was shown that for a given real form $\mathfrak{g}'$ and Θ-stable Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ of a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, there exist a Θ-stable Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{g}'$ and $l \in L^{\mathbb{C}}$ such that $\mathfrak{t}^{\mathbb{C}} = \mathrm{Ad}_l(\mathfrak{t}')^{\mathbb{C}}$. Hence $\mathfrak{g} = \mathrm{Ad}_l\, \mathfrak{g}'$ is a real form of $\mathfrak{g}^{\mathbb{C}}$ for which $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{t}^{\mathbb{C}}$ is a Θ-stable real form of $\mathfrak{t}^{\mathbb{C}}$ and Θ is a Cartan involution of $\mathfrak{g}$.

By Proposition 3.1 the Coxeter automorphism corresponding to the choice of simple roots $\alpha_1, \ldots, \alpha_N$ preserves the real form $\mathfrak{g}$ and in particular the Cartan subalgebra $\mathfrak{t}$. □

Corollary 3.3. *The complex Lie algebras $\mathfrak{g}^{\mathbb{C}}$ for which we can choose simple roots $\alpha_1, \ldots, \alpha_N$ and a real form $\mathfrak{g}$ such that the corresponding Coxeter automorphism preserves $\mathfrak{g}$ and for which the corresponding Cartan subalgebra $\mathfrak{t}$ is not maximally compact are those of type A_N, B_N, C_N, D_N, E_6 and E_7. In particular if $\mathfrak{g}$ is non-compact then $\mathfrak{g}^{\mathbb{C}}$ must be of one of these types.*

Proof. Inspection of the extended Dynkin diagrams for these Lie algebras shows that they possess involutions which act nontrivially on the lowest root α_0, whereas the extended Dynkin diagrams for the Lie algebras E_8, F_4 and G_2 do not. □

4. Toda frame

We now explore the relationship between cyclic primitive maps and the affine Toda field equations. Henceforth G shall denote a simple real Lie group, T a Cartan subgroup and $\alpha_1, \ldots, \alpha_N$ simple roots such that the resulting Coxeter automorphism σ preserves the real group G. This Coxeter automorphism then gives G/T the structure of a k-symmetric space, where $k-1$ is the maximum height of a root of $\mathfrak{g}^{\mathbb{C}}$. We shall consider cyclic primitive maps ψ from the complex plane into G/T and will see that cyclic primitive maps $\psi : \mathbb{C} \to G/T$ arise from and give rise to solutions of the two-dimensional affine Toda field equations for $\mathfrak{g}$. Our results also apply to maps from a simply-connected coordinate neighbourhood of any Riemann surface.

The famous Toda equations arose originally as a model for particle interactions within a one-dimensional crystal, with the affine model corresponding to the particles being arranged in a circle. They have been the subject of extensive study, both as a completely integrable Hamiltonian system and in the context of Toda field theories. The standard form of the two-dimensional affine Toda field equation for a compact simple Lie algebra $\mathfrak{g}$ is

$$2\Omega_{z\bar{z}} = \sum_{j=0}^{N} m_j e^{2\alpha_j(\Omega)} \alpha_j^{\sharp}.$$

Here $\Omega : \mathbb{C} \to i\mathfrak{t}$ is a smooth map, the m_j are chosen so that $\alpha_0 = -\sum_{j=1}^{N} m_j \alpha_j$ is the lowest root, we set $m_0 = 1$ and $\alpha_j^{\sharp}$ is the dual of α_j with respect to the Killing form. For compact $\mathfrak{g}$, one may choose root vectors R_{α_j} such that $\overline{R_{\alpha_j}} = R_{-\alpha_j}$ and $[R_{\alpha_j}, R_{-\alpha_j}] = \alpha_j^{\sharp}$.

Returning to the case of the general simple real Lie algebra $\mathfrak{g}$ and using (3), since the Coxeter automorphism preserves the real form $\mathfrak{g}$ there exists a permutation π of the extended Dynkin diagram such that

$$\overline{\alpha_j} = -\alpha_{\pi(j)}. \tag{22}$$

We assume henceforth that we have chosen such an involution π and root vectors R_{α_j} satisfying $\overline{R_{\alpha_j}} = R_{-\alpha_{\pi(j)}}$.

