LARGE DEVIATIONS AND TRANSITION BETWEEN EQUILIBRIA FOR STOCHASTIC LANDAU-LIFSHITZ EQUATION

ZDZISŁAW BRZEŹNIAK, BEN GOLDYS, AND T. JEGARAJ

Abstract. We study a stochastic Landau-Lifshitz equation on a bounded interval and with finite dimensional noise; this could be a simple model of magnetisation in a needle-shaped domain in magnetic media. We obtain a large deviation principle for small noise asymptotic of solutions using the weak convergence method. We then apply this large deviation principle to show that small noise in the field can cause magnetisation reversal and also to show the importance of the shape anisotropy parameter for reducing the disturbance of the magnetisation caused by small noise in the field.

Contents

1. Introduction

The aim of this paper is to study the stochastic Landau-Lifshitz model of magnetization for needle-shaped ferromagnetic domains. It is natural to describe such a domain as a bounded interval $\Lambda \subset \mathbb{R}$ filled in by a ferromagnetic material. Let $m(x) \in \mathbb{R}^3$ denote the magnetisation vector at the point $x \in \Lambda$. For temperatures below the Curie point the length of the

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magnetisation vector is constant over the domain, hence it can be assumed that

$$|m(x)| = 1, \quad x \in \Lambda.$$

In this paper we will always assume that m satisfies the Neumann boundary condition

$$\frac{\partial m}{\partial x}(t,x) = 0, \quad x \in \partial\Lambda.$$

This assumption is standard in physical models of ferromagnetism. According to the Landau theory of ferromagnetism any stable magnetisation vector m should minimise the Landau energy functional $\mathscr{E}(m)$. In this paper we will consider the energy functional of the form

$$\mathscr{E}(m) = \underbrace{\frac{1}{2}\int_\Lambda |\nabla m(x)|^2\, dx}_{exchange\ energy} + \underbrace{\frac{\beta}{2}\int_\Lambda \left(m_2^2(x) + m_3^2(x)\right)\, dx}_{anistropy\ energy} - \underbrace{\int_\Lambda F\cdot m(x)\, dx}_{energy\ of\ applied\ field}, \tag{1.1}$$

where $F \in \mathbb{R}^3$ is a fixed vector and $F \cdot m$ is an inner product in $\mathbb{R}^3$. Stable points $m \in H^1\left(\Lambda, \mathbb{R}^3\right)$ of the energy $\mathscr{E}$ satisfy the conditions

$$\nabla_{\mathbb{L}^2}\mathscr{E}(m) = 0, \quad |m(x)| = 1, \quad \left.\frac{\partial m}{\partial x}\right|_{\partial\Lambda} = 0,$$

where the gradient $\nabla_{\mathbb{L}^2}\mathscr{E}$ of $\mathscr{E}$ is understood with respect to the Hilbert space $L^2(\Lambda, \mathbb{R}^3)$. If m is not a stable configuration then according to the theory of Landau and Lifschitz modified later by Gilbert, the magnetisation will evolve in time subject to the equation (LLG in what follows):

$$\frac{\partial m}{\partial t}(t,x) = -m(t,x) \times \nabla_{\mathbb{L}^2}\mathscr{E}(m)(t,x) + \alpha m(t,x) \times (m(t,x) \times \nabla_{\mathbb{L}^2}\mathscr{E}(m)(t,x)),$$

where $\alpha > 0$. One may expect that the solution to this equation tends to a stationary point of the energy for $t \to \infty$. Let us recall that some magnetic memories (for more information see for example [21]) are made of single-domain, needle-shaped particles of iron or chromium oxide about 300 to 700 nm long and about 50 nm in diameter. Each magnetic domain has two stable states of opposite magnetisation and these states are used to represent the bits 0 and 1. The anisotropy energy is minimised when the magnetisation points along the needle axis. It is observed that under the influence of thermal noise the magnetisation can spontaneously reverse, causing loss of data. To investigate thermally induced magnetisation reversal, it is natural to use a stochastic version of the LLG equation. Informally speaking, we want to consider the LLG equation which corresponds to the energy functional with the applied field $K = F + noise$. More precisely, our aim will be to analyse transitions between equilibria under the influence of a small noise $\sqrt{\epsilon}d\xi(t,x)$, where

$$d\xi(t,x) = \sum_{i=1}^{3} h^i(x)dW_i(t),$$

ε represents dimensionless temperature, $h^i : \Lambda \to \mathbb{R}^3$ and $W = (W_i)$ is a standard Wiener process in $\mathbb{R}^3$. Then it is natural to model the dynamics of the magnetisation M using the Stratonovitch stochastic differential equation

$$dM = -M \times \left(\nabla_{\mathbb{L}^2}\mathscr{E}(M)dt - \circ\sqrt{\epsilon}dW\right) + \alpha M \times \left(M \times \left(\nabla_{\mathbb{L}^2}\mathscr{E}(M)dt - \circ\sqrt{\epsilon}dW,\right)\right) \tag{1.2}$$

where $\circ$ denotes the Stratonovitch integral. Let us note that the Stratonovitch integral allows us to preserve the constraint $|M(t,x)|=1$ for all times.

In order to formulate the final version of the equation we are going to study, we need some additional notations. Let (f_i) be an orthonormal basis in $\mathbb{R}^3$,

$$f(y)=(y\cdot f_2)\,f_2+(y\cdot f_3)\,f_3 \quad y\in\mathbb{R}^3,$$

and

$$G(M)h=M\times h-\alpha M\times(M\times h),\quad h:\Lambda\to\mathbb{R}^3,$$

see Section 2 for more details. Then, using (1.1) we obtain the stochastic LLG to be studied in this paper:

$$\begin{cases} dM=G(M)\left[(\Delta M-\beta f(M)+F\right]dt+\sqrt{\epsilon}G(M)\circ d\xi, & t\in(0,T],\\ \frac{\partial M}{\partial x}(t)\big|_{\partial\Lambda}=0, & t\in(0,T],\\ M(0)=M_0. \end{cases} \tag{1.3}$$

Note that since $G(y)h$ is orthogonal to $y\in\mathbb{R}^3$ for every $h\in\mathbb{R}^3$, the formal application of the Ito formula yields $|M(t,x)|=1$.
To the best of our knowledge the stochastic LLG in the form (1.3) has not been studied before. The existence of weak martingale solutions is proved for a similar equation in a three-dimensional domain in our earlier work [3]. Kohn, Reznikoff and vanden-Eijnden [14] modelled the magnetization M in a thin film, assuming that M is constant across the domain for all times and the energy functional contains the applied field only. In this case equation (1.2) reduces to an ordinary stochastic differential equation

$$dM=M\times(F+\sqrt{\varepsilon}\circ dW)-\alpha M\times(M\times(F+\sqrt{\varepsilon}\circ dW)).$$

where W is now a standard Wiener process in $\mathbb{R}^3$. Kohn, Reznikoff and Vanden-Eijnden used the large deviations theory to make a detailed computational and theoretical study of the behaviour of the solution. At the end of their paper, they remark that little is known about the behaviour of the solutions of stochastic Landau-Lifshitz equations which do not assume that the magnetization is uniform on the space domain.
In this work we study a stochastic LLG equation (1.3) on a bounded interval of the real line. We start with the existence of weak martingale solution stated in Theorem 3.1. The main ideas in the proof of Theorem 3.1 are essentially the same as those in the proof of the existence theorem in [3]. Since the full proof of Theorem 3.1 is rather long and adds little to [3], in the proof of Theorem 3.1 we only outline the main ideas and points of difference; a detailed proof of Theorem 3.1 is in [4].

Since the domain Λ is one-dimensional, we are able to prove that weak solutions are in fact strong and unique. In Theorem 4.2 in Section 4, we state a pathwise uniqueness result for solutions of equation (1.3); the proof is straightforward and details are omitted. In Theorem 4.4, we assert the uniqueness in law of weak martingale solutions of equation (1.3) with paths in $S:=C([0,T];\mathbb{H})\cap L^4\left(0,T;H^{1,2}\left(\Lambda,\mathbb{R}^3\right)\right)$; we also assert the existence of a measurable mapping $J:C\left([0,T];\mathbb{R}^3\right)\to S$ which maps the Wiener process W to a solution, $y:=J(W)$, of (1.3). Theorem 4.4 is a consequence of a very general version of the Yamada and Watanabe theorem proved in [16].

Next, we prove strong regularity properties of solutions to (1.3). Namely, we show that

$$\int_0^T \int_\Lambda |\nabla M(t)|^4 \, dx \, dt + \int_0^T \int_\Lambda |\Delta M(t)|^2 \, dx \, dt < \infty,$$

see Theorem 5.2 for a precise formulation.

In Section 6 we prove the Large Deviations Principle for equation (1.3) with $F = 0$.

We first identify the rate function and prove in Lemma 6.1 that it has compact level sets in the space

$$\mathscr{X} = C\left([0,T]; H^{1,2}\left(\Lambda;\mathbb{R}^3\right)\right) \cap L^2\left(0,T; H^{2,2}\left(\Lambda;\mathbb{R}^3\right)\right).$$

The proof of this result follows from the maximal regularity and ultracontractivity properties of the heat semigroup generated by the Neumann Laplacian and the estimates for weak solutions of equation (1.3) obtained in Theorem 3.1 below.

The Large Deviations Principle is proved in Theorem 6.5. To prove this theorem, we use the weak convergence method of Budhiraja and Dupuis [5, Theorem 4.4]. Following their work we show that the two conditions of Budhiraja and Dupuis, see Statements 1 and 2, Section 6, are satisfied and then Theorem 6.5 easily follows. A long proof that these two conditions are satisfied is split into a number of lemmas. We note that our proof is simpler than the corresponding proofs in [7] and [10]. In particular, we do not need to partition the time interval $[0,T]$ into small subintervals.

In Section 7, we apply the Large Deviations pPrinciple to a simple stochastic model of magnetization in a needle-shaped domain. We show that small noise in the applied field causes magnetization reversal with positive probability. We also obtain an estimate which shows the importance of the shape anisotropy parameter β, for reducing the disturbance of the magnetization caused by small noise in the field. The results we obtain partially answer a question posed in [14] and provide a foundation for the computational study of stochastic Landau-Lifshitz models with one dimensional domain.

1.1. **Notations.** The inner product of vectors $x, y \in \mathbb{R}^3$ will be denoted by $x \cdot y$ and $|x|$ will denote the Euclidean norm of x. We will use the standard notation $x \times y$ for the vector product in $\mathbb{R}^3$.

For a domain Λ we will use the notation $\mathbb{L}^p$ for the space $L^p\left(\Lambda;\mathbb{R}^3\right)$, $\mathbb{W}^{1,p}$ for the Sobolev space $W^{1,p}\left(\Lambda;\mathbb{R}^3\right)$ and so on. For $p = 2$ we will often write $\mathbb{H}$ instead of $\mathbb{L}^2$, $\mathbb{H}^1$ instead of $\mathbb{W}^{1,2}$ and $\mathbb{H}^2$ instead of $\mathbb{W}^{2,2}$. We will always emphasize the norm of the corresponding space writing $|f|_{\mathbb{H}}$, $|f|_{\mathbb{H}^1}$ and so on.

We will also need the spaces $L^p(0,T;E)$ and $C([0,T];E)$ of p-integrable, respectively continuous, functions $f : [0,T] \to E$ with values in a Banach space E. If $E = \mathbb{R}$ then we write simply $L^p(0,T)$ and $C([0,T])$.

Throughout the paper C stands for a positive real constant whose actual value may vary from line to line. We include an argument list, $C(a_1, \ldots, a_m)$, if we wish to emphasize that the constant depends only on the values of the arguments a_1 to a_m.

2. Preliminaries

We start with some basic concepts and notation that will be in constant use throughout the paper.

Let us recall that $\Lambda \subset \mathbb{R}$ is a bounded interval. We define a linear operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by

$$\begin{cases} D(A) & := \quad \{u \in \mathbb{H}^2 : \nabla u(x) = 0 \text{ for } x \in \partial\Lambda\}, \\ Au & := \quad -\Delta u \text{ for } u \in D(A). \end{cases} \tag{2.1}$$

Let us recall that the operator A is selfadjoint and nonnegative and $D\left(A^{1/2}\right)$ when endowed with the graph norm coincides with $\mathbb{H}^1$. Moreover, the operator $(A+I)^{-1}$ is compact. In what follows we will need the following, well known, interpolation inequality:

$$|u|^2_{\mathbb{L}^\infty} \leqslant k^2 |u|_{\mathbb{H}} |u|_{\mathbb{H}^1} \quad \forall u \in \mathbb{H}^1, \tag{2.2}$$

where the optimal value of the constant k is

$$k = 2\max\left(1, \frac{1}{\sqrt{|\Lambda|}}\right).$$

For $v, w, z \in \mathbb{H}^1$ by the expressions $w \times \Delta v$ and $z \times (w \times \Delta v)$ we understand the unique elements of the dual space $(\mathbb{H}^1)'$ of $\mathbb{H}^1$ such that for any $\phi \in \mathbb{H}^1$

$$_{(\mathbb{H}^1)'}\langle w \times \Delta v,\, \phi\rangle_{\mathbb{H}^1} = -\langle \nabla(\phi \times w),\, \nabla v\rangle_{\mathbb{H}} \tag{2.3}$$

and

$$_{(\mathbb{H}^1)'}\langle z \times (w \times \Delta v)\,, \phi\rangle_{\mathbb{H}^1} = -\langle \nabla((\phi \times z) \times w),\, \nabla v\rangle_{\mathbb{H}}, \tag{2.4}$$

respectively. Note that the space $H^1(\Lambda)$ is an algebra, hence for $v, w, z \in \mathbb{H}^1$, linear functionals $\mathbb{H}^1 \ni \phi \mapsto$ RHS of (2.3) (or (2.4)) are continuous. In particular, since $\langle a \times b, a\rangle = 0$ for $a, b \in \mathbb{R}^3$, we obtain

$$_{(\mathbb{H}^1)'}\langle w \times \Delta v,\, v\rangle_{\mathbb{H}^1} \;=\; -\langle v \times \nabla w,\, \nabla v\rangle_{\mathbb{H}} \tag{2.5}$$

$$_{(\mathbb{H}^1)'}\langle z \times (v \times \Delta v)\,, \phi\rangle_{\mathbb{H}^1} \;=\; -\langle \nabla(\phi \times z) \times v,\, \nabla v\rangle_{\mathbb{H}}, \tag{2.6}$$

and since $a \times a = 0$ for $a \in \mathbb{R}^3$, equation (2.3) yields

$$_{(\mathbb{H}^1)'}\langle \phi \times \Delta v,\, \phi\rangle_{\mathbb{H}^1} = -\langle \nabla(\phi \times \phi),\, \nabla v\rangle_{\mathbb{H}} = 0. \tag{2.7}$$

The maps $\mathbb{H}^1 \ni y \mapsto y \times \Delta y \in (\mathbb{H}^1)'$ and $\mathbb{H}^1 \ni y \mapsto y \times (y \times \Delta y) \in (\mathbb{H}^1)'$ are continuous homogenous polynomials of degree 2, resp. 3 hence they are locally Lipschitz continuous.
For any real number $\beta > 0$, we write $\mathbb{X}^\beta$ for the domain of the fractional power operator $D\left(A^\beta\right)$ endowed with the norm $|x|_{\mathbb{X}^\beta} = |(I+A)^\beta x|$ and $\mathbb{X}^{-\beta}$ denotes the dual space of $\mathbb{X}^\beta$ so that $\mathbb{X}^\beta \subset \mathbb{H} = \mathbb{H}' \subset \mathbb{X}^{-\beta}$ is a Gelfand triple.
Let $\mathscr{C}_T = \left(C_0\left([0,T];\mathbb{R}^3\right), \mathbb{F}^W_T, \mathscr{F}^W_T, \mu^W\right)$ denote the classical Wiener space, where μ^W stands for the Wiener measure on $C_0\left([0,T];\mathbb{R}^3\right)$ and $\mathbb{F}^W_T = \left(\mathscr{F}^W_t\right)$ is a μ^W-completion of the natural filtration $\mathbb{F}^0_T = \left(\mathscr{F}^0_t\right)_{t\in[0,T]}$ of the Wiener process. We will say that an $\mathbb{F}^0_T$ predictable function $F : [0,T] \times C_0\left([0,T];\mathbb{R}^3\right) \to \mathbb{R}^3$ belongs to $\mathscr{P}_T$ if

$$\|F\|_T = \sup_{\omega \in C_0([0,T];\mathbb{R}^3)} \left(\int_0^T |F(t,\omega)|^2 dt\right)^{1/2} < \infty.$$

Let us recall that if a given process Z is $\mathbb{F}^W_T$-predictable then it possesses an indistinguishable version that is $\mathbb{F}^0_T$-predictable.