We shall consider the generalisation of the affine Toda field equation

$$2\Omega_{z\bar{z}} = \sum_{j=0}^{N} n_j e^{2\alpha_j(\Omega)} [R_{\alpha_j}, R_{-\alpha_j}] \tag{23}$$

obtained by allowing $n_0, n_1, \ldots, n_N$ to be any positive real numbers such that $n_{\pi(j)} = n_j$.

Given a cyclic element $W = \sum_{j=0}^{N} r_j R_{\alpha_j}$ of $\mathfrak{g}_1^{\sigma}$, we say that a lift $F : \mathbb{C} \to G$ of $\psi : \mathbb{C} \to G/T$ is a *Toda frame* with respect to W if there exists a smooth map $\Omega : \mathbb{C} \to i\mathfrak{t}$ such that

$$\varphi' = (\Omega_z + \mathrm{Ad}_{\exp \Omega} W) dz \tag{24}$$

where $\varphi = F^*\omega$ is the pull-back of the Maurer-Cartan form and $\varphi = \varphi' + \varphi''$ is the decomposition of φ into $(1,0)$ and $(0,1)$ forms.

We call Ω an *affine Toda field* with respect to W. The motivation for this nomenclature is

Lemma 4.1. *Fix a cyclic element* $W = \sum_{j=0}^{N} r_j R_{\alpha_j}$ *of* $\mathfrak{g}_1^\sigma$ *such that* $r_{\pi(j)} = \overline{r_j}$ *and* $\overline{R_{\alpha_j}} = R_{-\alpha_{\pi(j)}}$.

The affine Toda field equation (23) *is the integrability condition for the existence of a Toda frame with respect to* W *where we take* $n_j = r_j\overline{r_j}$ *for* $j = 0, \dots, N$.

Proof. Using $[R_{\alpha_j}, R_{-\alpha_l}] = 0$ whenever $j \neq l$, we can rewrite the Toda field equation (23) as

$$\begin{aligned} 2\Omega_{z\bar{z}} &= \sum_{j,l=0}^{N} r_j \overline{r_l} e^{\alpha_j(\Omega)} e^{\alpha_j(\Omega)} [R_{\alpha_j}, R_{-\alpha_l}] \\ &= \left[\sum_{j=0}^{N} r_j e^{\alpha_j(\Omega)} R_{\alpha_j}, \sum_{l=0}^{N} \overline{r_l} e^{\alpha_l(\Omega)} R_{-\alpha_l} \right]. \end{aligned}$$

From equation (15) we know $e^{\alpha_j(\Omega)} R_{\alpha_j} = \mathrm{Ad}_{\exp \Omega} R_{\alpha_j}$ and also

$$e^{\alpha_l(\Omega)} R_{-\alpha_l} = e^{-\alpha_l(-\Omega)} R_{-\alpha_l} = \mathrm{Ad}_{\exp -\Omega} R_{-\alpha_l}.$$

If we set $W := \sum_{j=0}^{N} r_j R_{\alpha_j}$ with the normalisations described in the lemma then since $\sum_{j=0}^{N} r_j R_{\alpha_j} = \sum_{j=0}^{N} r_j \overline{R_{-\alpha_j}}$, the Toda field equation becomes

$$2\Omega_{z\bar{z}} = [\mathrm{Ad}_{\exp \Omega} W, \mathrm{Ad}_{\exp -\Omega} \overline{W}].$$

Now for any given $\Omega : \mathbb{C} \to i\mathfrak{t}$ the integrability condition for the existence of a Toda frame with respect to W is the Maurer-Cartan equation (1) for

$$\varphi = (\Omega_z + \mathrm{Ad}_{\exp \Omega} W) dz + (-\Omega_{\bar{z}} + \mathrm{Ad}_{\exp -\Omega} \bar{W}) d\bar{z}.$$

Namely, this integrability condition is

$$\begin{aligned} 0 &= (-\Omega_{\bar{z}} + \mathrm{Ad}_{\exp -(\Omega)} \overline{W})_z - (\Omega_z + \mathrm{Ad}_{\exp \Omega} W)_{\bar{z}} \\ &\quad + [\Omega_z + \mathrm{Ad}_{\exp \Omega} W, -\Omega_{\bar{z}} + \mathrm{Ad}_{\exp -(\Omega)} \overline{W}] \\ &= -2\Omega_{z\bar{z}} + [\mathrm{Ad}_{\exp \Omega} W, \mathrm{Ad}_{\exp -\Omega} \overline{W}], \end{aligned}$$

which is precisely the Toda field equation. $\square$

Given $\tilde{F} : \mathbb{C} \to G$ with

$$\tilde{F}^{-1} \tilde{F}_z|_{\mathfrak{g}_1^\sigma} = \sum_{j=0}^{N} c_j R_{\alpha_j}, \tag{25}$$

we say that a cyclic element

$$W = \sum_{j=0}^{N} r_j R_{\alpha_j}$$

$\mathfrak{g}_1^\sigma$ is *normalised with respect to* $\tilde{F}:\mathbb{C}\to G$ if

$$r_0\prod_{j=1}^N r_j^{m_j} = c_0\prod_{j=1}^N c_j^{m_j}.$$

Theorem 4.2. *A map $\psi:\mathbb{C}\to G/T$ possesses a Toda frame if and only if it has a cyclic primitive frame for which $c_0\prod_{j=1}^N c_j^{m_j}$ is constant.*

More precisely, let $\psi:\mathbb{C}\to G/T$ be a cyclic primitive map possessing a frame $\tilde{F}:\mathbb{C}\to G$ such that $c_0\prod_{j=1}^N c_j^{m_j}$ is constant, where c_j are the root coefficients defined in (25) *and $m_1,\dots m_N$ are as defined in equation* (18)*. Then for any cyclic element W of $\mathfrak{g}_1^\sigma$ which is normalised with respect to $\tilde{F}$ there exists a Toda frame $F:\mathbb{C}\to G$ of ψ with respect to W.*

Conversely, if $\psi:\mathbb{C}\to G/T$ has a Toda frame F with respect to cyclic $W\in\mathfrak{g}_1^\sigma$ then ψ is cyclic primitive and W is normalised with respect to F. In particular then the root coefficients c_j are such that $c_0\prod_{j=1}^N c_j^{m_j}$ is constant.

Proof. Consider the frames $F:=\tilde{F}\exp X$ of ψ where $X:\mathbb{C}\to\mathfrak{t}$. For such F we have $F^{-1}F_z = \mathrm{Ad}_{\exp -X}\tilde{F}^{-1}\tilde{F}_z + X_z$ and so

$$F^{-1}F_z|_{\mathfrak{g}_1^\sigma} = \mathrm{Ad}_{\exp -X}\tilde{F}^{-1}\tilde{F}_z|_{\mathfrak{g}_1^\sigma}.$$

This implies the Toda condition of $\mathrm{Ad}_{\exp\Omega}W = F^{-1}F_z|_{\mathfrak{g}_1^\sigma}$ is equivalent to

$$\mathrm{Ad}_{\exp(X+\Omega)}W = \tilde{F}^{-1}\tilde{F}_z|_{\mathfrak{g}_1^\sigma} = \sum_{j=0}^N c_jR_{\alpha_j}. \tag{26}$$

Using equation (15) we can rewrite this as

$$\sum_{j=0}^N r_je^{\alpha_j(X+\Omega)}R_{\alpha_j} = \sum_{j=0}^N c_jR_{\alpha_j}.$$

Comparing root space coefficients implies that

$$e^{\alpha_j(X+\Omega)} = \frac{c_j}{r_j} \text{ for } j=1,\dots k \tag{27}$$

and $r_0\prod_{j=1}^N(e^{\alpha_j(X+\Omega)})^{-m_j} = c_0$. Since W is normalised with respect to $\tilde{F}$ and $\mathbb{C}$ is simply connected, we can solve for $X+\Omega$. We can then find Ω and X from $X+\Omega$ by taking its $\mathfrak{t}$ and $i\mathfrak{t}$ components respectively.