We will denote by g the mapping $g:\mathbb{R}^3\to\mathscr{L}\left(\mathbb{R}^3\right)$ defined as

$$g(y)h = y\times h-\alpha y\times(y\times h)\,. \tag{2.8}$$

The function g is of class C^2. In particular, we have

$$\left[g'(y)h\right]z=\nabla\left[g(y)h\right]z=z\times h-\alpha[z\times(y\times h)+y\times(z\times h)],\quad h,y,z\in\mathbb{R}^3, \tag{2.9}$$

and for every $r>0$

$$\sup_{|y|\leqslant r}\left(\|g(y)\|+\left\|g'(y)\right\|\right)<\infty. \tag{2.10}$$

Clearly, we can define a mapping $(f,h)\to g(f)h$ for f,h from functions spaces of $\mathbb{R}^3$-valued functions. If $f\in\mathbb{L}^\infty$ and $h\in\mathbb{L}^2$ then $g(f)h$ is a well defined element of $\mathbb{L}^2$. We will use the notation G for a mapping $G(f):\mathbb{L}^2\to\mathbb{L}^2$ given by

$$G(f)h=f\times h-\alpha f\times(f\times h)\,. \tag{2.11}$$

For fixed $h_i\in\mathbb{L}^2$, $i=1,2,3$, let $B:\mathbb{R}^3\to\mathbb{L}^2$ be defined as

$$By=\sum_{i=1}^3 y_ih_i\,.$$

In the next lemma we use the notation $h=(h_i)$ and

$$|h|_{\mathbb{L}^\infty}=\max_{i\leqslant 3}|h_i|_{\mathbb{L}^\infty}\,.$$

Lemma 2.1. *Assume that $f_i\in\mathbb{L}^\infty$, $i=1,2$ and $|f_i|_{\mathbb{L}^\infty}\leqslant r$. Then there exists $C_r>0$ such that*

$$|G(f_1)h-G(f_2)h|_{\mathbb{L}^2}\leqslant C_r|h|_{\mathbb{L}^\infty}|f_1-f_2|_{\mathbb{L}^2}\,,\quad h\in\mathbb{L}^\infty\,.$$

3. Existence of solutions

We will be concerned with the following stochastic integral equation

$$\begin{aligned}M(t)=M_0&+\int_0^t[M\times\Delta M)-\alpha M\times(M\times\Delta M)]\,ds\\&+\sqrt{\varepsilon}\int_0^t G(M)B\,dW(s)+\frac{\varepsilon}{2}\sum_{i=1}^3\int_0^t\left[G'(M)h_i\right]\left(G(M)h_i\right)ds\\&-\beta\int_0^t G(M)f(M)\,ds+\int_0^t G(M)BF(s,W)\,ds,\quad t\in[0,T].\end{aligned} \tag{3.1}$$

Note, that the expresssion

$$\sqrt{\varepsilon}\int_0^t G(M(s))B\,dW(s)+\frac{\varepsilon}{2}\sum_{i=1}^3\int_0^t\left[G'(M(s))h_i\right]\left(G(M(s))h_i\right)ds$$

can be identified with the Stratonovich integral

$$\sqrt{\varepsilon}\int_0^t G(M(s))B\circ dW(s)$$

but we will not use this concept in the paper.

We will now formulate the main result of this Section.

Theorem 3.1 (Existence of a weak martingale solution in $\mathbb{H}^1$)**.** *Assume that* $h = (h_i)_{i=1}^3 \in \left(\mathbb{H}^1\right)^3$. *Let* $T > 0$ *be fixed and assume that* $F \in \mathscr{P}_T$. *Then for every* $M_0 \in \mathbb{H}^1$ *there exists a filtered probability space* $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, *an* $(\mathscr{F}_t)$*-adapted Wiener process* W *in* $\mathbb{R}^3$ *and an* $(\mathscr{F}_t)$*-adapted process* M *such that*

(1) *for each* $\beta < \frac{1}{2}$ *the paths of* M *are continuous* $\mathbb{X}^\beta$*-valued functions* $\mathbb{P}$*-a.s.;*
(2) *For every* $p \geqslant 1$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^p_{\mathbb{H}^1}\right] \leqslant C\left(T, p, \alpha, \|F\|_T, \|M_0\|, \|h\|\right);$$

(3) *For almost every* $t \in [0,T]$ *we have* $M(t) \times \Delta M(t) \in \mathbb{L}^2$ *and*

$$\mathbb{E}\left(\int_0^T |M(s) \times \Delta M(s)|^2_{\mathbb{H}}\, ds\right)^p \leqslant C\left(T, \alpha, \|F\|_T, \|M_0\|, \|h\|\right)$$

(4) $|M(t)(x)|_{\mathbb{R}^3} = 1$ *for all* $x \in \Lambda$ *and for all* $t \in [0,T]$, $\mathbb{P}$*-a.s.;*
(5) *For every* $t \in [0,T]$ *equation* (3.1) *holds* $\mathbb{P}$*-a.s.*

Note that in Theorem 3.1, M is an $\mathbb{H}^1$-valued process, hence the expressions $M(s) \times \Delta M(s)$ and $M(s) \times (M(s) \times \Delta M(s))$ are interpreted in the sense of (2.3) and (2.4) respectively.

Proof. The proof of Theorem 3.1 is very similar to the proof of Theorem 2.7 in [3] and here we only sketch the main arguments putting for simplicity $h_1 = h$ and $h_2 = h_3 = 0$. We start with some auxiliary definitions. Let $e_1, e_2, e_3, \dots$ be an orthonormal basis of $\mathbb{H}$ composed of eigenvectors of A. For each $n \geqslant 1$, let $\mathbb{H}_n$ be the linear span of $\{e_1, \dots, e_n\}$ and let

$$\pi_n u = \sum_{i=1}^n \langle u, e_i\rangle_{\mathbb{H}}\, e_i, \quad u \in \mathbb{H}.$$

Let

$$G_n(f) = \pi_n G\left(\pi_n f\right)\pi_n, \quad f \in \mathbb{L}^\infty$$

and

$$G_n'(f) = \pi_n G'\left(\pi_n f\right)\pi_n, \quad f \in \mathbb{L}^\infty.$$

Formally, G_n' is a derivative of G_n.

For each $n \geqslant 1$, we define a process $M_n : [0,T] \times \Omega \to \mathbb{H}_n$ to be a solution of the following equation in $\mathbb{H}_n$:

$$\begin{aligned} M_n(t) &= \pi_n M_0 + \int_0^t \pi_n(M_n \times \Delta M_n)\, ds \\ &- \alpha \int_0^t \pi_n(M_n \times (M_n \times \Delta M_n))\, ds \\ &+ \sqrt{\varepsilon} \int_0^t G_n(M_n) B dW(s) + \frac{\varepsilon}{2} \sum_{i=1}^{3} \int_0^t \left[G_n'(M_n) h_i\right] (G_n(M_n) h_i)\, ds \\ &- \beta \int_0^t G_n(M_n) f(M_n)\, ds + \int_0^t G_n(M_n) BF(s,W)\, ds. \end{aligned} \tag{3.2}$$

The proof of Theorem 10.6 in [8] can be used to show that, for each $n \geqslant 1$, equation (3.2) has a unique strong (in the probabilistic sense) solution. Applying the Itô formula and the Gronwall lemma to the processes $|M_n(\cdot)|^2_{\mathbb{H}}$ and $|M_n(\cdot)|^2_{\mathbb{H}^1}$, one can obtain the following, uniform in $n \geqslant 1$, estimates.

Lemma 3.2. *Let the assumptions of Theorem 3.1 be satisfyied. Then for each $n \geqslant 1$*

$$|M_n(t)|_{\mathbb{H}} = |\pi_n u_0|_{\mathbb{H}}, \quad \text{for all} \quad t \in [0,T] \quad \mathbb{P} - a.s.$$

Moreover, for each $p \in [1,\infty)$ there exists a constant $C(\alpha, T, R, p, M_0, h)$ such that for every $n \geqslant 1$

$$\mathbb{E} \sup_{t\in[0,T]} |M_n(t)|^p_{\mathbb{H}^1} \leqslant C(\alpha, T, R, p, M_0, h), \tag{3.3}$$

$$\mathbb{E}\left(\int_0^T |M_n(s) \times \Delta M_n(s)|^2_{\mathbb{H}}\, ds\right)^p \leqslant C(\alpha, T, R, p, M_0, h)$$

and

$$\mathbb{E}\left(\int_0^T |M_n(s) \times (M_n(s) \times \Delta M_n(s))|^2_{\mathbb{H}}\, ds\right)^p \leqslant C(\alpha, T, R, p, M_0, h).$$

Proposition 3.3. *There exists a probability space $(\Omega', \mathscr{F}', \mathbb{P}')$ and there exists a sequence (W_n', M_n') of $C([0,T]) \times C([0,T]; \mathbb{X}^{-\frac{1}{2}})$-valued random variables defined on $(\Omega', \mathscr{F}', \mathbb{P}')$ such that the laws of (W, M_n) and (W_n', M_n') are equal for each $n \geqslant 1$ and (W_n', M_n') converges pointwise in $C([0,T]) \times C([0,T]; \mathbb{X}^{-\frac{1}{2}})$, $\mathbb{P}'$-a.s., to a limit (W', M') with distribution $\mathbb{P}^{W,M}$.*

Proof. Lemma 3.2 and the two key results of Flandoli and Gątarek [11, Lemma 2.1 and Theorem 2.2] can be used to obtain a subsequence of $((W, M_n))_{n\geqslant 1}$ which converges in law on $C([0,T]) \times C([0,T]; \mathbb{X}^{-1/2})$ to a limiting distribution $\mathbb{P}^{W,M}$. Then the proposition follows from the Skorohod theorem (see [12, Theorem 4.30]). □

It remains to show that the pointwise limit (W', M') defined on the probability space $(\Omega', \mathscr{F}', \mathbb{P}')$ satisfies all the claims of Theorem 3.1. For each $n \geqslant 1$, (W_n', M_n') satisfies an equation obtained from (3.2) by replacing W and M_n by W_n' and M_n', respectively. Then the processes M_n' satisfy the estimates of Lemma 3.2. These estimates together with the pointwise convergence of the sequence $((W_n', M_n'))_{n\geqslant 1}$ imply that the (W', M') satisfies equation (3.1). The proof of part 5 of Theorem 3.1 is analogous to the proofs of [3, Lemma 5.1 and Lemma 5.2] but an additional care must be taken to prove that for every $t \leqslant T$

$$\int_0^t G_n\left(M_n'\right) BF\left(s, W_n'\right)\, ds \longrightarrow \int_0^t G_n\left(M'\right) BF\left(s, W'\right)\, ds \quad \mathbb{P}' - a.s. \tag{3.4}$$

To this end we note first that the processes

$$[0,T] \times \Omega' \ni \left(s, \omega'\right) \to F\left(s, W'\left(\omega'\right)\right) \in \mathbb{R}^3$$

and

$$[0,T] \times \Omega' \ni \left(s, \omega'\right) \to F\left(s, W_n'\left(\omega'\right)\right) \in \mathbb{R}^3$$

are well defined and predictable on the space $(\Omega', \mathscr{F}', \mathbb{F}', \mathbb{P}')$. For any $t \in (0,T]$ we have

$$\begin{aligned}\int_0^t G_n\left(M_n'\right) BF\left(s, W_n'\right) ds &= \int_0^t \left(G_n\left(M_n'\right) - G\left(M_n'\right)\right) BF\left(s, W_n'\right) ds \\ &+ \int_0^t \left(G\left(M_n'\right) - G\left(M'\right)\right) BF\left(s, W_n'\right) ds + \int_0^t G\left(M'\right) BF\left(s, W_n'\right) ds\,.\end{aligned}$$

Using the estimate (3.3) and the definition of G it is easy to check that in the above identity the first two terms on the right-hand side converge to zero $\mathbb{P}'$-a.s. To complete the proof of (3.4) we use the fact that, given $\eta > 0$, there exists a bounded function $F_\eta : [0,T] \times C_0\left([0,T]; \mathbb{R}^3\right) \to \mathbb{R}^3$ measurable with respect to the completed Borel σ-algebra $\mathscr{B}\left([0,T] \times C_0\left([0,T], \mathbb{R}^3\right)\right)$ such that $F_\eta(s, \cdot)$ is continuous on $C_0\left([0,T]; \mathbb{R}^3\right)$ for each fixed $s \in [0,T]$ and

$$\int_{C_0([0,T];\mathbb{R}^3)} \int_0^T |F(s,f) - F_\eta(s,f)|^2\, ds\, \mu^W(df) < \eta^2.$$

This fact follows form the assumption that the associated mapping $F : C_0\left([0,T]; \mathbb{R}^3\right) \to L^2\left([0,T]; \mathbb{R}^3\right)$ is Borel-measurable and bounded and μ^W is a Radon measure. Finally,

$$\begin{aligned}\int_0^t G\left(M'\right) BF\left(s, W_n'\right) ds &= \int_0^t G\left(M'\right)\left(BF\left(s, W_n'\right) - BF_\eta\left(s, W_n'\right)\right) ds \\ &+ \int_0^t G\left(M'\right) BF_\eta\left(s, W_n'\right) ds\,.\end{aligned}$$

Since each process W_n' has distribution μ^W on the space $C_0\left([0,T]; \mathbb{R}^3\right)$ convergence (3.4) follows by taking first $n \to \infty$ and then $\eta \to 0$. □

4. Uniqueness and the existence of a strong solution

The main results in this section are Theorem 4.2, on pathwise uniqueness of solutions of equation (3.1) and Theorem 4.4 which is a version of the well known Yamada and Watanabe theorem, on uniqueness in law and the existence of a strong solution to equation (3.1). We start with a simple

Lemma 4.1. *Let u be an element of $\mathbb{H}^1$ such that*

$$|u(x)| = 1 \quad \text{for all } x \in \Lambda.$$

Then

$$u \times (u \times \Delta u) = -|\nabla u|^2 u - \Delta u, \tag{4.1}$$

where the equality holds in $(\mathbb{H}^1)'$.

Proof. Using definition (2.4) and invoking a well known identity

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c, \quad a, b, c \in \mathbb{R}^3,$$

we obtain for any $\phi \in \mathbb{H}^1$:

$$\begin{aligned} {}_{(\mathbb{H}^1)'}\langle u \times (u \times \Delta u), \phi \rangle_{\mathbb{H}^1} &= -\langle \nabla[(\phi \times u) \times u], \nabla u \rangle_{\mathbb{H}} \\ &= \left\langle |u|^2 \nabla \phi, \nabla u \right\rangle_{\mathbb{H}} - \langle \nabla[(u \cdot \phi)u], \nabla u \rangle_{\mathbb{H}}. \end{aligned}$$

Since by assumption $u \cdot \nabla u = \frac{1}{2}\nabla(|u|^2) = 0$ for a.e. $x \in \Lambda$, we obtain

$$\begin{aligned} {}_{(\mathbb{H}^1)'}\langle u \times (u \times \Delta u), \phi \rangle_{\mathbb{H}^1} &= \langle \nabla \phi, \nabla u \rangle_{\mathbb{H}} - \left\langle |\nabla u|^2 u, \phi \right\rangle_{\mathbb{H}} \\ &= {}_{(\mathbb{H}^1)'}\left\langle -|\nabla u|^2 u - \Delta u, \phi \right\rangle_{\mathbb{H}^1} \end{aligned}$$

□

Theorem 4.2 (Pathwise uniqueness)**.** *Assume that $(\Omega, \mathscr{F}, \mathbb{F}_T, \mathbb{P})$ is a filtered probability space with the filtration $\mathbb{F}_T = (\mathscr{F}_t)_{t\in[0,T]}$ and an $\mathbb{R}^3$-valued $\mathbb{F}$-Wiener process $(W(t))_{t\in[0,T]}$ defined on this space. Assume also that $F \in \mathscr{P}_T$. Let $M_1,\ M_2 : [0,T] \times \Omega \to \mathbb{H}$ be $\mathbb{F}$-progressively measurable continuous processes such that, for $i = 1, 2$, the paths of M_i lie in $L^4\left(0, T; \mathbb{H}^1\right)$, satisfy property (4) from Theorem 3.1 and each M_i satisfies the equation*

$$\begin{aligned} M_i(t) &= M_0 + \int_0^t M_i \times \Delta M_i \, ds - \alpha \int_0^t M_i \times (M_i \times \Delta M_i) \, ds \\ &\quad + \sqrt{\varepsilon} \int_0^t G(M_i) \, B dW(s) + \frac{\varepsilon}{2} \sum_{j=1}^3 \int_0^t \left[G'(M_i) h_j\right] G(M_i) h_j \, ds \\ &\quad - \beta \int_0^t G(M_i) f(M_i) \, ds + \int_0^t G(M_i) BF(s, W) \, ds \end{aligned} \tag{4.2}$$

for all $t \in [0,T]$, $\mathbb{P}$-almost everywhere. Then

$$M_1(\cdot, \omega) = M_2(\cdot, \omega), \quad \text{for} \quad \mathbb{P} - a.e.\ \omega \in \Omega.$$

Proof. Let us fix $F \in \mathscr{P}_T$ and let $R > 0$ be such that

$$\int_0^T |F(t)|^2\,dt \leqslant R^2, \quad \mathbb{P}-\text{a.s.}$$

Note that the above implies that

$$\int_0^T |F(t)|\,dt \leqslant R\sqrt{T}, \quad \mathbb{P}-\text{a.s.} \tag{4.3}$$

First, we note that by Lemma 4.1 the following equality holds in $\mathbb{X}^{-1/2}$.

$$M_i(s)\times(M_i(s)\times\Delta M_i(s)) = -|\nabla M_i(s)|^2 M_i(s) - \Delta M_i(s).$$

Let us assume that M_1 and M_2 are two solutions. Because $|M_i|$ are uniformly bounded by assumption, by the local Lipschitz property of G, G' and f, as well by the assumptions that each $h_i \in \mathbb{L}^\infty$, there exists a constant $C_1 > 0$, such that for all $t \in [0,T]$,

$$\sum_{i=1}^{3} |G(M_2(t))h_i - G(M_1(t))h_i|^2_{\mathbb{L}^2} \leqslant C_1|h|^2_{\mathbb{L}^\infty}|M_2(t)-M_1(t)|^2_{\mathbb{L}^2}, \tag{4.4}$$

$$\sum_{i=1}^{3} |G'(M_2(t))h_iG(M_2(t))h_i - G'(M_1(t))h_iG(M_1(t))h_i|^2_{\mathbb{L}^2} \leqslant C_1|h|^2_{\mathbb{L}^\infty}|M_2(t)-M_1(t)|^2_{\mathbb{L}^2} \tag{4.5}$$

$$\begin{aligned}&\langle\big[\big(G(M_2(t)) - G(M_1(t))\big)\big]BF(s,W), M_2(t)-M_1(t)\rangle_{\mathbb{L}^2} \\ &\qquad\qquad \leqslant C|F(t)||h|_{\mathbb{L}^\infty}|M_2(t)-M_1(t)|^2_{\mathbb{L}^2}\end{aligned} \tag{4.6}$$

$$\begin{aligned}&\langle G(M_2(t))\,f(M_2(t)) - G(M_1(t))\,f(M_1(t)), M_2(t)-M_1(t)\rangle_{\mathbb{L}^2} \\ &\qquad\qquad \leqslant C_1|M_2(t)-M_1(t)|^2_{\mathbb{L}^2}\end{aligned} \tag{4.7}$$

Let $Z = M_2 - M_1$. Then by Lemma 4.1 the process Z belongs to $\mathbb{M}^2(0,T;V)\cap L^2(\Omega, C([0,T];\mathbb{H})$ and Z is a weak solution of the problem

$$\begin{aligned} dZ(t) &= \alpha AZ\,dt + \alpha\big(|\nabla M_2|^2M_2 - |\nabla M_1|^2M_1\big)\,dt \\ &+ \big(M_2\times\Delta M_2 - M_1\times\Delta M_1\big)\,dt \\ &\quad + \sqrt{\varepsilon}\big(G(M_2) - G(M_1)\big)\,BdW(s) \\ &+ \frac{\varepsilon}{2}\sum_{j=1}^{3}\Big[G'(M_2)\,h_jG(M_2)\,h_j - G'(M_1)\,h_jG(M_1)\,h_j\Big]\,dt \\ &\quad - \beta\Big[G(M_2)\,f(M_2) - G(M_1)\,f(M_1)\Big]\,dt \\ &+ \Big[\big(G(M_2(t)) - G(M_1(t))\big)\Big]BF(s,W)\,dt \end{aligned} \tag{4.8}$$

We can check that all assumptions of the Itô Lemma from [17] are satisfied (to be done!) and therefore we have

$$
\begin{aligned}
\frac{1}{2} d|Z(t)|^2_{\mathrm{H}} &= -\langle AZ, Z\rangle \, dt \\
&+ \alpha\langle |\nabla M_2(t)|^2 M_2(t)\, dt - |\nabla M_1(t)|^2 M_1(t), Z\rangle \, dt \\
&+ \alpha\langle \big(\nabla M_1(t) + \nabla M_2(t)\big) M_1(t) \nabla Z, Z\rangle \, dt \\
&+ \Big[\langle M_2(t) \times \Delta Z, Z\rangle - \langle Z \times \Delta M_1(t), Z\rangle\Big] \, dt \\
&+ \frac{\varepsilon}{2} \sum_{j=1}^{3} \langle G'\left(M_2(t)\right) h_j G\left(M_2(t)\right) h_j - G'\left(M_1(t)\right) h_j G\left(M_1(t)\right) h_j, Z\rangle \, dt \\
&\quad - \beta\langle G\left(M_2(t)\right) f\left(M_2(t)\right) - G\left(M_1(t)\right) f\left(M_1(t)\right), Z\rangle \, dt \\
&+ \langle \big[\big(G\left(M_2(t)\right) - G\left(M_1(t)\right)\big)\big] BF(s, W), Z\rangle \, dt \\
&+ \frac{1}{2}\varepsilon \sum_{j=1}^{3} |\big(G\left(M_2(t)\right) - G\left(M_1(t)\right)\big) h_j|^2_{\mathrm{H}} dt \\
&\quad + \sqrt{\varepsilon} \sum_{j=1}^{3} \langle G\left(M_2(t)\right) - G\left(M_1(t)\right)\big) h_j, Z\rangle \, dW_j(s) \\
&= \sum_{i=1}^{8} I_i(t)\, dt + \sum_{j=1}^{3} I_{9,j}(t)\, dW_j(t)
\end{aligned} \tag{4.9}
$$

We will estimate all the terms in (4.9). In what follows we will often use inequality (2.2) and k is the constant from that inequality. Let us start with the 1^{st} term:

$$
I_1(t) = -\langle AZ(t), Z(t)\rangle = -|\nabla Z(t)|^2.
$$

As for the 2nd term we have

$$
\begin{aligned}
&\langle |\nabla M_2|^2 M_2 - |\nabla M_1|^2 M_1, Z\rangle \\
&\qquad = \langle |\nabla M_2|^2 Z, Z\rangle + \langle (\nabla M_1 + \nabla M_2) M_1 \nabla Z, Z\rangle =: I + \sum_{i=1}^{2} II_i \,.
\end{aligned}
$$

Next,

$$
\begin{aligned}
I &\leqslant |\nabla M_2|^2_{\mathbb{L}^2} |Z|^2_{\mathbb{L}^\infty} \\
&\leqslant k^2 |\nabla M_2|^2_{L^2} |Z|_{\mathbb{L}^2} |Z|_{\mathbb{H}^1} \\
&\leqslant k^2 |\nabla M_2|^2_{\mathbb{L}^2} |Z|_{\mathbb{L}^2} \big(|Z|_{\mathbb{L}^2} + |\nabla Z|_{\mathbb{L}^2}\big) \\
&\leqslant k^2 |\nabla M_2|^2_{\mathbb{L}^2} |Z|^2_{\mathbb{L}^2} + k^2 |\nabla M_2|^2_{\mathbb{L}^2} |Z|_{\mathbb{L}^2} |\nabla Z|_{\mathbb{L}^2} \\
&\leqslant k^2 |\nabla M_2|^2_{\mathbb{L}^2} |Z|^2_{\mathbb{L}^2} + \frac{k^4}{2\eta^2} |\nabla M_2|^4_{\mathbb{L}^2} |Z|^2_{\mathbb{L}^2} + \eta^2 |\nabla Z|^2_{\mathbb{L}^2} \,,
\end{aligned}
$$

and

$$
\begin{aligned}
II_i &\leqslant |\nabla M_i|_{\mathbb{L}^2}|M_1|_{\mathbb{L}^\infty}|\nabla Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^\infty} \leqslant |\nabla M_i|_{\mathbb{L}^2}|\nabla Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^\infty}\\
&\leqslant k|\nabla M_i|_{\mathbb{L}^2}|\nabla Z|_{\mathbb{L}^2}|Z|^{\frac12}_{\mathbb{L}^2}\big(|Z|^{\frac12}_{\mathbb{L}^2}+|\nabla Z|^{\frac12}_{\mathbb{L}^2}\big)\\
&\leqslant k|\nabla M_i|_{\mathbb{L}^2}|\nabla Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^2}+k|\nabla M_i|_{\mathbb{L}^2}|Z|^{\frac12}_{\mathbb{L}^2}|\nabla Z|^{\frac32}_{\mathbb{L}^2}\\
&\leqslant \frac{k^2}{\eta^2}|\nabla M_i|^2_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2}+\eta^2|\nabla Z|^2_{\mathbb{L}^2}+\frac{k^4}{4\eta^6}|\nabla M_i|^4_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2}+\frac34\eta^2|\nabla Z|^2_{\mathbb{L}^2}\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_2(t)=\langle|\nabla M_2|^2M_2-|\nabla M_1|^2M_1,Z\rangle &\leqslant k^2\Big[|\nabla M_2|^2_{\mathbb{L}^2}+\frac{k^2}{2\eta^2}|\nabla M_2|^4_{\mathbb{L}^2}\\
&+\sum_{i=1}^2\frac{1}{\eta^2}|\nabla M_i|^2_{\mathbb{L}^2}+\frac{k^2}{4\eta^6}\sum_{i=1}^2|\nabla M_i|^4_{\mathbb{L}^2}\Big]|Z|^2_{\mathbb{L}^2}+\frac52\eta^2|\nabla Z|^2_{\mathbb{L}^2}
\end{aligned}
$$

Let us note now that by (2.7), the 2^{nd} part of the 4^{th} term, i.e. $\langle Z\times\Delta M_1,Z\rangle$ is equal to 0. Next, by definition (2.5), similarly as the estimate of II_i above, we have the following estimates for the 1st part of the 4^{th} term using the bound $|Z|_{L^\infty}\leqslant 2$, we get

$$
\begin{aligned}
\langle M_2\times\Delta Z,Z\rangle &= -\langle Z\times\nabla M_2,\nabla Z\rangle \leqslant |Z|_{\mathbb{L}^\infty}|\nabla M_2|_{\mathbb{L}^2}|\nabla Z|_{\mathbb{L}^2}\\
&\leqslant \frac{k^2}{\eta^2}|\nabla M_i|^2_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2}+\eta^2|\nabla Z|^2_{\mathbb{L}^2}+\frac{k^4}{4\eta^6}|\nabla M_i|^4_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2}+\frac34\eta^2|\nabla Z|^2_{\mathbb{L}^2}
\end{aligned}
$$

Therefore, we get the following inequality for the 4^{th} term

$$
\begin{aligned}
I_4(t) &= \Big[\langle M_2(t)\times\Delta Z,Z\rangle-\langle Z\times\Delta M_1(t),Z\rangle\Big]\\
&\leqslant \frac{k^2}{\eta^2}|\nabla M_i|^2_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2}+\eta^2|\nabla Z|^2_{\mathbb{L}^2}+\frac{k^4}{4\eta^6}|\nabla M_i|^4_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2}+\frac34\eta^2|\nabla Z|^2_{\mathbb{L}^2}
\end{aligned}
$$

Next, we will deal with the 3^{rd} term. Since $|M_1|_{\mathbb{L}^\infty}=1$, the Hölder inequality yields

$$
\begin{aligned}
\langle\nabla M_jM_1\nabla Z,Z\rangle &\leqslant |\nabla M_j|_{\mathbb{L}^2}|M_1|_{\mathbb{L}^\infty}|\nabla Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^\infty}\leqslant \nabla M_j|_{\mathbb{L}^2}|\nabla Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^\infty}\\
&\leqslant \frac{k^2}{\eta^2}|\nabla M_i|^2_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2}+\eta^2|\nabla Z|^2_{\mathbb{L}^2}+\frac{k^4}{4\eta^6}|\nabla M_i|^4_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2}+\frac34\eta^2|\nabla Z|^2_{\mathbb{L}^2}.
\end{aligned}
$$

Therefore, we get the following inequality for the 3^{rd} term

$$
\begin{aligned}
I_3(t)=\langle\big(\nabla M_1+\nabla M_2\big)M_1\nabla Z,Z\rangle &= \sum_{j=1}^2\langle\nabla M_jM_1\nabla Z,Z\rangle\leqslant\frac{k^2}{\eta^2}\Big(\sum_{i=1}^2|\nabla M_i|^2_{\mathbb{L}^2}\Big)|Z|^2_{\mathbb{L}^2}\\
&+\eta^2|\nabla Z|^2_{\mathbb{L}^2}+\frac{k^4}{4\eta^6}\Big(\sum_{i=1}^2|\nabla M_i|^4_{\mathbb{L}^2}\Big)|Z|^2_{\mathbb{L}^2}+\frac32\eta^2|\nabla Z|^2_{\mathbb{L}^2}
\end{aligned}
$$

By inequalities (4.4), (4.6) and (4.7) we get the following bound for the 5^{th}, 6^{th} and 8^{th} terms

$$
\sum_{i=5,6,8}I_i(t)\leqslant C_1|Z(t)|^2_{\mathbb{L}^2}.
$$

Finally, for the last term we get by (4.6)

$$I_7(t) \leqslant C_1 |F(t)||Z(t)|^2_{\mathbb{L}^2}.$$

Finally, let us define an $\mathbb{R}$-valued process

$$\xi_9(t) := \int_0^t \sum_{j=1}^3 I_{9,j}(s)\, dW_j(s) \quad t \in [0,T].$$

Obviously, ξ_9 is an $\mathbb{L}^2$-valued martingale. Next we add together the terms containing $\eta^2|\nabla Z|^2_{\mathbb{L}^2}$ to obtain

$$\frac{19}{4}\eta^2|\nabla Z|^2_{\mathbb{L}^2} \leqslant 5\eta^2|\nabla Z|^2_{\mathbb{L}^2}\,.$$

Choosing η in such a way that $5\eta^2 = \frac{1}{2}$, we introduce a process

$$\begin{aligned}
\varphi(t) \;&=\; C + |F(t)| + k^2\Big[|\nabla M_2|^2_{\mathbb{L}^2} + \frac{k^2}{2\eta^2}|\nabla M_2(t)|^4_{\mathbb{L}^2} \\
&+\; \sum_{i=1}^2 \frac{1}{\eta^2}|\nabla M_i(t)|^2_{\mathbb{L}^2} + \frac{k^2}{4\eta^6}\sum_{i=1}^2 |\nabla M_i(t)|^4_{\mathbb{L}^2}\Big] \\
&+\; \frac{k^2}{\eta^2}|\nabla M_i(t)|^2_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}|\nabla M_i(t)|^4_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}\sum_{i=1}^2 |\nabla M_i(t)|^4_{\mathbb{L}^2}, \quad t \in [0,T].
\end{aligned}$$

Therefore, from all our inequalities we infer that for some constant $C > 0$

$$|Z(t)|^2_{\mathbb{L}^2} \leqslant \int_0^t \varphi(s)\, |Z(s)|^2_{\mathbb{L}^2}\, ds + \xi_9(t), \quad t \in [0,T] \tag{4.10}$$

Then by the Itô Lemma applied to an $\mathbb{R}$-valued process (see [20] for a similar idea)

$$Y(t) := |Z(t)|^2_{\mathbb{L}^2} e^{-\int_0^t \varphi(s)\, ds}, \quad t \in [0,T],$$

we infer that

$$\begin{aligned}
Y(t) \;&\leqslant\; \int_0^t e^{-\int_0^t \varphi(s)\, ds}\, d\xi_9(s) \\
&=\; \sqrt{\varepsilon}\sum_{j=1}^3 \int_0^t e^{-\int_0^t \varphi(s)\, ds} \langle G\left(M_2(t)\right) - G\left(M_1(t)\right)\big)h_j, Z\rangle\, dW_j(s), \quad t \in [0,T].
\end{aligned}$$

Since M_1, M_2 and Z are uniformly bounded and G is locally Lipschitz the process defined by the RHS of the last inequality is an $\mathbb{F}$-martingale. Thus, we infer that

$$\mathbb{E}Y(t) \leqslant 0, \quad t \in [0,T]\,,$$

and since Y is nonnegative, we deduce that $Y(t) = 0$, $\mathbb{P}$-a.s., for every $t \in [0,T]$. Finally, the definition of Y yields

$$Z(t) = 0 \quad \mathbb{P} - a.s., \quad \text{for every} \quad t \in [0,T]\,.$$

This completes the proof. $\square$

Remark 4.3. Our uniqueness result should also hold in a stronger sense as follows. Suppose that M_1 is a solution in the sense of the last theorem and M_2 a solution in the sense of Theorem 3.1, defined on the same filtered probability space. Then $M_1 = M_2$ as in Theorem 4.2.

By an infinite-dimensional version of the Yamada and Watanabe theorem [16] pathwise uniqueness and the existence of weak solutions implies uniqueness in law and the existence of a strong solution. In Theorem 4.4 below, we state such a result for equation (3.1). In what follows we say 'weak solution' instead of 'weak martingale solution' and, to simplify notation, we set

$$S_T := C([0,T];\mathbb{H}) \cap L^4(0,T;\mathbb{H}^1). \tag{4.11}$$

Theorem 4.4. *Let assumptions of Theorem 4.2 be satisfied. Then uniqueness in law and the existence of a strong solution holds for equation (3.1) in the following sense:*

1. *if (W,M) and (W',M') are two weak solutions of equation (3.1) with W and W' being two Wiener processes, defined on possibly different probability spaces and M and M' are S_T-valued random variables, then M and M' have the same law on S_T;*
2. *there exists a measurable function $J : \left(C\left([0,T];\mathbb{R}^3\right), \mathscr{B}_{C([0,T];\mathbb{R}^3)}\right) \to (S_T, \mathscr{B}_{S_T})$ with the following property: for any filtered probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t)_{t\in[0,T]}, \tilde{\mathbb{P}})$, where the filtration $(\tilde{\mathscr{F}}_t)$ is such that $\tilde{\mathscr{F}}_0$ contains all sets in $\tilde{\mathscr{F}}$ of $\tilde{\mathbb{P}}$-measure zero, and for any $(\tilde{\mathscr{F}}_t)$-Wiener process $(\tilde{W}(t))_{t\in[0,T]}$ defined on this space, the process $J(\tilde{W})$ is $(\tilde{\mathscr{F}}_t)$-adapted and the pair $(\tilde{W}, J(\tilde{W}))$ is a weak solution of equation (3.1).*

By Theorem 4.2 it is enough to apply in the space S_T the Yamada-Wantanbe theorem in the version proved in [16].