It remains to show that $\Omega_zdz = F^{-1}\partial F|_{\mathfrak{t}} = \varphi'_{\mathfrak{t}}$. From the $\mathfrak{g}_1^\sigma$ component (8) of the Maurer-Cartan equation for φ we have

$$\partial(\mathrm{Ad}_{\exp\Omega}W) - [\mathrm{Ad}_{\exp\Omega}W,\varphi'_{\mathfrak{t}}] = 0$$

or equivalently

$$[\mathrm{Ad}_{\exp\Omega}W,\varphi'_{\mathfrak{t}} - \partial\Omega] = 0.$$

Since W is cyclic so is $\mathrm{Ad}_{\exp\Omega}W$ and thus $\varphi'_{\mathfrak{t}} = \partial\Omega$.

Conversely, given W and a solution Ω to the corresponding affine Toda field equation, the resulting Toda frame F is primitive. Furthermore the equation

$$r_0(e^{-\sum_{j=1}^N m_j\alpha_j(X+\Omega)}R_{\alpha_0} + \sum_{j=1}^N r_j e^{\alpha_j(X+\Omega)}R_{\alpha_j} = \sum_{j=0}^N c_j R_{\alpha_j}$$

implies that $r_0\prod_{j=1}^N r_j^{m_j} = c_0\prod_{j=1}^N c_j^{m_j}$ and hence $c_0\prod_{j=1}^N c_j^{m_j}$ is a non-zero constant. This implies that the c_j are nowhere zero and ψ is cyclic primitive.

□

Our chief interest lies in cyclic primitive ψ which are doubly-periodic, as it is these we shall show are of finite type. We henceforth restrict our attention to doubly-periodic maps and denote by $\mathbb{C}/\Lambda$ any genus one Riemann surface. Let W be a cyclic element of $\mathfrak{g}_1^\sigma$ as before. We say that a frame $F:\mathbb{C}/\Lambda\to G$ of $\psi:\mathbb{C}/\Lambda\to G/T$ is a Toda frame with respect to W if F is a Toda frame of ψ when both are considered as maps from $\mathbb{C}$.

The following lemma shows that for doubly-periodic maps, the requirement in Theorem 4.2 that $c_0\prod_{i=1}^N c_i^{m_i}$ be constant is automatically satisfied.

Lemma 4.3. *Let $\psi:\mathbb{C}\to G/T$ be a primitive map, $F:\mathbb{C}\to G$ be a lift of ψ and $\varphi = F^*\omega$ be the pull-back of the Maurer-Cartan form. Let $c_0,\dots c_N$ be the coefficients of the simple and lowest roots in φ', that is*

$$\varphi'_{\mathfrak{p}} = \sum_{i=0}^N c_i R_{\alpha_i} dz.$$

Then the function

$$p(F) = c_0\prod_{i=1}^N c_i^{m_i}$$

is holomorphic.

Proof. Let $m_0 = 1$, so $p(F) = \prod_{k=0}^N c_k^{m_k}$. Now

$$\frac{d}{d\bar z}\left(\prod_{k=0}^N c_k^{m_k}\right) = \sum_{k=0}^N m_k (c_k)_{\bar z} c_k^{m_k-1}\prod_{j\neq k} c_j^{m_j} \tag{28}$$

and so we need to show that this quantity is zero.

We have assumed that ψ is primitive. This implies that ψ satisfies the harmonic equation $d\varphi'_{\mathfrak{p}} + [\varphi'_{\mathfrak{t}}\wedge\varphi''_{\mathfrak{p}}] = 0$, as shown in (8).

Using the root decomposition we can write $\varphi' = \sum_{i=1}^N s_i\eta_i dz + \sum_{i=0}^N c_i R_{\alpha_i} dz$. With this decomposition $\varphi'_{\mathfrak{p}} = \sum_{i=0}^N c_i R_{\alpha_i} dz$ and $\varphi''_{\mathfrak{t}} = \sum_{i=1}^N \overline{s_i\eta_i}$. Substituting these into

(8) tells us that

$$\sum_{i=0}^{N}(c_i)_{\bar z}R_{\alpha_i}+\left[\sum_{i=1}^{N}\overline{s_i\eta_i},\,\sum_{i=0}^{N}c_iR_{\alpha_i}\right]=0. \tag{29}$$