5. Further regularity

In this section, we assume that (W,M) is a given weak solution of equation (3.1) such that M has paths in the space S_T defined in (4.11). Recall that, by Theorem 4.4, the law of M is unique. Some regularity properties of M are listed in Theorem 3.1. The main result of this section is Theorem 5.2, where we prove stronger regularity of the solution. In Proposition 5.4, we use this estimate to show that paths of M lie in $C([0,T];\mathbb{H}^1)$, $\mathbb{P}$-almost everywhere; this improves upon the continuity property in Theorem 3.1.
We start with a lemma that expresses M in a form which allows us to exploit the regularizing properties of the semigroup (e^{-tA}). The proof of this well known fact is omitted.

Lemma 5.1. *For each $t \in [0,T]$ $\mathbb{P}$-a.s.*

$$\begin{aligned} M(t) = e^{-\alpha t A}M_0 &+ \int_0^t e^{-\alpha(t-s)A}(M\times\Delta M)ds + \alpha\int_0^t e^{-\alpha(t-s)A}\left(|\nabla M|^2 M\right)\,ds \\ &+ \int_0^t e^{-\alpha(t-s)A}G(M)BF(s,W)\,ds + \varepsilon^{\frac{1}{2}}\int_0^t e^{-\alpha(t-s)A}G(M)B\,dW(s) \\ &- \beta\int_0^t e^{-\alpha(t-s)A}f(M)\,ds + \frac{\varepsilon}{2}\sum_{i=1}^3\int_0^t e^{-\alpha(t-s)A}G'(M)h_iG(M)h_i\,ds. \end{aligned} \tag{5.1}$$

Theorem 5.2. *Assume that* $M_0 \in \mathbb{H}^1$*,* $F \in \mathscr{P}_T$ *and* $h = \left(h_i\right)_{i=1}^3 \in (\mathbb{H}^1)^3$ *and let* M *be the corresponding weak solution. Then there exists a constant* $C = C\left(\alpha, T, \|F\|, \|M_0\|, |h|\right)$ *such that*

$$\mathbb{E}\left(\int_0^T \left|\Delta M(t)\right|^2_{\mathbb{H}}\, dt + \int_0^T \left|\nabla M(t)\right|^4_{\mathbb{L}^4} dt\right) \leqslant C.$$

Proof. We will use repeatedly the following well known properties of the semigroup $\left(e^{-tA}\right)$. The semigroup $\left(e^{-tA}\right)$, where A is defined in 2.1, is ultracontractive (see, for example, [2]), that is, there exists $C > 0$ such that for $1 \leqslant p \leqslant q \leqslant \infty$

$$|\mathrm{e}^{-tA} f|_{\mathbb{L}^q} \leqslant \frac{C}{t^{\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}} |f|_{\mathbb{L}^p}, \quad f \in \mathbb{L}^p,\ t > 0. \tag{5.2}$$

It is also well known that A has maximal regularity property, that is, there exists $C > 0$ such that for any $f \in L^2\left(0, T; \mathbb{H}\right)$ and

$$u(t) = \int_0^t e^{-(t-s)A} f(s) ds, \quad t \in [0, T],$$

we have

$$\int_0^T \left|Au(t)\right|^2_{\mathbb{H}}\, dt \leqslant C \int_0^T \left|f(t)\right|^2_{\mathbb{H}} dt. \tag{5.3}$$

Without loss of generality we may assume that $h_2 = h_3 = 0$ and put $h_1 = h$. Then by Lemma 5.1 M can be written as a sum of seven terms:

$$M(t) = \sum_{i=0}^6 m_i(t),$$

and we will consider each term separately. Let us fix for the rest of the proof $T > 0$ and $\delta \in \left(\frac{5}{8}, \frac{3}{4}\right)$. In what follows, C stands for a generic constant that depends on T only. In order to simplify notation, we put (without loss of generality) $\epsilon = \alpha = 1$.
We will show first that

$$\mathbb{E} \int_0^T \left|M(t)\right|^4_{\mathbb{W}^{1,4}} dt \leqslant C\left(\alpha, T, R, M_0, h\right). \tag{5.4}$$

To this end we will prove a stronger estimate:

$$\mathbb{E} \int_0^T \left|A^\delta M(t)\right|^4_{\mathbb{H}} dt \leqslant C\left(\alpha, T, R, M_0, h\right). \tag{5.5}$$

Then (5.4) will follow from the Sobolev imbedding $\mathbb{X}^\delta \hookrightarrow \mathbb{W}^{1,4}$.
We start with m_0. For each $t \in (0, T]$, we have

$$\left|A^\delta e^{-tA} M_0\right|^4_{\mathbb{H}} \leqslant \frac{C}{t^{4\delta - 2}} \left|M_0\right|^4_{\mathbb{H}^1},$$

and therefore,

$$\int_0^T \left|A^\delta m_0(t)\right|_{\mathbb{H}}^4 \,dt \leqslant C\,|M_0|_{\mathbb{H}^1}^4\,. \tag{5.6}$$

We will consider m_1. Putting $f = M \times \Delta M$ we have

$$|A^\delta e^{-(t-s)A} f(s)|_{\mathbb{H}} \leqslant C(t-s)^{-\delta}|f(s)|_{\mathbb{H}}, \quad 0 < s < t < T,$$

hence applying the Young inequality we obtain

$$\begin{aligned}\int_0^T |A^\delta m_1(t)|_{\mathbb{H}}^4\,dt \leqslant \quad & C\int_0^T \left(\int_0^t (t-s)^{-\delta}|f(s)|_{\mathbb{H}}\,ds\right)^4 dt \\ \leqslant \quad & C\left(\int_0^T s^{-4\delta/3}\,ds\right)^3 \left(\int_0^T |f(s)|_{\mathbb{H}}^2\,ds\right)^2 .\end{aligned}$$

Thereby, since $\frac{4}{3}\delta < 1$, part (3) of Theorem 3.1 yields

$$\mathbb{E}\int_0^T |A^\delta m_1(t)|_{\mathbb{H}}^4\,dt \leqslant C(R,T,M_0,h). \tag{5.7}$$

Since for every $t \in [0,T]$ we have $|M(t,x)| = 1$- x-a.e. and $h \in \mathbb{H}^1$, the estimate (**??**) implies that there exists deterministic $c > 0$ such that

$$|G(M)| + |G'(M)hG(M)h| \leqslant c.$$

Therefore, the same arguments as for m_1 yield

$$\mathbb{E}\int_0^T |A^\delta m_3(t)|_{\mathbb{H}}^4\,dt \leqslant C(R,T,M_0,h). \tag{5.8}$$

and

$$\mathbb{E}\int_0^T |A^\delta m_5(t)|_{\mathbb{H}}^4\,dt \leqslant C(R,T,M_0,h). \tag{5.9}$$

We will consider the next term m_2 using the fact that $f = |\nabla M|^2 M \in \mathbb{L}^1(0,T;\mathbb{H})$. Invoking the semigroup property of e^{-tA} and the ultracontractive estimate (5.2) with $p = 1$ and $q = 2$ we find that there exists $C > 0$ such that obtain for s,t such that $\mathbb{P}$-a.s.

$$|A^\delta e^{-(t-s)A} f(s)|_{\mathbb{H}} \leqslant \frac{C}{(t-s)^{\delta+\frac{1}{4}}} \sup_{r\leqslant T} |M\,(r)\,|_{\mathbb{H}^1}^2, \quad 0 < s < t \in [0,T].$$

Therefore,

$$\int_0^T \left|\int_0^t A^\delta e^{-(t-s)A} f(s)\,ds\right|_{\mathbb{H}}^4 dt \leqslant C \sup_{r\leqslant T}|M\,(r)\,|_{\mathbb{H}^1}^8 \int_0^T \left(\int_0^t \frac{ds}{(t-s)^{\delta+\frac{1}{4}}}\,ds\right)^4 dt$$

hence (since $\delta+\frac{1}{4}<1$) Theorem 3.1 yields

$$\mathbb{E}\int_0^T \left|A^\delta m_2(t)\right|^4_{\mathbb{H}}\,dt \leqslant C\left(R,T,M_0,h\right). \tag{5.10}$$

In order to estimate m_4 we recall that

$$|\nabla G(M(s))h| \leqslant a+b|\nabla M(s)|, \tag{5.11}$$

where $a,b>0$ are constants depending on h only. Invoking Lemma 7.2 in [9] we find that for any $t\in[0,T]$

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t A^\delta e^{-(t-s)A}G(M(s))h\,d\widehat{W}(s)\right|^4_{\mathbb{H}} \\
&\leqslant C(T)\mathbb{E}\left(\int_0^t |A^\delta e^{-\alpha(t-s)A}G(M(s))h|^2_{\mathbb{H}}\,ds\right)^2 \\
&= C(T)\mathbb{E}\left(\int_0^t |A^{\delta-\frac{1}{2}}e^{-(t-s)A}A^{\frac{1}{2}}G(M(s))h|^2_{\mathbb{H}}\,ds\right)^2 \\
&\leqslant C(T)\mathbb{E}\left(\int_0^t \frac{|G(M(s))h|^2_{\mathbb{H}^1}}{(t-s)^{2\delta-1}}\,ds\right)^2 \\
&\leqslant C(T)\mathbb{E}\sup_{r\leqslant T}|M\left(r\right)|^8_{\mathbb{H}^1}.
\end{aligned}$$

Theorem 3.1 now yields

$$\mathbb{E}\int_0^T |A^\delta m_4(t)|^4_{\mathbb{H}}\,dt \leqslant C(T,R,M_0,h). \tag{5.12}$$

Finally, combining estimates (5.6) to (5.12) we obtain (5.5) and (5.4) follows.
It remains to prove that

$$\mathbb{E}\int_0^T |AM(t)|^2_{\mathbb{H}}\,dt \leqslant C(T,R,M_0,h). \tag{5.13}$$

To this end we note first that using the maximal inequality (5.13) and the first part of the proof it is easy to see that

$$\mathbb{E}\int_0^T |Am_i(t)|^2_{\mathbb{H}}\;dt \leqslant C(T,R,M_0,h), \quad i=1,2,3,5. \tag{5.14}$$

The estimate

$$\int_0^T |Am_0(t)|^2_{\mathbb{H}}\;dt \leqslant C\left(M_0\right), \tag{5.15}$$

is an immediate consequence of the fact that $M_0 \in \mathbb{H}^1 = D\left(A^{1/2}\right)$.
We will consider now the stochastic term m_4. Using (5.11), a result of Pardoux in [17] and part 1 of Theorem 3.1 we find that

$$\begin{aligned}\mathbb{E}\int_0^T |Am_4(t)|^2_{\mathbb{H}} dt &\leqslant \mathbb{E}\int_0^T \left(a|\nabla(M(t))|^2_{\mathbb{H}} + b\right)\, dt \\ &\leqslant C\left(\alpha, T, R, M_0, h\right).\end{aligned} \tag{5.16}$$

Combining (5.14), (5.15) and (5.16) we obtain (5.13) and the proof is complete. □

Remark 5.3. By estimate (5.4) the vector $AM(t,x) \in \mathbb{R}^3$ is well defined t,x-a.e. and $|M(t,x)|^2 = 1$ t,x-a.e., hence

$$M(t,x)\cdot \Delta M(t,x) = -|\nabla M(t,x)|^2, \quad t,x-\ a.e.$$

Therefore, an elementary identity

$$|a\times b| + |a\cdot b|^2 = |a|^2\cdot|b|^2, \quad a,b\in\mathbb{R}^3,$$

yields

$$|M(t,x)\times \Delta M(t,x)|^2 + |\nabla M(t,x)|^4 = |\Delta M(t,x)|^2, \quad t,x-a.e.$$

Proposition 5.4. *Paths of M lie in $C([0,T];\mathbb{H}^1)$, $\mathbb{P}$-a.s.*

Proof. The proposition follows easily from the results in [17]. □

Corollary 5.5. *Let $h_i \in \mathbb{W}^{1,3}$, $i=1,2,3$, and let $F\in\mathscr{P}_T$. Let W be an $\mathbb{F}_T$-adapted Wiener process probability space $(\Omega,\mathscr{F},\mathbb{F}_T,\mathbb{P})$ Then, for every $M_0\in\mathbb{H}^1$ and $\epsilon>0$, there exists a unique pathwise solution $M^\epsilon \in C\left([0,T];\mathbb{H}^1\right)\cap L^2\left(0,T;D(A)\right)$ of the equation*

$$\begin{aligned}M(t) = M_0 + \alpha\int_0^t \Delta M\, ds + \alpha\int_0^t |\nabla M|^2 M\, ds + \int_0^t M\times\Delta M\, ds \\ \frac{\varepsilon}{2}\sum_{i=1}^3\int_0^t G'(M)h_i G(M)h_i\, ds + \sqrt{\varepsilon}\int_0^t G(M)\, d\widehat{W}(s) \\ -\beta\int_0^t G(M)f(M)\, ds + \int_0^t G(M)\widehat{F}(s,W)\, ds,\end{aligned}$$

where all the integrals,except the Itô integral, are the Bochner integrals in $\mathbb{H}$.

In what follows we will denote by $\mathscr{X}_T$ the Banach space

$$\mathscr{X}_T = C\left([0,T];\mathbb{H}^1\right)\cap L^2(0,T;D(A)).$$

6. Small noise asymptotics

In this section we will prove the large deviation principle for the family of laws of the solutions of equation (3.1) with F identically zero and the parameter $\varepsilon\in(0,1]$ approaching zero.

The setting in this section is given by a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ carrying an $\mathbb{R}^3$-valued Wiener process $(W(t))_{t\in[0,T]}$. We will denote by $\left(\mathscr{F}_t^0\right)$ the natural filtration of this Wiener process. For each $\varepsilon \in [0,1]$ and $F \in \mathscr{P}_T$, let

$$J_F^\varepsilon : C\left([0,T];\mathbb{R}^3\right) \to S$$

denote the Borel mapping defined in Theorem 4.4 that acts on a Wiener process to give a weak solution of equation (3.1). By Corollary 5.5 the image of the Wiener process under the map J_F^ε has paths in $\mathscr{X}_T$ almost surely. In what follows, we consider J_F^ε as a Borel map

$$J_F^\varepsilon : C\left([0,T];\mathbb{R}^3\right) \to \mathscr{X}_T\,.$$

If $F(t,W) = v(t)$ with $v \in L^2\left(0,T;\mathbb{R}^3\right)$ is deterministic, then by Corollary 5.5 the function $y_v = J_v^0$ is the unique solution of the equation

$$\begin{aligned} y_v(t) \quad = \quad & M_0 + \alpha\int_0^t \Delta y_v\,ds + \alpha\int_0^t \left|\nabla y_v\right|^2 y_v\,ds + \int_0^t y_v \times \Delta y_v\,ds \\ & - \beta\int_0^t G\left(y_v\right) f\left(y_v\right)ds + \int_0^t G\left(y_v\right)Bv(s)\,ds, \end{aligned} \tag{6.1}$$

where the mapping G has been defined in (2.11) and $h_i \in \mathbb{W}^{1,3}$ for $i = 1,2,3$. Hence the map

$$L^2\left(0,T;\mathbb{R}^3\right) \ni v \longrightarrow J_v^0 \in \mathscr{X}_T$$

is well defined. We will define now the rate function $I : \mathscr{X}_T \to [0,\infty]$ by the formula

$$I(u) := \inf\left\{\frac{1}{2}\int_0^T |\phi(s)|^2\,ds : \ \phi \in L^2\left(0,T;\mathbb{R}^3\right) \text{ and } u = J_\phi^0\right\}$$

where the infimum of the empty set is taken to be infinity.
For $R > 0$, define

$$B_R := \left\{\phi \in L^2\left(0,T;\mathbb{R}^3\right) : \int_0^T |\phi(s)|^2\,ds \leqslant R^2\right\} \text{ with the weak topology of } L^2\left(0,T;\mathbb{R}^3\right).$$

In order to show that the family of laws $\{\mathscr{L}(J_0^\varepsilon(W)) : \varepsilon \in (0,1]\}$ satisfies the large deviation principle with the rate function I we will follow the weak convergence method of Budhiraja and Dupuis [5], see also Duan and Millet [10] and Chueshov and Millet [7]. To this end we need to show that the following two statements are true.

Statement 1. *For each $R > 0$, the set $\left\{J_v^0 : v \in B_R\right\}$ is a compact subset of $\mathscr{X}_T$.*

Statement 2. *Let (ε_n) be any sequence from $(0,1]$ convergent to 0 and let (v_n) be any sequence of $\mathbb{R}$-valued $\left(\mathscr{F}_t^0\right)$-predictable processes defined on $[0,T]$ such that $\|v\|_T \leqslant R$ for a certain $R > 0$. Then if (v_n) converges in law on B_R to v, then $(J_{v_n}^{\varepsilon_n}(W)$ converges in law on $\mathscr{X}_T$ to J_v^0.*

The remaining part of this section is devoted to the proof of thsese two statements.

Lemma 6.1. *Suppose that $(w_n) \subset L^2\left(0,T;\mathbb{R}^3\right)$ is a sequence converging weakly to w. Then the sequence y_{w_n} converges strongly to y_w in $\mathscr{X}_T$. In particular, the mapping*

$$B_R \ni w \longrightarrow J^0_w \in \mathscr{X}_T$$

is Borel.

Proof of Lemma 6.1. To simplify notation, we write y_n for y_{w_n}, y for y_w and set $u_n = y_n - y$. Let

$$R^2 = \sup_{n\in\mathbb{N}} \int_0^T |w_n|^2(s)\,ds.$$

Let us recall the notation $h=(h_i)$. By Theorem 3.1, Theorem 5.2 and uniqueness of solutions, there exists a finite constant $C(T,\alpha,R,M_0,h)$ such that for all $n \in \overline{\mathbb{N}}$:

$$\sup_{t\in[0,T]} |y_n(t)|_{\mathbb{H}^1} \leqslant C(T,\alpha,R,M_0,h), \tag{6.2}$$

$$\int_0^T \left(|\Delta y_n(s)|^2_{\mathbb{H}} + |\nabla y_n|^4_{\mathbb{L}^4}\right) ds \leqslant C(T,\alpha,R,M_0,h) \tag{6.3}$$

and clearly

$$|y_n(t)(x)| = 1, \quad x\in\Lambda,\ t\in[0,T]. \tag{6.4}$$

The same estimates hold for y.

Step 1. We note first that each subsequence of (y_n) has a further subsequence that converges in $C([0,T];\mathbb{H})$ strongly. Indeed, since $\mathbb{H}^1$ is compactly embedded in $\mathbb{H}$, the estimate (6.2) implies that for each fixed $t\in[0,T]$, $\{y_n(t) : n\geqslant 1\}$ is relatively compact in $\mathbb{H}$. Hence, it is enough to show that $\{y_n : n\geqslant 1\}$ is a uniformly equicontinuous subset of $C([0,T];\mathbb{H})$. To this end we write for $0\leqslant t<t'\leqslant T$:

$$\begin{aligned} y_n(t') - y_n(t) &= \int_t^{t'} y_n\times\Delta y_n\,ds + \alpha\int_t^{t'}\Delta y_n\,ds + \alpha\int_t^{t'}|\nabla y_n|^2 y_n\,ds \\ &\quad - \beta\int_t^{t'} G(y_n)\,f(y_n)\,ds + \int_t^{t'} G(y_n)\,Bw_n\,ds. \end{aligned}$$

Then (6.2), (6.3) and the Cauchy-Schwarz inequality yield immediately

$$|y_n(t')-y_n(t)|_{\mathbb{H}} \leqslant (1+\alpha)\left(\sqrt{C(T,\alpha,R,M_0,h)} + R|M_0|_{\mathbb{H}}|h|_{\mathbb{L}^\infty}\right)\sqrt{t'-t}.$$

This completes the proof of *Step 1.*

Step 2. Let $q\in L^2(0,T;\mathbb{H})$. Recall that $w_n\to w$ weakly in $L^2\left(0,T;\mathbb{R}^3\right)$. We will show that

$$\lim_{n\to\infty}\left(\sup_{t\in[0,T]}\left|\int_0^t \langle q(s),u_n(s)\rangle_{\mathbb{H}}(w_n(s)-w(s))\,ds\right|\right) = 0. \tag{6.5}$$

By *Step 1* we can assume that $u_n \to u_\infty$ in $C([0,T];H)$. For $n \in \mathbb{N}\cup\{\infty\}$ we define operators $\mathscr{K}_n : L^2\left(0,T;\mathbb{R}^3\right) \to C\left([0,T];\mathbb{R}^3\right)$ by the formula

$$\mathscr{K}_n v(t) = \int_0^t \langle q(s), u_n(s)\rangle_{\mathbb{H}}\, v(s)\, ds, \quad t \in [0,T], \ \ v \in L^2\left(0,T;\mathbb{R}^3\right).$$

The operators $\mathscr{K}_n$, are compact because the functions $\langle q(\cdot), u_n(\cdot)\rangle_{\mathbb{H}}$ belong to $L^2(0,T;\mathbb{R})$. Moreover, since the sequence $\langle q(\cdot), u_n(\cdot)\rangle_{\mathbb{H}}$ converges strongly in $L^2(0,T;\mathbb{R})$ to a function $\langle q(\cdot), u_\infty(\cdot)\rangle_{\mathbb{H}}$ we infer that

$$\lim_{n\to\infty} \|\mathscr{K}_n - \mathscr{K}_\infty\| = 0.$$

Since

$$\left|\mathscr{K}_n(w_n - w)\right|_{C([0,T];\mathbb{R}^3))} \leqslant \|\mathscr{K}_n - \mathscr{K}_\infty\| \cdot |w_n - w|_{L^2([0,T];\mathbb{R}^3)} + \left|\mathscr{K}_\infty(w_n - w)\right|_{C([0,T];\mathbb{R}^3)},$$

the claim follows immediately by the compactness of K_∞ because $w_n \to w$ weakly in $L^2\left([0,T];\mathbb{R}^3\right)$.

Step 3. We will establish uniform estimates on u_n. More precisely, we will show that

$$\sup_{n\in\overline{\mathbb{N}}}\left[\sup_{t\in[0,T]} |u_n(t)|^2_{\mathbb{H}^1} + \alpha \int_0^T |\Delta u_n|^2_{\mathbb{H}}\, ds\right] < \infty. \tag{6.6}$$

Without loss of generality we may assume that $h_2 = h_3 = 0$ and put $h_1 = h$ and $w_n^1 = w_n \in \mathbb{R}$. For each $n \geqslant 1$, we have

$$\begin{aligned} u_n(t) = &\int_0^t u_n \times \Delta y_n\, ds + \int_0^t y \times \Delta u_n\, ds \\ &+ \alpha \int_0^t (|\nabla y_n| - |\nabla y|)(|\nabla y_n| + |\nabla y|) y_n\, ds \\ &+ \alpha \int_0^t |\nabla y|^2 u_n\, ds + \alpha \int_0^t \Delta u_n\, ds \\ &- \beta \int_0^t G(y_n)\, f(u_n)\, ds - \beta \int_0^t \left(G(y_n) - G(y)\right) f(y)\, ds \\ &+ \int_0^t (u_n \times h)\, w_n\, ds + \int_0^t G(y) h (w_n - w)\, ds \\ &- \alpha \left[\int_0^t u_n \times (y_n \times h) w_n\, ds + \int_0^t y \times (u_n \times h) w_n\, ds\right]. \end{aligned} \tag{6.7}$$

Note that each integrand on the right hand side of (6.7) is square integrable in $\mathbb{H}$, because of (6.2), (6.3) and (6.4). Therefore,

$$\frac{d}{dt}|u_n(t)|^2_{\mathbb{H}^1} = 2\left\langle \frac{d}{dt}u_n(t), (I + A)u_n(t)\right\rangle_{\mathbb{H}}, \quad t \in [0,T]. \tag{6.8}$$

Let N be an arbitrary positive integer. The functions $s\mapsto y(s)$, $s\mapsto y(s)\times\pi_N h$ and $s\mapsto y(s)\times(y(s)\times\pi_N h)$ all belong to $L^2(0,T;D(A))$. Therefore, from (6.8) we have:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u_n(t)|^2_{\mathbb{H}^1} &= \langle u_n\times\Delta y_n,-\Delta u_n\rangle_{\mathbb{H}}+\langle y(s)\times\Delta u_n,u_n\rangle_{\mathbb{H}}\\
&+ \alpha\langle(|\nabla y_n|-|\nabla y|)(|\nabla y_n|+|\nabla y|)y_n,u_n-\Delta u_n\rangle_{\mathbb{H}}\\
&+ \alpha\langle|\nabla y|^2u_n,u_n-\Delta u_n\rangle_{\mathbb{H}}\,ds-\alpha|\nabla u_n|^2_{\mathbb{H}}-\alpha|\Delta u_n|^2_{\mathbb{H}}\\
&- \beta\left\langle G\left(y_n\right)f\left(u_n\right),u_n-\Delta u_n\right\rangle_{\mathbb{H}}-\beta\left\langle\left(G\left(y_n\right)-G(y)\right)f(y),u_n-\Delta u_n\right\rangle_{\mathbb{H}} \qquad (6.9)\\
&- \langle u_n\times h,\Delta u_n\rangle_{\mathbb{H}}w_n+\langle G(y)h,u_n\rangle_{\mathbb{H}}(w_n-w)\\
&- \langle\Delta(y\times\pi_N h),u_n\rangle_{\mathbb{H}}(w_n-w)-\langle y\times(h-\pi_N h),\Delta u_n\rangle_{\mathbb{H}}(w_n-w)\\
&+ \alpha\langle u_n\times(y_n\times h),\Delta u_n\rangle_{\mathbb{H}}w_n-\alpha\langle y\times(u_n\times h),u_n-\Delta u_n\rangle_{\mathbb{H}}w_n\,ds\\
&+ \alpha\langle\Delta(y\times(y\times\pi_N h)),u_n\rangle_{\mathbb{H}}(w_n-w)\\
&+ \alpha\langle y\times(y\times(h-\pi_N h)),\Delta u_n\rangle_{\mathbb{H}}(w_n-w) \qquad (6.10)
\end{aligned}$$

for all $t\in[0,T]$. In the rest of this proof, C and C_1 denote positive real constants whose value may depend only on α, T, R, M_0 and h; the actual value of the constant may be different in different instances. For each $\eta>0$ we estimate the terms on the right hand side of (6.10) using the Cauchy-Schwarz inequality and (2.2) and the Young inequality to obtain:

$$\begin{aligned}
|u_n(t)|^2_{\mathbb{H}^1} &+ 2\alpha\int_0^t|\Delta u_n|^2_{\mathbb{H}}\,ds\leqslant C_1\eta^2\int_0^t|\Delta u_n|^2_{\mathbb{H}}\,ds+b_{n,N}+C\int_0^t|u_n|^2_{\mathbb{H}^1}\psi_n(s)\,ds\\
&+ C|h-\pi_N h|_{\mathbb{H}}\left(\int_0^T|\Delta u_n(s)|^2_{\mathbb{H}}\,ds\right)^{\frac{1}{2}}\left(\int_0^T(w_n(s)-w(s))^2\,ds\right)^{\frac{1}{2}} \qquad (6.11)
\end{aligned}$$

where, for $s\in[0,T]$, we put[1]

$$\begin{aligned}
\psi_n(s) &= 1+\frac{1}{\eta^2}+\frac{1}{\eta^2}\,|\Delta y_n(s)|^2_{\mathbb{H}}\\
&+\left(1+\frac{1}{\eta^2}\right)\left[|y_n(s)|_{\mathbb{H}^1}\left(|y_n(s)|_{\mathbb{H}^1}+|\Delta y_n(s)|_{\mathbb{H}}\right)\right.\\
&+|y(s)|_{\mathbb{H}^1}(|y(s)|_{\mathbb{H}^1}+|\Delta y(s)|_{\mathbb{H}})\\
&\left.+|y(s)|^2_{\mathbb{H}^1}\left(|y(s)|^2_{\mathbb{H}^1}+|\Delta y(s)|^2_{\mathbb{H}}\right)\right]+\frac{1}{\eta^2}w_n^2+|w_n(s)|
\end{aligned}$$

[1]Note that the functions y_n satisfy the uniform estimates (6.3) and so

$$\sup_n\int_0^T\psi_n(s)\,ds<\infty.$$

.

and

$$\begin{aligned} b_{n,N} &= \sup_{r\leqslant T}\left|\int_0^r \langle G(y(s))h(s), u_n\rangle_{\mathbb{H}}(w_n(s)-w(s))\,ds\right| \\ &+ \sup_{r\leqslant T}\left|\int_0^r \langle \Delta(y(s)\times \pi_N h(s)), u_n(s)\rangle_{\mathbb{H}}(w_n(s)-w(s))\,ds\right| \\ &+ C\sup_{r\leqslant T}\left|\int_0^r \langle \Delta(y(s)\times (y(s)\times \pi_N h(s))), u_n(s)\rangle_{\mathbb{H}}(w_n(s)-w(s))\,ds\right| \\ &+ C\int_0^T |u_n(s)|^2_{\mathbb{H}}\,ds. \end{aligned}$$

We choose $\eta>0$ such that $C_1\eta^2=\frac{\alpha}{2}$ in (6.11) and use the boundedness of the sequence w_n in $L^2\left(0,T;\mathbb{R}^3\right)$ together with the Hölder inequality to estimate the product of square roots in the second line of (6.11) to obtain

$$|u_n(t)|^2_{\mathbb{H}^1}+\alpha\int_0^t |\Delta u_n(s)|^2_{\mathbb{H}}\,ds \leqslant C\int_0^t \psi_n(s)|u_n(s)|^2_{\mathbb{H}^1}\,ds + b_{n,N}. \tag{6.12}$$

By estimates (6.2) and (6.3)

$$\gamma := \sup_{n\geqslant 1}\int_0^T \psi_n(s)\,ds < \infty$$

and γ depends on α, T, R, M_0 and h only. Applying the Gronwall lemma we obtain

$$\sup_{t\in[0,T]}|u_n(t)|^2_{\mathbb{H}^1}+\alpha\int_0^T |\Delta u_n|^2_{\mathbb{H}}\,ds \leqslant b_{n,N}e^{\gamma T} \tag{6.13}$$

what completes the proof of estimates (6.13). Therefore, since by Claim(6.5) $b_{n,N}\to 0$ as $N\to\infty$ we infer that

$$\limsup_{n\to\infty}\left(\sup_{t\in[0,T]}|u_n(t)|^2_{\mathbb{H}^1}+\alpha\int_0^T |\Delta u_n|^2_{\mathbb{H}}\,ds\right) \leqslant C\,|h-\pi_N h|^2_{\mathbb{H}}. \tag{6.14}$$

We complete the proof of Lemma 6.1 by taking the limit as $N\to\infty$. □

Note, that Statement 1 follows Lemma 6.1.

Now we will occupy ourselves with the proof of that Statement 2. For this purpose let us chhose and fix the following processes:

$$Y_n = J^{\epsilon_n}_{v_n} \text{ and } y_n = J^0_{v_n}.$$

Let $N>|M_0|_{\mathbb{H}^1}$ be fixed. For each $n\geqslant 1$ we define an $(\mathscr{F}_t)$-stopping time

$$\tau_n = \inf\{t>0 : |Y_n(t)|_{\mathbb{H}^1}\geqslant N\}\wedge T. \tag{6.15}$$

Lemma 6.2. *For τ_n as defined in (6.15) we have*

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{t\in[0,T]}\left|Y_n\left(t\wedge\tau_n\right)-y_n\left(t\wedge\tau_n\right)\right|^2_{\mathbb{H}}+\int_0^{\tau_N}\left|Y_n-y_n\right|^2_{\mathbb{H}^1}\,ds\right)=0.$$

Proof. Let $X_n=Y_n-y_n$. We assume without loss of generality that $\beta=0$, $h_2=h_3=0$ and $h_1=h$. Then for any $n\geqslant 1$ we have

$$\begin{aligned}dX_n=&\,\alpha\Delta X_n dt\\&+\alpha\left(\nabla X_n\right)\cdot\left(\nabla\left(Y_n+y_n\right)\right)Y_n dt+\alpha\left|\nabla y_n\right|^2X_n dt\\&+X_n\times\Delta Y_n dt+y_n\times\Delta X_n dt\\&+\left(G\left(Y_n\right)-G\left(y_n\right)\right)hv_n dt\\&+\sqrt{\epsilon_n}G\left(Y_n\right)hdW+\frac{\epsilon_n}{2}G'\left(Y_n\right)G\left(Y_n\right)hdt\end{aligned}\tag{6.16}$$

Using a version of the Itô formula given in [17] and integration by parts we obtain

$$\begin{aligned}\frac{1}{2}d\left|X_n\right|^2_{\mathbb{H}}=&-\alpha\left|X_n\right|^2_{\mathbb{H}^1}dt+\alpha\left|\left|\nabla y_n\right|X_n\right|^2_{\mathbb{H}}dt\\&+\alpha\left\langle X_n,\left(\nabla X_n\right)\cdot\left(\nabla\left(Y_n+y_n\right)\right)Y_n\right\rangle_{\mathbb{H}}dt\\&-\left\langle\nabla X_n,X_n\times\nabla y_n\right\rangle_{\mathbb{H}}dt\\&+\left\langle\left(G\left(Y_n\right)-G\left(y_n\right)\right)h,X_n\right\rangle_{\mathbb{H}}v_n dt\\&+\frac{\epsilon_n}{2}z_n dt+\sqrt{\epsilon_n}\left\langle G\left(Y_n\right)h,X_n\right\rangle_{\mathbb{H}}dW\end{aligned}$$

where z_n is a process defined by

$$z_n=\left\langle G'\left(Y_n\right)G\left(Y_n\right)h,X_n\right\rangle_{\mathbb{H}}+\left|G\left(Y_n\right)h\right|^2_{\mathbb{H}}.$$

Therefore

$$\begin{aligned}|X_n(t)|^2_{\mathbb{H}}+2\alpha\int_0^t\left|X_n\right|^2_{\mathbb{H}^1}ds\leqslant&\, C\int_0^t\left|X_n\right|_{\mathbb{H}}\left|X_n\right|_{\mathbb{H}^1}\left|y_n\right|^2_{\mathbb{H}^1}ds\\&+C\int_0^t\left|X_n\right|^{3/2}_{\mathbb{H}^1}\left|X_n\right|_{\mathbb{H}}\left(\left|y_n\right|_{\mathbb{H}^1}+\left|y_n\right|_{\mathbb{H}^1}\right)ds\\&+C\int_0^t\left|X_n\right|^{3/2}_{\mathbb{H}^1}\left|X_n\right|_{\mathbb{H}}\left|y_n\right|_{\mathbb{H}^1}ds\\&+C\int_0^t\left|X_n\right|^2_{\mathbb{H}}\left|v_n\right|ds\\&+C\epsilon_n+\sqrt{\epsilon_n}\left|\int_0^t\left\langle G\left(Y_n\right)h,X_n\right\rangle_{\mathbb{H}}dW\right|.\end{aligned}$$