Now $[\overline{\eta_i},R_{\alpha_k}]=\alpha_k(\overline{\eta_i})R_{\alpha_k}=\overline{\overline{\alpha_k}(\eta_i)}R_{\alpha_k}=\overline{-\alpha_{\pi(k)}(\eta_i)}R_{\alpha_k}$. If $\pi(k)\neq 0$ (that is $k\neq\pi(0)$) then $\alpha_{\pi(k)}(\eta_i)=\delta_{ik}$ and if $\pi(k)=0$ then $\alpha_{\pi(k)}(\eta_i)=-m_i$. Together these imply that

$$\left[\sum_{i=1}^{N}\overline{s_i\eta_i},\,c_kR_{\alpha_k}\right]=\begin{cases}-\overline{s_{\pi(k)}}c_kR_{\alpha_k} & \text{if } k\neq\pi(0)\\ \sum_{i=1}^{N}m_i\overline{s_i}c_kR_{\alpha_k} & \text{if } k=\pi(0).\end{cases} \tag{30}$$

We now can consider separately the coefficients of each of the root vectors R_{α_k} in (29) using (30). These tell us that $(c_k)_{\bar z}=s_{\pi(k)}c_k$ when $k\neq\pi(0)$ and that $(c_{\pi(0)})_{\bar z}=-\sum_{i=1}^{N}m_i\overline{s_i}c_{\pi(0)}$. Using these substitutions in (28) we make the following calculation.

$$\begin{aligned}\frac{d}{d\bar z}p(F)&=\sum_{\substack{k\in\{0,1,\dots,N\}\\ k\neq\pi(0)}}m_k(\overline{s_{\pi(k)}}c_k)c_k^{m_k-1}\prod_{j\neq k}c_j^{m_j}-m_{\pi(0)}\sum_{k=1}^{N}m_k\overline{s_k}c_{\pi(0)}c_{\pi(0)}^{m_{\pi(0)}-1}\prod_{j\neq\pi(0)}c_j^{m_j}\\&=\left(\sum_{\substack{k\in\{0,1,\dots,N\}\\ k\neq\pi(0)}}m_k\overline{s_{\pi(k)}}-m_{\pi(0)}\sum_{k=1}^{N}m_k\overline{s_k}\right)\prod_j c_j^{m_j}\\&=\left(\sum_{k\in\{1,\dots,N\}}m_{\pi(k)}\overline{s_k}-m_{\pi(0)}\sum_{k=1}^{N}m_k\overline{s_k}\right)\prod_j c_j^{m_j}=0.\end{aligned}$$

The last equality uses $m_{\pi(k)}=m_k$ and $m_{\pi(0)}=m_0=1$. By directly calculating $\frac{d}{d\bar z}p(F)=0$ we have shown that $p(F)$ is holomorphic. □

Theorem 4.4. *A map $\psi:\mathbb{C}/\Lambda\to G/T$ possesses a Toda frame if and only if it is cyclic primitive.*

More precisely, let $\psi:\mathbb{C}\to G/T$ be a map possessing a so cyclic primitive frame $\tilde F:\mathbb{C}\to G$. Then for any cyclic element W of $\mathfrak{g}_1^\sigma$ which is normalised with respect to $\tilde F$ there exists a Toda frame $F:\mathbb{C}\to G$ of ψ with respect to W. Furthermore if ψ and $\tilde F$ are doubly periodic with lattice Λ then so is the Toda frame F.

Conversely, if $\psi:\mathbb{C}\to G/T$ has a Toda frame F with respect to cyclic $W\in\mathfrak{g}_1^\sigma$ then ψ is cyclic primitive and W is normalised with respect to F. If $F:\mathbb{C}/\Lambda\to G$ is a Toda frame of $\psi:\mathbb{C}/\Lambda\to G/T$ then the corresponding affine Toda field $\Omega:\mathbb{C}\to i\mathfrak{t}$ has the property that $\exp\Omega$ and Ω_z are doubly periodic with lattice Λ.

Proof. From Theorem 4.2 and Lemma 4.3, only the periodicity statements require proof. Assume then that $\tilde F$ is doubly periodic with respect to a lattice Λ. Then for

$j = 1, \dots N$, from(27) we see that $e^{\alpha_j(X+\Omega)}$ is doubly periodic with respect to Λ and so

$$\exp(X + \Omega) = \exp(\sum_{j=1}^{N} \alpha_j(X + \Omega)\eta_j)$$

is also. Given any $\Gamma \in \Lambda$ it follows that

$$\exp(X(z + \Gamma) - X(z)) = \exp(\Omega(z) - \Omega(z + \Gamma)). \tag{31}$$

Using the conjugation map $\mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ which fixes $\mathfrak{g}$, we obtain from (31) that

$$\exp(X(z + \Gamma) - X(z)) = \exp(-\Omega(z) + \Omega(z + \Gamma). \tag{32}$$

When combined, (31) and (32) imply that $\exp(X(z + \Gamma)) = \exp(z))$ for all z and hence $\exp X$ is doubly periodic with lattice Λ.

Since $\tilde{F}$ and $\exp X$ are both doubly periodic with lattice Λ we know $F = \tilde{F} \exp X$ is also. $\square$

5. Finite type result

We will now show that all cyclic primitive ψ from a 2-torus $\mathbb{C}/\Lambda$ into the k-symmetric space G/T frame are of finite type. Hence all such maps can be constructed from a pair of commuting ordinary differential equations on a finite-dimensional vector subspace of a loop algebra. In [4] it was shown that all semisimple adapted harmonic maps of a 2-torus into a compact semisimple Lie group are of finite type. We prove our finite type result by adapting the methods of that paper. Note that being cyclic primitive is equivalent to possessing a Toda frame, by Theorem 4.4.

A map $Y : \mathbb{C}/\Lambda \to \mathfrak{g}^{\mathbb{C}}$ is called a *Jacobi field* if there exists $\dot{\Omega} : \mathbb{C}/\Lambda \to \mathfrak{t}^{\mathbb{C}}$ such that

$$dY + [F^{-1}dF, Y] = \Big(\dot{\Omega}_z + [\dot{\Omega}, F^{-1}F_z]\Big)\, dz + \Big(-\dot{\Omega}_{\bar{z}} - [\dot{\Omega}, F^{-1}F_{\bar{z}}]\Big)\, d\bar{z}. \tag{33}$$

If F_t is a family of Toda frames with corresponding $\Omega_t : \mathbb{C} \to i\mathfrak{t}$ then $\frac{d}{dt}F_t|_{t=0}$ is a Jacobi field with $\dot{\Omega} = \frac{d}{dt}\Omega_t|_{t=0}$. Note that if $\dot{\Omega} = 0$ the Jacobi equation is the Killing field equation.

Let F be a Toda frame for $\psi : \mathbb{C} \to G/T$. We have

$$F^{-1}dF = (\Omega_z + \mathrm{Ad}_{\exp \Omega} W)dz + (-\Omega_{\bar{z}} + \mathrm{Ad}_{\exp -\Omega} \overline{W})d\bar{z}$$

for some $\Omega : T^2 \to i\mathfrak{t}$ and cyclic $W \in \mathfrak{g}_1^{\sigma}$. Let Y be a Jacobi field with corresponding $\dot{\Omega} : T^2 \to i\mathfrak{t}$. Then Y must satisfy

$$Y_z + [\Omega_z + \mathrm{Ad}_{\exp \Omega} W, Y] = \dot{\Omega}_z + [\dot{\Omega}, \mathrm{Ad}_{\exp \Omega} W] \tag{34}$$

$$Y_{\bar{z}} + [-\Omega_{\bar{z}} + \mathrm{Ad}_{\exp -\Omega} \overline{W}, Y] = -\dot{\Omega}_{\bar{z}} - [\dot{\Omega}, \mathrm{Ad}_{\exp -\Omega} \overline{W}]. \tag{35}$$

Taking $(34)_{\bar{z}} - (35)_z$ we obtain

$$2\dot{\Omega}_{z\bar{z}} = -\big[\mathrm{Ad}_{\exp \Omega} W, [\dot{\Omega}, \mathrm{Ad}_{\exp -\Omega} \overline{W}]\big] - \big[\mathrm{Ad}_{\exp -\Omega} \overline{W}, [\dot{\Omega}, \mathrm{Ad}_{\exp \Omega} W]\big].$$

Since Ω and W are fixed, we see that $\dot{\Omega}$ satisfies a linear elliptic partial differential equation. As the torus is compact, the space of possible $\dot{\Omega}$ is finite dimensional.