By (6.2) we have $\sup_n |y_n|_{\mathbb{H}^1} < \infty$ and therefore, using repeatedly the Young inequality we find that there exists $C > 0$ such that for all $t \in [0,T]$

$$|X_n(t)|^2_{\mathbb{H}} + \alpha \int_0^t |X_n|^2_{\mathbb{H}^1} \, ds \leqslant C \int_0^t |X_n|^2_{\mathbb{H}} \left(1 + |v_n| + \beta |y_n|^4_{\mathbb{H}^1}\right) ds$$
$$+ C\epsilon_n + \sqrt{\epsilon_n} \left| \int_0^t \langle G(Y_n) h, X_n \rangle_{\mathbb{H}} \, dW \right|.$$

Denoting the left hand side of the above inequality by L_t and using the definition of τ_n we have

$$\begin{aligned} L_{t\wedge\tau_n} &\leqslant C \int_0^{t\wedge\tau_n} |X_n|^2_{\mathbb{H}} \left(1 + |v_n| + \beta |y_n|^4_{\mathbb{H}^1}\right) ds \\ &\quad + C\epsilon_n + \sqrt{\epsilon_n} \left| \int_0^{t\wedge\tau_n} \langle G(Y_n) h, X_n \rangle_{\mathbb{H}} \, dW \right| \\ &\leqslant \int_0^{t\wedge\tau_n} |X_n|^2_{\mathbb{H}} \psi_{n,N} ds + C\epsilon_n \\ &\quad + \sqrt{\epsilon_n} \left| \int_0^{t\wedge\tau_n} \langle G(Y_n) h, X_n \rangle_{\mathbb{H}} \, dW \right|, \end{aligned}$$

where

$$\psi_{n,N}(s) = 1 + |v_n(s)| + \beta N^4, \quad s \leqslant T.$$

Since

$$\sup_n \sup_{t\in[0,T]} \langle G(Y_n(t)), X_n(t) \rangle^2 \leqslant C, \quad \mathbb{P} - a.s.,$$

the Burkoholder-Davis-Gundy inequality yields

$$\mathbb{E} \sup_{s\leqslant t} L_{s\wedge\tau_n} \leqslant C\sqrt{\epsilon_n} + \int_0^t \mathbb{E} \sup_{r\leqslant s} |X_n(r\wedge\tau_n)|^2_{\mathbb{H}} \psi_{n,N} ds, \tag{6.17}$$

and therefore

$$\mathbb{E} \sup_{r\leqslant t} |X_n(r\wedge\tau_n)|^2_{\mathbb{H}} \leqslant C\sqrt{\epsilon_n} + \int_0^t \mathbb{E} \sup_{r\leqslant s} |X_n(r\wedge\tau_n)|^2_{\mathbb{H}} \psi_{n,N} ds.$$

Clearly,

$$\sup_{n\geqslant 1} \int_0^T \psi_{n,N} ds < \infty,$$

hence the Gronwall Lemma implies

$$\mathbb{E}\sup_{r\leqslant T}|X_n(r\wedge\tau_n)|^2_{\mathbb{H}}\leqslant C\sqrt{\epsilon_n}\mathrm{e}^{\int_0^T\psi_{n,N}ds}\to 0 \text{ as } n\to\infty.$$

Returning now to (6.17), we also have

$$\mathbb{E}\int_0^{\tau_n}|X_n(s)|^2_{\mathbb{H}^1}\,ds\leqslant C\sqrt{\varepsilon_n}\mathrm{e}^{\int_0^T\psi_{n,N}ds}\to 0 \text{ as } n\to\infty.$$

This completes the proof of Lemma 6.2. □

Lemma 6.3. *For the stopping time τ_n defined in (6.15) we have*

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{t\in[0,T]}|\nabla(Y_n(t\wedge\tau_n)-y_n(t\wedge\tau_n))|^2_{\mathbb{H}}+\int_0^{\tau_n}|\Delta(Y_n-y_n)|^2_{\mathbb{H}}\,ds\right)=0.$$

Proof. By a version of the Itô formula, see [17],

$$\frac{1}{2}d|\nabla(Y_n(t)-y_n(t))|^2_{\mathbb{H}}=-\left\langle\Delta\left(Y_n-y_n\right),d\left(Y_n-y_n\right)\right\rangle_{\mathbb{H}}+\epsilon_n\left|\nabla G\left(Y_n\right)h\right|^2_{\mathbb{H}}dt.$$

Therefore, putting $X_n=Y_n-y_n$ and invoking equality (6.16) we obtain for any $\eta>0$

$$\begin{aligned}\frac{1}{2}d\left|\nabla X_n(t)\right|^2_{\mathbb{H}}=&-\alpha\left|\Delta X_n\right|^2_{\mathbb{H}}\\&-\alpha\left\langle\Delta X_n,\nabla X_n\cdot\left(\nabla Y_n+\nabla y_n\right)Y_n\right\rangle_{\mathbb{H}}dt\\&-\alpha\left\langle\Delta X_n,\left|\nabla y_n\right|^2X_n\right\rangle_{\mathbb{H}}dt\\&-\left\langle X_n\times\Delta y_n,\Delta X_n\right\rangle_{\mathbb{H}}dt\\&-\left\langle\left(G\left(Y_n\right)-G\left(y_n\right)\right)h,\Delta X_n\right\rangle_{\mathbb{H}}F_ndt\\&-\sqrt{\epsilon_n}\left\langle\nabla G\left(Y_n\right)h,\Delta X_n\right\rangle_{\mathbb{H}}dW\\&-\frac{\epsilon_n}{2}\left\langle G'\left(Y_n\right)G\left(Y_n\right)h,\Delta X_n\right\rangle_{\mathbb{H}}dt\\&+\epsilon_n\left|\nabla G\left(Y_n\right)h\right|^2_{\mathbb{H}}dt.\end{aligned}\tag{6.18}$$

We will estimate the terms in (6.18). First, noting that

$$\left\langle X_n\times\Delta Y_n,\Delta X_n\right\rangle_{\mathbb{H}}=\left\langle X_n\times\Delta u_n,\Delta X_n\right\rangle_{\mathbb{H}}$$

we find that

$$\left|\left\langle X_n\times\Delta Y_n,\Delta X_n\right\rangle_{\mathbb{H}}\right|\leqslant C\eta^2\left|\Delta X_n\right|^2_{\mathbb{H}}+\frac{C}{\eta^2}\left|X_n\right|_{\mathbb{H}}\left|X_n\right|_{\mathbb{H}^1}.\tag{6.19}$$

Next, by the Young inequality and the interpolation inequality (2.2)

$$
\begin{aligned}
|\langle \Delta X_n, \nabla X_n \cdot (\nabla Y_n + \nabla y_n)\, Y_n\rangle_{\mathbb{H}}| &\leqslant C\eta^2 |\Delta X_n|^2_{\mathbb{H}} \\
&\quad + \frac{C}{\eta^2} \int_\Lambda |\nabla X_n|^2 \left(|\nabla Y_n|^2 + |\nabla y_n|^2\right) dx \\
&\leqslant C\eta^2 |\Delta X_n|^2_{\mathbb{H}} \\
&\quad + \frac{C}{\eta^2} |\nabla X_n|^2_\infty \int_\Lambda \left(|\nabla Y_n|^2 + |\nabla y_n|^2\right) dx \\
&\leqslant C\eta^2 |\Delta X_n|^2_{\mathbb{H}} \\
&\quad + \frac{C}{\eta^2} |X_n|_{\mathbb{H}^1} \left(|X_n|_{\mathbb{H}^1} + |\Delta X_n|_{\mathbb{H}}\right) \left(|Y_n|^2_{\mathbb{H}^1} + |y_n|^2_{\mathbb{H}^1}\right)
\end{aligned},
$$

and thereby

$$
\begin{aligned}
|\langle \Delta X_n, \nabla X_n \cdot (\nabla Y_n + \nabla y_n)\, Y_n\rangle_{\mathbb{H}}| &\leqslant C\eta^2 |\Delta X_n|^2_{\mathbb{H}} \\
&\quad + \frac{C}{\eta^2} |X_n|^2_{\mathbb{H}^1} \left(|Y_n|^2_{\mathbb{H}^1} + |y_n|^2_{\mathbb{H}^1}\right) \\
&\quad + \frac{C}{\eta^6} |X_n|^2_{\mathbb{H}^1} \left(|Y_n|^4_{\mathbb{H}^1} + |y_n|^4_{\mathbb{H}^1}\right) \\
&\leqslant C\eta^2 |\Delta X_n|^2_{\mathbb{H}} + C_\eta |X_n|^2_{\mathbb{H}^1} \left(1 + |Y_n|^4_{\mathbb{H}^1}\right).
\end{aligned} \tag{6.20}
$$

Finally, using (2.2) we obtain

$$
\begin{aligned}
|\left\langle \Delta X_n, |\nabla y_n|^2 X_n \right\rangle_{\mathbb{H}}| &\leqslant |\Delta X_n|_{\mathbb{H}} |\nabla y_n|_{\mathbb{L}^\infty} |\nabla y_n|_{\mathbb{H}} |X_n|_{\mathbb{L}^\infty} \\
&\leqslant C\eta^2 |\Delta X_n|^2_{\mathbb{H}} \\
&\quad + \frac{C}{\eta^2} |y_n|_{\mathbb{H}^1} \left(|y_n|_{\mathbb{H}^1} + |\Delta y_n|_{\mathbb{H}}\right) |y_n|^2_{\mathbb{H}^1} |X_n|_{\mathbb{H}} |X_n|_{\mathbb{H}^1}.
\end{aligned} \tag{6.21}
$$

Taking into account (6.19), (6.20) and (6.21) we obtain from (6.18)

$$
\begin{aligned}
|\nabla X_n(t)|^2_{\mathbb{H}} + 2\alpha \int_0^t |\Delta X_n|^2_{\mathbb{H}}\, ds \leqslant C\eta^2 \int_0^t |\Delta X_n|^2_{\mathbb{H}}\, ds \\
&+ C_\eta \sup_{r\leqslant t}\left(1+|Y_n|^4_{\mathbb{H}^1}\right)\int_0^t |X_n|^2_{\mathbb{H}^1}\, ds \\
&+ C\eta^2 \int_0^t |\Delta X_n|^2_{\mathbb{H}}\, ds + C_\eta\left(\sup_{r\leqslant t}|X_n(r)|_{\mathbb{H}}\right)\left(\sup_{r\leqslant t}|X_n(r)|_{\mathbb{H}^1}\right) \\
&+ C\eta^2 \int_0^t |\Delta X_n|^2_{\mathbb{H}}\, ds + C_\eta\left(\sup_{r\leqslant t}|X_n(r)|_{\mathbb{H}}\right)\left(\sup_{r\leqslant t}|X_n(r)|_{\mathbb{H}^1}\right) \\
&+ C\eta^2 \int_0^t |\Delta X_n|^2_{\mathbb{H}}\, ds + C_\eta \sup_{r\leqslant t}|X_n(r)|^2_{\mathbb{H}} \\
&+ \sqrt{\epsilon_n}\left|\int_0^t \langle \nabla G(Y_n)\, h, \Delta X_n\rangle_{\mathbb{H}}\, dW\right| \\
&+ C\epsilon_n \int_0^t \left(1+|\Delta X_n|^2_{\mathbb{H}}\right) ds.
\end{aligned}
\tag{6.22}
$$

Choosing η in such a way that $4C\eta^2=\alpha$ we obtain

$$
\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]}|\nabla X_n(t\wedge\tau_n)|^2_{\mathbb{H}} + \alpha\int_0^{t\wedge\tau_n}|\Delta X_n|^2_{\mathbb{H}}\, ds\right) \leqslant C_\eta\left(1+N^4\right)\mathbb{E}\int_0^{\tau_n}|X_n|^2_{\mathbb{H}^1}\, ds \\
&+ C_\eta(1+N)\,\mathbb{E}\sup_{t\in[0,T]}|X_n(t\wedge\tau_n)|_{\mathbb{H}} \\
&+ \sqrt{\epsilon_n}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^{t\wedge\tau_n}\langle\nabla G(Y_n)\, h, \Delta X_n\rangle_{\mathbb{H}}\, dW\right| \\
&+ C\epsilon_n\mathbb{E}\int_0^T\left(1+|\Delta X_n|^2_{\mathbb{H}}\right) ds.
\end{aligned}
$$

By Theorem 5.2 there exists a finite constant C, depending on T, α, R, M_0 and h only, such that for each $n\geqslant 1$

$$
\mathbb{E}\int_0^T |\Delta Y_n(s)|^2_{\mathbb{H}}\, ds \leqslant C(T,\alpha,M,u_0,h),
$$

hence invoking the Burkholder-Davis-Gundy inequality we find that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|\nabla X_n(t\wedge\tau_n)|^2_{\mathbb{H}}+\alpha\int_0^{t\wedge\tau_n}|\Delta X_n|^2_{\mathbb{H}}\,ds\right)\leqslant C_\eta\left(1+N^4\right)\mathbb{E}\int_0^{\tau_n}|X_n|^2_{\mathbb{H}^1}\,ds$$
$$+C_\eta(1+N)\,\mathbb{E}\sup_{t\in[0,T]}|X_n(t\wedge\tau_n)|_{\mathbb{H}}$$
$$+C(1+N)\sqrt{\epsilon_n}.$$

Finally, Lemma 6.3 follows from Lemma 6.2. □

Lemma 6.4. *Let* $(\epsilon_n)\subset(0,1]$ *with* $\epsilon_n\to 0$ *and let* $(F_n)\subset\mathscr{P}_T$ *be such a sequence that*

$$\sup_{n\geqslant 1}\|F_n\|_T\leqslant R,\quad \text{for every } \omega\in\Omega,$$

and F_n *converges to* F *in law on* B_R. *Then the sequence of* $\mathscr{X}_T$*-valued random elements* $(J^{\varepsilon_n}_{F_n}(W)-J^0_{F_n})$ *converges in probability to* 0.

Proof. We will use the same notation as in the proof of Lemma (6.3). Let $\delta>0$ and $\nu>0$. Invoking part (2) of Theorem (3.1) we can find $N>|M_0|_{\mathbb{H}^1}$ such that

$$\frac{1}{N}\sup_{n\geqslant 1}\mathbb{E}\sup_{t\in[0,T]}|Y_n(t)|_{\mathbb{H}^1}<\frac{\nu}{2}.$$

Then invoking Lemma 6.3 we find that for all n sufficiently large

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]}|Y_n(t)-u_n(t)|^2_{\mathbb{H}^1}+\int_0^T|Y_n-u_n|^2_{D(A)}\,ds\geqslant\delta\right)\\
\leqslant\;&\mathbb{P}\left(\sup_{t\in[0,T]}|Y_n(t\wedge\tau_n)-u_n(t\wedge\tau_n)|^2_{\mathbb{H}^1}+\int_0^{\tau_n}|Y_n-u_n|^2_{D(A)}\,ds\geqslant\delta,\ \tau_n=T\right)\\
&+\mathbb{P}\left(\sup_{t\in[0,T]}|Y_n(t)|_{\mathbb{H}^1}\geqslant N\right)\\
\leqslant\;&\frac{1}{\delta}\mathbb{E}\left(\sup_{t\in[0,T]}|Y_n(t\wedge\tau_n)-u_n(t\wedge\tau_n)|^2_{\mathbb{H}^1}+\int_0^{\tau_n}|Y_n-u_n|^2_{D(A)}\,ds\right)\\
&+\frac{1}{N}\mathbb{E}\sup_{t\in[0,T]}|Y_n(t)|_{\mathbb{H}^1}\\
<\;&\nu.
\end{aligned}$$

□

Theorem 6.5. *The family of laws* $\{\mathscr{L}(J^\varepsilon_0(W)):\varepsilon\in(0,1]\}$ *on* $\mathscr{X}_T$ *satisfies the large deviation principle with rate function* I.