Lemma 5.1. *Suppose $\psi : \mathbb{C}/\Lambda \to G/T$ is a cyclic primitive map possessing a formal Killing field $Y = \sum_{j\leq 1} \lambda^j Y_j \in \Omega^\sigma \mathfrak{g}^{\mathbb{C}}$. Then ψ has a (real) polynomial Killing field with highest term Y_1.*

Proof. We will find an infinite number of linearly independent Jacobi fields for which some linear combination must be a formal Killing field. Since Y is a formal Killing field, we have (13).

$$\sum_{j\leq 1} \lambda^j dY_j = \left[\sum_{j\leq 1} \lambda^j Y_j, \varphi_\lambda\right].$$

Comparing coefficients of λ^j gives the equations

$$(Y_j)_z dz + [\varphi'_{\mathfrak{k}}, Y_j] + [\varphi'_{\mathfrak{p}}, Y_{j-1}] = 0,$$
$$(Y_j)_{\bar{z}} d\bar{z} + [\varphi''_{\mathfrak{k}}, Y_j] + [\varphi''_{\mathfrak{p}}, Y_{j+1}] = 0.$$

For each $l \in \mathbb{Z}^+$ set

$$Y^l := \frac{1}{2} Y_{-kl} + \sum_{-kl<j\leq 1} \lambda^{j+kl} Y_j.$$

We will show that the Y^l are all Jacobi fields. Considering the coefficients separately gives

$$(Y^l)_z dz + [\lambda\varphi'_{\mathfrak{p}} + \varphi'_{\mathfrak{k}}, Y^l] = \frac{1}{2}(Y_{-kl})_z dz + \left[\frac{1}{2} Y_{-kl}, \lambda\varphi'_{\mathfrak{p}}\right]$$
$$(Y^l)_{\bar{z}} d\bar{z} + \left[\varphi''_{\mathfrak{k}} + \lambda^{-1}\varphi''_{\mathfrak{p}}, Y^l\right] = -\frac{1}{2}(Y_{-kl})_{\bar{z}} d\bar{z} - \left[\frac{1}{2} Y_{-kl}, \lambda^{-1}\varphi''_{\mathfrak{p}}\right].$$

Since $Y_{-kl} \in \mathfrak{g}_0 = \mathfrak{t}^{\mathbb{C}}$ we can set $\dot{\Omega}^l := \frac{1}{2} Y_{-kl}$. With this choice of $\dot{\Omega}$, Y is a solution to (33) and hence is a Jacobi field. The space of potential $\dot{\Omega}$ is finite dimensional, so there must be a non-trivial finite linear combination of the $\dot{\Omega}^l$ which equals 0. The corresponding finite linear combination of the Y^l is a formal Killing field. Since the highest order terms of the Y^l are each Y_1 we can rescale this formal Killing field to one with highest order term Y_1. After multiplying by an appropriate power of λ^k we may also assume that the degree of the lowest term has smaller absolute value than the degree of the highest term. Then $\overline{\xi} + \xi$ is a polynomial Killing field for ξ and by construction has highest order term Y_1. □

Theorem 5.2. *Suppose $\psi : \mathbb{C}/\Lambda \to G/T$ has a cyclic primitive frame $F : \mathbb{C}/\Lambda \to G$. Then ψ is of finite type.*

Proof. Using Theorem 4.4, we may take a Toda frame $F : \mathbb{C}/\Lambda \to G$ of ψ and corresponding $\Omega : \mathbb{C}/\Lambda \to i\mathfrak{t}$ and $W \in \mathfrak{g}_1^\sigma$. Recall that ψ is of finite type if it has an adapted polynomial Killing field ξ, that is a $\xi = \sum_{j=-d}^{d} \lambda^j \xi_j$ in the real twisted loop algebra $\Omega^\sigma \mathfrak{g}$ satisfying the Killing field equation (13) and such that

$$\xi_d = Ad_{\exp \Omega} W, \quad \xi_{d-1} = 2\Omega_z.$$

Since G was assumed simple, the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is simple and hence has a faithful linear representation so can be regarded as a subalgebra of some $\mathfrak{gl}(m, \mathbb{C})$. A recursive argument similar to that employed in the proof of Theorem 7.1, [4] shows that there exists a formal Killing field $Y = \sum_{j \le 0} \lambda^j Y_j$ in $\mathfrak{gl}(m, \mathbb{C})$ and furthermore $Y_0 = \mathrm{Ad}_{\exp \Omega} W$. We omit the details of this argument, but explain how to project this Y onto $\Omega^\sigma(\mathfrak{g}^{\mathbb{C}})$ to get a solution to the Killing field equation in the correct loop algebra.