Proof. We interpret each process $J^{\varepsilon_n}_{F_n}(W)$ as the solution of equation (3.1) with ε_n and F_n in place of ε and F. In order to use Theorem 4.4 we note that any $\left(\mathscr{F}^0_t\right)$-predictable process

$u : [0,T] \times \Omega \to \mathbb{R}$ such that

$$\int_0^T u^2(t,\omega)\, dt \leqslant R^2, \quad \text{for all} \quad \omega \in \Omega$$

can be written as

$$u(t,\omega) = F(t, W(\omega)) \;\; \forall (t,\omega) \in [0,T] \times \Omega, \tag{6.23}$$

for some $(\mathscr{H}_t^0)$-predictable function $F : [0,T] \times C_0([0,T]) \to \mathbb{R}$ satisfying $\int_0^T F^2(s,z)\, ds \leqslant R^2$ $\forall z \in C_0([0,T])$. Statement 1 follows from Lemma 6.1.
Next, we will show that Statement 2 holds true. Let (ε_n) be a sequence from $(0,1]$ that converges to 0 and let $(F_n : [0,T] \times \Omega \to \mathbb{R})$ be a sequence of $\left(\mathscr{F}_t^0\right)$-predictable processes that converges in law on B_R, for some $R > 0$, to F. By Lemma 6.4 $J^{\varepsilon_n}_{F_n}(W) - J^0_{F_n}$ converges in probability (as a sequence of random variables in $\mathscr{X}_T$) to 0. Lemma 6.1 implies that $J^0_{F_n}$ converges in distribution on $\mathscr{X}_T$ to J^0_F. Indeed, since B_R is a separable metric space, the Skorohod Theorem (see, for example, [12, Theorem 4.30]) allows us to work with a sequence of processes that converges in B_R almost surely. These two convergence results imply that $\left(J^{\varepsilon_n}_{F_n}(W)\right)$ converges in law on $\mathscr{X}_T$ to J^0_F. Thus, for any uniformly continuous and bounded function $f : \mathscr{X}_T \to \mathbb{R}$ we have

$$\begin{aligned}
&\left| \int_\Omega f\left(J^{\varepsilon_n}_{F_n}(W)\right) d\mathbb{P} - \int_{\mathscr{X}_T} f(x)\, d\mathscr{L}(J^0_F)(x) \right| \\
\leqslant\; & \int_\Omega |f(J^{\varepsilon_n}_{F_n}(W)) - f(J^0_{F_n})|\, d\mathbb{P} \\
& + \left| \int_{\mathscr{X}_T} f(x)\, d\mathscr{L}(J^0_{F_n})(x) - \int_{\mathscr{X}_T} f(x)\, d\mathscr{L}(J^0_F)(x) \right| \\
\to\; & 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, Statement 2 is true as well and thus we conclude the proof of Theorem 6.5. $\square$

7. Application to a model of a ferromagnetic needle

In this section we will use the large deviation principle established in the previous section to investigate the dynamics of a stochastic Landau-Lifshitz model of magnetization in a needle-shaped particle. Here the shape anisotropy energy is crucial. When there is no applied field and no noise in the field, the shape anisotropy energy gives rise to two locally stable stationary states of opposite magnetization. We add a small noise term to the field and use the large deviation principle to show that noise induced magnetization reversal occurs and to quantify the effect of material parameters on sensitivity to noise.

The axis of the needle is represented by the interval Λ and at each $x \in \Lambda$ the magnetization $u(x) \in \mathbb{S}^2$ is assumed to be constant over the cross-section of the needle. We define the total magnetic energy of magnetization $u \in \mathbb{H}^1$ of the needle by

$$E_t(u) = \frac{1}{2}\int_\Lambda |\nabla u(x)|^2\,dx + \beta\int_\Lambda \Phi(u(x))\,dx - \int_\Lambda \mathscr{K}(t,x)\cdot u(x)\,dx, \tag{7.1}$$

where

$$\Phi(u) = \Phi(u_1,u_2,u_3) = \frac{1}{2}\left(u_2^2+u_3^2\right),$$

β is the positive real shape anisotropy parameter and $\mathscr{K}$ is the externally applied magnetic field, such that $\mathscr{K}(t) \in \mathbb{H}$ for each t.

With this magnetic energy, the deterministic Landau-Lifshitz equation becomes:

$$\frac{\partial y}{\partial t}(t) = y\times\Delta y - \alpha y\times(y\times\Delta y) + G(y)\left(-\beta f(y) + \mathscr{K}(t)\right) \tag{7.2}$$

where $f(y) = \nabla\Phi(y)$, $y \in \mathbb{R}^3$. We assume, as before, that the initial state $u_0 \in \mathbb{H}^1$ and $|u_0(x)|_{\mathbb{R}^3} = 1$ for all $x \in \Lambda$. We also assume that the applied field $\mathscr{K}(t) : \Lambda \to \mathbb{R}^3$ is constant on Λ at each time t. Equation (7.2) has nice features: the dynamics of the solution can be studied using elementary techniques and, when the externally applied field $\mathscr{K}$ is zero, the equation has two stable stationary states, $e_1 = (1,0,0)$ and $-e_1$.

We now outline the structure of this example. In Proposition 7.2, we show that if the applied field $\mathscr{K}$ is zero and the initial state u_0 satisfies

$$|u_0 \pm e_1|_{\mathbb{H}^1} < \frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha},$$

then the solution $y(t)$ of (7.2) converges to $\pm e_1$ in $\mathbb{H}^1$ as t goes to ∞. In Lemma 7.3, we show that if $\mathscr{H} = (\mathscr{H}_1,\mathscr{H}_2,\mathscr{H}_3)^{\mathrm{T}} \in \mathbb{R}^3$ has norm exceeding a certain value (depending on α and β) and the applied field is $\mathscr{K} = \mathscr{H} + \frac{\beta}{|\mathscr{H}|}f(\mathscr{H})$ and $|u_0 - \frac{\mathscr{H}}{|\mathscr{H}|}|_{\mathbb{H}^1} < \frac{1}{k}$, then $y(t)$ converges in $\mathbb{H}^1$ to $\frac{\mathscr{H}}{|\mathscr{H}|}$ as t goes to ∞. Lemma 7.3 is used to show that, given $\delta \in (0,\infty)$ and $T \in (0,\infty)$, there is a piecewise constant (in time) externally applied field, $\mathscr{K}$, which drives the magnetization from the initial state $-e_1$ to the $\mathbb{H}^1$-ball centred at e_1 and of radius δ by time T; in short, in the deterministic system, this applied field causes magnetization reversal by time T(see Definition 7.4). What we are really interested in is the effect of adding a small noise term to the field. We will show that if $\mathscr{K}$ is zero but a noise term multiplied by $\sqrt{\varepsilon}$ is added to the field, then the solution of the resulting stochastic equation exhibits magnetization reversal by time T with positive probability for all sufficiently small positive ε. This result, in Proposition 7.5, is obtained using the lower bound of the large deviation principle. Finally, in Proposition 7.7, the upper bound of the large deviation principle is used: we obtain an exponential estimate of the probability that, in time interval $[0,T]$, the stochastic magnetization leaves a given $\mathbb{H}^1$-ball centred at the initial state $-e_1$ and of radius less than or equal to $\frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}$. This estimate emphasizes the importance of a large value of β for reducing the disturbance in the magnetization caused by noise in the field.

7.1. **Stable stationary states of the deterministic equation.** In this subsection, we identify stable stationary states of the deterministic equation (7.2) when the applied field $\mathscr{K}$ does not vary with time.

Let $\zeta \in \mathbb{S}^2$. Since the time derivative $\frac{dy}{dt}$ of the solution y to (7.2), belongs to $L^2(0,T;\mathbb{H})$ and y belongs to $L^2(0,T;D(A))$, we have for all $t \geqslant 0$:

$$\begin{aligned} |y(t)-\zeta|^2_{\mathbb{H}} &= |u_0-\zeta|^2_{\mathbb{H}} \\ &+ 2\int_0^t \langle y-\zeta, y\times\Delta y - \alpha y\times(y\times\Delta y) \\ &\qquad\qquad + G(y)(-\beta f(y)+\mathscr{K}\rangle_{\mathbb{H}}\, ds \\ &= |u_0-\zeta|^2_{\mathbb{H}} + 2\int_0^t \langle -\zeta, y\times\Delta y - \alpha y\times(y\times\Delta y) \\ &\qquad\qquad + G(y)(-\beta f(y)+\mathscr{K}\rangle_{\mathbb{H}}\, ds \end{aligned} \tag{7.3}$$

$$\begin{aligned} |\nabla y(t)|^2_{\mathbb{H}} &= |\nabla u_0|^2_{\mathbb{H}} - 2\int_0^t \langle \Delta y, G(y)\,(-\beta f(y)+\mathscr{K}) \\ &\qquad\qquad - \alpha y\times(y\times\Delta y)\rangle_{\mathbb{H}}\, ds \end{aligned} \tag{7.4}$$

Lemma 7.1. *Let $u \in \mathbb{H}^1$ be such that $u(x) \in \mathbb{S}^2$ and*

$$|u \pm e_1|_{\mathbb{H}^1} \leqslant \frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}.$$

Then for all $x \in \Lambda$

(1) $\frac{1-u_1^2(x)}{u_1^2(x)} + \alpha\frac{(1-u_1^2(x))^2}{u_1^2(x)} - \alpha u_1^2(x) \leqslant 0$ *for all $x \in \Lambda$,*

(2) $\langle u(x), \pm e_1\rangle \geqslant \frac{3}{4}$ *and*

(3) $\frac{7}{8}|u(x)\pm e_1|^2 \leqslant |u(x)\times e_1|^2$.

Proof. By (2.2)

$$\begin{aligned} \sup_{x\in\Lambda}|u(x)\pm e_1|^2 &\leqslant k^2|u\pm e_1|_{\mathbb{H}}|u-\zeta|_{\mathbb{H}^1}, \\ &\leqslant k^2\, 2\sqrt{|\Lambda|}\,\frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha} = \frac{\alpha}{1+2\alpha}. \end{aligned} \tag{7.5}$$

Invoking (7.5), we find that

$$u_1^2(x) = 1-(u_2^2(x)+u_3^2(x)) \geqslant 1-|u(x)-\zeta|^2 \geqslant \frac{1+\alpha}{1+2\alpha}, \quad x\in\Lambda. \tag{7.6}$$

Hence one can use (7.6) and straightforward algebraic manipulations to verify that

$$\frac{1-u_1^2(x)}{u_1^2(x)} + \alpha\frac{(1-u_1^2(x))^2}{u_1^2(x)} - \alpha u_1^2(x) \leqslant 0.$$

Statements 2 and 3 of Lemma 7.1 follow easily from (7.5). □

Proposition 7.2. *Let the applied field $\mathscr{K}$ be zero and let $u_0 \in \mathbb{H}^1$ satisfy*

$$|u_0 \pm e_1|_{\mathbb{H}^1} < \frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}. \tag{7.7}$$

Let the process y be the solution to (7.2). Then $y(t)$ converges to $\pm e_1$ in $\mathbb{H}^1$ as $t\to\infty$.

Proof. Using some algebraic manipulation and the fact that $\langle \nabla y(s), y(s)\rangle = 0$ a.e. on Λ for each $s \geqslant 0$, one may simplify equations (7.4) and (7.4).
We obtain from (7.4):

$$|y(t) \pm e_1|^2_{\mathbb{H}} = |u_0 \pm e_1|^2_{\mathbb{H}} - 2\alpha \int_0^t \int_\Lambda |\nabla y(s)|^2 \langle y(s), \pm e_1\rangle \, dx\, ds - 2\alpha\beta \int_0^t \int_\Lambda \langle y(s), \pm e_1\rangle |y(s) \times e_1|^2 \, dx\, ds \quad \forall t \geqslant 0, \tag{7.8}$$

and

$$|\nabla y(t)|^2_{\mathbb{H}} = |\nabla u_0|^2_{\mathbb{H}} - 2\alpha \int_0^t |y(s) \times \Delta y(s)|^2_{\mathbb{H}} \, ds + 2\beta \int_0^t \int_\Lambda R(s)\, dx\, ds \quad \forall t \geqslant 0, \tag{7.9}$$

where

$$\begin{aligned} R &= \nabla y_1 (y_3 \nabla y_2 - y_2 \nabla y_3) + \alpha (\nabla y_1)^2 - \alpha y_1^2 |\nabla y|^2 \\ &= \frac{-y_2 \nabla y_2 - y_3 \nabla y_3}{y_1}(y_3 \nabla y_2 - y_2 \nabla y_3) \\ &\quad + \alpha(1 - y_1^2)\left(\frac{y_2 \nabla y_2 + y_3 \nabla y_3}{y_1}\right)^2 - \alpha y_1^2((\nabla y_2)^2 + (\nabla y_3)^2. \end{aligned} \tag{7.10}$$

Define

$$\tau = \inf\left\{ t \geqslant 0 : \, |y(t) \pm e_1|_{\mathbb{H}^1} \geqslant \frac{1}{2k^2\sqrt{|\Lambda|}} \frac{\alpha}{1+2\alpha} \right\}.$$

Then, by our choice of u_0, $\tau > 0$. For each $s \in [0, \tau)$, $y(s)$ satisfies the hypotheses of Lemma 7.1, hence and

$$y(s)(x) \cdot (\pm e_1) \geqslant \frac{3}{4}, \quad x \in \Lambda,$$

$$|y(s)(x) \times (\pm e_1)\,|^2 \geqslant \frac{7}{8} |y(s)(x) \pm e_1|^2, \quad x \in \Lambda.$$

and, invoking the Cauchy-Schwartz inequality

$$R \leqslant \left(\frac{1 - y_1^2}{y_1^2} + \alpha \frac{(1 - y_1^2)^2}{y_1^2} - \alpha y_1^2 \right) ((\nabla y_2)^2 + (\nabla y_3)^2) \leqslant 0, \quad x \in \Lambda. \tag{7.11}$$

Consequently, from (7.8) and (7.9) we deduce that the functions $|y(\cdot) \pm e_1|^2_{\mathbb{H}}$ and $|\nabla y(\cdot)|^2_{\mathbb{H}}$ are nonincreasing on $[0, \tau)$. Furthermore, we have

$$|y(t) \pm e_1|^2_{\mathbb{H}} \leqslant |u_0 \pm e_1|^2_{\mathbb{H}} - \frac{3}{2}\alpha \int_0^t |\nabla y(s)|^2_{\mathbb{H}} \, ds - \frac{21}{16} \alpha\beta \int_0^t |y(s) \pm e_1|^2_{\mathbb{H}} \, ds, \quad t < \tau, \tag{7.12}$$

and

$$|\nabla y(t)|^2_{\mathbb{H}} \leqslant |\nabla u_0|^2_{\mathbb{H}}, \quad t < \tau. \tag{7.13}$$

Suppose, to get a contradiction, that $\tau < \infty$. Then, from (7.12) and (7.13), we have

$$|y(\tau) \pm e_1|_{\mathbb{H}^1} \leqslant |u_0 \pm e_1|_{\mathbb{H}^1} < \frac{1}{2k^2\sqrt{l(\Lambda)}} \frac{\alpha}{1+2\alpha},$$

which contradicts the definition of τ. Therefore, $\tau = \infty$. Since (7.12) holds for all $t \geqslant 0$, we have

$$\int_0^\infty |\nabla y(s)|^2_{\mathbb{H}}\, ds + \int_0^\infty |y(s) \pm e_1|^2_{\mathbb{H}}\, ds < \infty.$$

Since both integrands are nonincreasing

$$\lim_{t\to\infty} \left(|\nabla y(t)|_{\mathbb{H}} + |y(t) \pm e_1|_{\mathbb{H}}\right) = 0.$$

□

We will show next, that if the applied field has sufficiently large magnitude, then there exists a stable stationary state that is roughly in the direction of the applied field.

Lemma 7.3. *Assume that $\mathscr{H} \in \mathbb{S}^2$ and a real number λ satisfies*

$$\lambda > \left(\frac{4\beta + 4\alpha\beta}{3\alpha} \vee \frac{2\beta + 4\alpha\beta - \alpha}{\alpha}\right). \tag{7.14}$$

Let the applied field be[2]

$$\mathscr{K} := \lambda\mathscr{H} + \beta f(\mathscr{H}).$$

Let y be a solution to the problem (7.2) with initial data u_0 satisfying $|u_0 - \mathscr{H}|_{\mathbb{H}^1} < \frac{1}{k}$. Then

$$|y(t) - \mathscr{H}|_{\mathbb{H}^1} \leqslant |u_0 - \mathscr{H}|_{\mathbb{H}^1}\, \mathrm{e}^{-\frac{1}{2}\gamma t} \quad \forall t \geqslant 0, \tag{7.15}$$

where

$$\gamma := (\alpha\lambda + \alpha - 2\beta - 4\alpha\beta) \wedge \left(\frac{3}{2}\alpha\lambda - 2\beta - 2\alpha\beta\right) > 0$$

is positive, by condition (7.14).

Proof. We have, from (7.3) and (7.4) with ζ replaced by $\mathscr{H}$:

$$\begin{aligned} &|y(t) - \mathscr{H}|^2_{\mathbb{H}} \qquad (7.16)\\ =\ & |u_0 - \mathscr{H}|^2_{\mathbb{H}} - 2\beta\int_0^t \langle y - \mathscr{H}, y \times f(y - \mathscr{H})\rangle_{\mathbb{H}}\, ds \\ & - 2\alpha \int_0^t \int_\Lambda |\nabla y|^2 (y \cdot \mathscr{H})\, dx\, ds \\ & - 2\alpha \int_0^t |y \times \mathscr{H}|^2_{\mathbb{H}}\, ds \\ & + 2\alpha\beta \int_0^t \langle y \times \mathscr{H}, y \times f(y - \mathscr{H})\rangle_{\mathbb{H}}\, ds \quad \forall t \geqslant 0. \end{aligned}$$

[2]Note that a constant function $\mathscr{H}$ is a stationary solution to the problem (7.2).