Representations of simple Lie algebras are completely reducible and we have identified $\mathfrak{g}^{\mathbb{C}}$ with a subalgebra of $\mathfrak{gl}(m, \mathbb{C})$ so it must have a complementary subspace in $\mathfrak{gl}(m, \mathbb{C})$ which is invariant under the adjoint action of $\mathfrak{g}^{\mathbb{C}}$. This means there exists a projection map $\pi : \Omega(\mathfrak{gl}(m, \mathbb{C})) \to \Omega(\mathfrak{g}^{\mathbb{C}})$ such that

$$d\pi(Y) = \pi(dY) = \pi([Y, \varphi_\lambda]) = [\pi(Y), \varphi_\lambda].$$

Thus we have that $\pi(Y) \in \Omega(\mathfrak{g}^{\mathbb{C}})$ satisfies the Killing field equation. Furthermore $Y_0 = \pi(\mathrm{Ad}_{\exp \Omega} W) = \mathrm{Ad}_{\exp \Omega} W$. Set $\tilde{Y} = \lambda Y = \sum_{j \le 1} \lambda^j Y_{j-1}$ and note that $\tilde{Y}_1 = Y_0 = \mathrm{Ad}_{\exp \Omega} W$.

We want to project $\tilde{Y}$ onto $\Omega^\sigma(\mathfrak{g}^{\mathbb{C}})$. Consider the map

$$\pi_j^\sigma := \frac{1}{k}(\mathrm{Id} + \epsilon^{-j}\sigma + \epsilon^{-2j}\sigma^2 + \ldots + \epsilon^{-(k-1)j}\sigma^{k-1})$$

where ϵ is the k-th primitive root of unity. This map π_j^σ projects any element in $\mathfrak{g}^{\mathbb{C}}$ to its part in $\mathfrak{g}_j$. Thus we can define $\pi^\sigma : \Omega(\mathfrak{g}^{\mathbb{C}}) \to \Omega^\sigma(\mathfrak{g}^{\mathbb{C}})$ by

$$\pi^\sigma(\sum_j \lambda^j \xi_j) = \sum_j \lambda^j \pi_j^\sigma(\xi_j).$$

Note that this map is a correction of the “averaging map” used in the proof of Theorem 3.6, [3], which projected everything to the eigenspace $\mathfrak{g}_0$. Then $\tilde{\xi} = \pi^\sigma(\tilde{Y})$ satisfies

$$d\tilde{\xi} = [\tilde{\xi}, \Omega_z + \lambda \,\mathrm{Ad}_{\exp \Omega} W)dz + (-\Omega_{\bar{z}} + \lambda^{-1} \mathrm{Ad}_{\exp -\Omega} \overline{W})d\bar{z}]$$

and $\tilde{\xi}_1 = \tilde{Y}_1 = \mathrm{Ad}_{\exp \Omega} W$.

Now we may apply Lemma 5.1 to $\tilde{\xi}$ to conclude the existence of a (real) polynomial Killing field ξ whose top term, ξ_d, is $\mathrm{Ad}_{\exp \Omega} W$.

The $d-1$ coefficient of $\xi_z = [\xi, \Omega_z + \lambda \,\mathrm{Ad}_{\exp \Omega} W]$ is

$$(\mathrm{Ad}_{\exp \Omega} W)_z = [\mathrm{Ad}_{\exp \Omega} W, \Omega_z] + [\xi_{d-1}, \mathrm{Ad}_{\exp \Omega} W]$$

which implies

$$[\xi_{d-1} - 2\Omega_z, \mathrm{Ad}_{\exp \Omega} W] = 0.$$

Since W is a cyclic element and $\xi_{d-1} - 2\Omega_z \in \mathfrak{t}$ we conclude $\xi_{d-1} - 2\Omega_z = 0$ and hence ξ is an adapted polynomial Killing field. $\square$

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