From (7.4) we have:

$$\begin{aligned}|\nabla y(t)|^2_{\mathbb{H}} &= |\nabla u_0|^2_{\mathbb{H}} + 2\beta \int_0^t \langle \Delta y, y \times f(y - \mathscr{H})\rangle_{\mathbb{H}}\, ds \\ &\quad - 2\alpha \int_0^t |y \times \Delta y|^2_{\mathbb{H}}\, ds \\ &\quad - 2\alpha \int_0^t \int_\Lambda |\nabla y|^2 (y \cdot \mathscr{H})\, dx\, ds \\ &\quad - 2\alpha\beta \int_0^t \langle \Delta y, y \times (y \times f(y - \mathscr{H}))\rangle_{\mathbb{H}}\, ds \quad \forall t \geqslant 0.\end{aligned}$$

Define

$$\tau_1 := \inf\{t \geqslant 0 : |y(t) - \mathscr{H}|_{\mathbb{H}^1} \geqslant \tfrac{1}{k}\}. \tag{7.17}$$

By our choice of u_0, τ_1 is greater than zero. Observe that

$$\sup_{x \in \Lambda} |y(t)(x) - \mathscr{H}|_{\mathbb{R}^3} < 1 \quad \text{for all } t < \tau_1. \tag{7.18}$$

It is easy to check that for every $t < \tau_1$

$$\frac{3}{4}|y(t) - \mathscr{H}|^2_{\mathbb{H}} \leqslant |y(t) \times \mathscr{H}|^2_{\mathbb{H}} \leqslant |y(t) - \mathscr{H}|^2_{\mathbb{H}}, \tag{7.19}$$

and

$$y(t,x) \cdot \mathscr{H} \geqslant \tfrac{1}{2}, \quad x \in \Lambda. \tag{7.20}$$

Adding equalities (7.16) and (7.17) we obtain for $t > 0$

$$
\begin{aligned}
&|y(t) - \mathscr{H}|^2_{\mathbb{H}^1} \\
&= |u_0 - \mathscr{H}|^2_{\mathbb{H}^1} - 4\alpha \int_0^t \int_\Lambda |\nabla y|^2 \,(y \cdot \mathscr{H})\; dx\, ds \\
&\quad + 2\beta \int_0^t \langle \Delta y, y \times f(y - \mathscr{H})\rangle_{\mathbb{H}}\, ds \\
&\quad - 2\alpha\beta \int_0^t \langle \Delta y, y \times (y \times f(y - \mathscr{H}))\rangle_{\mathbb{H}}\, ds \\
&\quad - 2\alpha \int_0^t |y \times \mathscr{H}|^2_{\mathbb{H}}\, ds \\
&\quad - 2\beta \int_0^t \langle y - \mathscr{H}, y \times f(y - \mathscr{H})\rangle_{\mathbb{H}}\, ds \\
&\quad + 2\alpha\beta \int_0^t \langle y \times \mathscr{H}, y \times f(y - \mathscr{H})\rangle_{\mathbb{H}}\, ds \\
&\quad - 2\alpha \int_0^t |y \times \Delta y|^2_{\mathbb{H}}\, ds\,.
\end{aligned}
\tag{7.21}
$$

Therefore for every $t < \tau_1$

$$
\begin{aligned}
|y(t) - \mathscr{H}|^2_{\mathbb{H}^1} \;&\leqslant\; |u_0 - \mathscr{H}|^2_{\mathbb{H}^1} - (2\alpha - 2\beta - 4\alpha\beta) \int_0^t |\nabla y|^2_{\mathbb{H}}\, ds \\
&\quad - (\tfrac{3}{2}\alpha - 2\beta - 2\alpha\beta) \int_0^t |y - \mathscr{H}|^2_{\mathbb{H}}\, ds \\
&\quad - 2\alpha \int_0^t |y \times \Delta y|^2_{\mathbb{H}}\, ds\,,
\end{aligned}
\tag{7.22}
$$

where we used (7.18), (7.19) and (7.20). Because of hypothesis (7.14), the two expressions $(2\alpha - 2\beta - 4\alpha\beta)$ and $(\frac{3}{2}\alpha - 2\beta - 2\alpha\beta)$ on the right hand side of (7.22) are positive numbers. Suppose, to get a contradiction, that $\tau_1 < \infty$. Then, from (7.22), we have

$$
|y(\tau_1) - \mathscr{H}|_{\mathbb{H}^1} \leqslant |u_0 - \mathscr{H}|_{\mathbb{H}^1} < \tfrac{1}{k},
$$

which contradicts the definition of τ_1 in (7.17). Hence $\tau_1 = \infty$. It now follows from (7.22) that

$$\int_0^\infty |\nabla y(s)|^2_{\mathbb{H}}\, ds \quad < \quad \infty, \tag{7.23}$$

$$\int_0^\infty |y(s) - \mathscr{H}|^2_{\mathbb{H}}\, ds \quad < \quad \infty \tag{7.24}$$

$$\text{and} \quad \int_0^\infty |y(s) \times \Delta y(s)|^2_{\mathbb{H}}\, ds \quad < \quad \infty. \tag{7.25}$$

From (7.21) and these three inequalities, the function $t \in [0,\infty) \mapsto |y(t) - \mathscr{H}|^2_{\mathbb{H}^1}$ is absolutely continuous and, for almost every $t \geqslant 0$, its derivative is:

$$\begin{aligned}
\frac{d}{dt}|y - \mathscr{H}|^2_{\mathbb{H}^1}(t) \quad &= \quad -4\alpha \int_\Lambda |\nabla y(t)|^2 (y(t)\cdot \mathscr{H})\, dx \\
&\quad + 2\beta\langle \Delta y(t), y(t) \times f(y - \mathscr{H})\rangle_{\mathbb{H}} \\
&\quad - 2\alpha\beta\langle \Delta y(t), y(t) \times (y(t) \times f(y - \mathscr{H}))\rangle_{\mathbb{H}} \\
&\quad - 2\alpha |y(t) \times \mathscr{H}|^2_{\mathbb{H}} \\
&\quad - 2\beta\big\langle y(t) - \mathscr{H}, y(t) \times f(y - \mathscr{H})\big\rangle_{\mathbb{H}} \\
&\quad + 2\alpha\beta\langle y(t) \times \mathscr{H}, y(t) \times f(y - \mathscr{H})\rangle_{\mathbb{H}} \\
&\quad - 2\alpha |y(t) \times \Delta y(t)|^2_{\mathbb{H}} \\
\quad &\leqslant \quad -(2\alpha - 2\beta - 4\alpha\beta)|\nabla y(t)|^2_{\mathbb{H}} \\
&\quad - (\tfrac{3}{2}\alpha - 2\beta - 2\alpha\beta)|y(t) - \mathscr{H}|^2_{\mathbb{H}} \\
&\quad - 2\alpha |y(t) \times \Delta y(t)|^2_{\mathbb{H}} \\
\quad &\leqslant \quad -\gamma |y(t) - \mathscr{H}|^2_{\mathbb{H}^1},
\end{aligned} \tag{7.26}$$

where

$$\gamma := (\alpha + \alpha\lambda - 2\beta - 4\alpha\beta) \wedge \left(\frac{3}{2}\alpha\lambda - 2\beta - 2\alpha\beta\right) > 0.$$

Now the lemma follows by a standard argument. □

7.2. **Noise induced instability and magnetization reversal.** In Proposition 7.2 we showed that the states e_1 and $-e_1$ are stable stationary states of the deterministic Landau-Lifshitz equation (7.2) when the externally applied field $\mathscr{K}$ is zero. In this section we show that a small noise term in the field may drive the magnetization from the initial state $-e_1$ to any given $\mathbb{H}^1$-ball centred at e_1 in any given time interval $[0,T]$. We also find an exponential upper bound for the probability that small noise in the field drives the magnetization outside a given $\mathbb{H}^1$-ball centred at the initial state $-e_1$ in time interval $[0,T]$. Firstly we need a definition.

Definition 7.4. *Let δ be a given small positive real number. Suppose that the initial magnetization is $-e_1$ and that at some time T the magnetization lies in the open $\mathbb{H}^1$-ball centred at e_1 and of radius δ. Then we say that* magnetization reversal *has occurred by time T.*

We consider a stochastic equation for the magnetization, obtained by setting $\mathscr{K}$ to zero and adding a three dimensional noise term to the field. Denoting the magnetization by Y, the equation is:

$$\left.\begin{aligned} dY &= (Y\times\Delta Y-\alpha Y\times(Y\times\Delta Y)+\beta G(Y)f(Y))\;dt \\ &\quad+\sqrt{\varepsilon}G(Y)B\circ dW(t) \\ Y(0) &= -e_1. \end{aligned}\right\} \tag{7.27}$$

In (7.27), we assume that the vectors $h^1,h^2,h^3\in\mathbb{R}^3$ are linearly independent. The parameter $\varepsilon>0$ corresponds to the 'dimensionless temperature' parameter appearing in the stochastic differential equation (**??**) of Kohn, Reznikoff and Vanden-Eijnden []

Fix $T>0$. There is no deterministic applied field in (7.27) but, as we will see, the lower bound of the large deviation principle satisfied by the solutions Y^ε ($\varepsilon\in(0,1)$) of (7.27) implies that, for all sufficiently small positive ε, the probability of magnetization reversal by time T is positive.

Firstly, we shall use Lemma 7.3 to construct a piecewise constant (in time) deterministic applied field, $\mathscr{K}$, such that the solution y of (7.2), with initial state $(-1,0,0)^{\mathrm{T}}$, undergoes magnetization reversal by time T.
Take points $u^i\in\mathbb{S}^2$, $i=0,1,\dots,N$, such that $u^0=-e_1$ and $u^N=e_1$ and

$$|u^i-u^{i+1}|_{\mathbb{H}^1}=|u^i-u^{i+1}|_{\mathbb{R}^3}\sqrt{|\Lambda|}<\frac{1}{k}\quad\text{for } i=0,1,\dots,N-1.$$

Let

$$\eta:=\min\left\{\frac{1}{k}-|u^i-u^{i+1}|_{\mathbb{H}^1}:i=1,\dots,N-1\right\}\wedge\frac{\delta}{2}.$$

Using Lemma 7.3, we can take the applied field to be

$$\mathscr{K}(t):=\sum_{i=0}^{N-1}1_{(i\frac{T}{N},(i+1)\frac{T}{N}]}(t)\left(Ru^{i+1}+\beta f(u^{i+1})\right),\quad t\geqslant 0, \tag{7.28}$$

with the positive real number R chosen to ensure that, as t varies from $i\frac{T}{N}$ to $(i+1)\frac{T}{N}$, $y(t)$ starts at a distance of less than η from u^i (i.e. $|y(i\frac{T}{N})-u^i|_{\mathbb{H}^1}<\eta$) and moves to a distance of less than η from u^{i+1} (i.e. $|y((i+1)\frac{T}{N})-u^{i+1}|_{\mathbb{H}^1}<\eta$). Specifically, we take $R\in(0,\infty)$ such that

$$\frac{1}{k}\mathrm{e}^{-\frac{1}{2}[(\alpha R+\alpha-2\beta-4\alpha\beta)\wedge(\frac{3}{2}\alpha R-2\beta-2\alpha\beta)]\frac{T}{N}}<\eta.$$

For each $i=0,1,\dots,N-1$, let $\phi^{i+1}=(\phi_1^{i+1},\phi_2^{i+1},\phi_3^{i+1})^{\mathrm{T}}\in\mathbb{R}^3$ be the vector of scalar coefficients satisfying the equality

$$\phi_1^{i+1}a^1+\phi_2^{i+1}a^2+\phi_3^{i+1}a^3=Ru^{i+1}+\beta f(u^{i+1}),$$

and define

$$\phi(t):=\sum_{i=0}^{N-1}1_{(i\frac{T}{N},(i+1)\frac{T}{N}]}(t)\;\phi^{i+1},\quad t\in[0,T]. \tag{7.29}$$

We remark that the function ϕ depends on the chosen values of δ and T, the material parameters Λ, α and β and the noise parameters a^1, a^2 and a^3.

Recall that Y^ε denotes the solution of (7.27). By an argument very much like that leading to Theorem 6.5, the family of laws $\{\mathscr{L}(Y^\varepsilon) : \varepsilon \in (0,1)\}$ on $\mathscr{X}_T$ satisfies a large deviation principle. In order to define the rate function, we introduce an equation

$$\begin{aligned} y_\psi(t) &= -e_1 + \int_0^t y_\psi \times \Delta y_\psi \, ds - \alpha \int_0^t y_\psi \times (y_\psi \times \Delta y_\psi)\, ds \\ &\quad - \beta \int_0^t G(y_\psi) f(y_\psi)\, ds + \int_0^t G(y_\psi) B\psi \, ds. \end{aligned} \tag{7.30}$$

By Corollary 5.5 this equation has unique solution $y_\psi \in \mathscr{X}_T$ for every $\psi \in L^2(0,T;\mathbb{R}^3)$. The rate function $I : \mathscr{X}_T \to [0,\infty]$, is defined by:

$$I_T(v) := \inf \left\{ \frac{1}{2} \int_0^T |\psi(s)|^2 \, ds : \psi \in L^2(0,T;\mathbb{R}^3) \text{ and } v = y_\psi \right\}, \tag{7.31}$$

where the infimum of the empty set is taken to be ∞.
Let y be the solution of equation (7.2) with $u_0 = -e_1$ and $\mathscr{K}$ as defined in (7.28). Using the notation in (7.30), we have $y = y_\phi$, for ϕ defined in (7.29). Therefore

$$I_T(y) \leqslant \frac{1}{2} \int_0^T |\phi(s)|^2 \, ds < \infty.$$

Since y undergoes magnetization reversal by time T, paths of Y^ϵ which lie close to y also undergo magnetization reversal by time T. In particular, by the Freidlin-Wentsell formulation of the lower bound of the large deviation principle (see, for example, [9, Proposition 12.2]), given $\xi > 0$, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in[0,T]} |Y^\epsilon(t) - y(t)|_{\mathbb{H}^1} + \left(\int_0^T |Y^\varepsilon(s) - y(s)|^2_{D(A)}\, ds \right)^{\frac{1}{2}} < \frac{\delta}{2} \right) \\ &\geqslant \exp\left(\frac{-I_T(y) - \xi}{\varepsilon} \right) \\ &\geqslant \exp\left(\frac{-\frac{1}{2}\int_0^T |\phi(s)|^2\, ds - \xi}{\varepsilon} \right). \end{aligned} \tag{7.32}$$

Since we have $|y(T) - e_1|_{\mathbb{H}^1} < \frac{\delta}{2}$, the right hand side of (7.32) provides a lower bound for the probability that Y^ε undergoes magnetization reversal by time T. We summarize our conclusions in the following proposition.

Proposition 7.5. *For all sufficiently small $\varepsilon > 0$, the probability that the solution Y^ε of (7.27) undergoes magnetization reversal by time T is bounded below by the expression on the right hand side of (7.32); in particular, it is positive.*

We shall now use the upper bound of the large deviation principle satisfied by $\{\mathscr{L}(Y^\varepsilon) : \varepsilon \in (0,1)\}$ to find an exponential upper bound for the probability that small noise in the field drives the magnetization outside a given $\mathbb{H}^1$-ball centred at the initial state $-e_1$ in time interval $[0,T]$. This is done in Proposition 7.7 below; the proof of the proposition uses Lemma 7.6. In Lemma 7.6 and Proposition 7.7, for ψ an arbitrary element of $L^2(0,T;\mathbb{R}^3)$, y_ψ denotes the function in $\mathscr{X}_T$ which satisfies equality (7.30) and τ_ψ is defined by

$$\tau_\psi := \inf\left\{t\in[0,T] : |y_\psi(t)+e_1|_{\mathbb{H}^1} \geqslant \frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}\right\}.$$

Lemma 7.6. *For each $\psi\in L^2(0,T;\mathbb{R}^3)$, we have $|\nabla y_\psi(t)|_{\mathbb{H}}=0$ for all $t\in[0,\tau_\psi\wedge T)$.*

Proof. Let $\psi\in L^2(0,T;\mathbb{R}^3)$. To simplify notation in this proof, we write y instead of y_ψ. Proceeding as in the derivation of (7.9), we obtain

$$\begin{aligned}|\nabla y(t)|^2_{\mathbb{H}} &= -2\alpha\int_0^t |y\times\Delta y|^2_{\mathbb{H}}\,ds + 2\beta\int_0^t\int_\Lambda R\,dx\,ds\\ &\quad - 2\alpha\sum_{i=1}^3\int_0^t \langle\nabla y, y\times(\nabla y\times a^i)\rangle_{\mathbb{H}}\psi_i\,ds,\quad t\in[0,T],\end{aligned}\tag{7.33}$$

where $R(s)$ defined in (7.10) satisfies inequality (7.11). For each $s\in[0,\tau_\psi\wedge T)$, $y(s)$ satisfies the hypotheses of Lemma 7.1, thus we have $R(s)(x)\leqslant 0$ for all $x\in\Lambda$. It follows from (7.33) that for all $t\in[0,\tau_\psi\wedge T)$:

$$|\nabla y(t)|^2_{\mathbb{H}} \leqslant 2\alpha\int_0^t |\nabla y|^2_{\mathbb{H}}\sum_{i=1}^3 |a^i|\cdot|\psi_i|\,ds.\tag{7.34}$$

By the Gronwall lemma applied to (7.34), $|\nabla y(t)|^2_{\mathbb{H}}=0$ for all $t\in[0,\tau_\psi\wedge T)$. ☐

Proposition 7.7. *Let*

$$0<r<\rho\leqslant\frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}.$$

The for any $\xi>0$, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$:

$$\mathbb{P}\left(\sup_{t\in[0,T]}|Y^\varepsilon(t)+e_1|_{\mathbb{H}^1}\geqslant\rho\right)\leqslant\exp\left(\frac{-\kappa r^2+\xi}{\varepsilon}\right),\tag{7.35}$$

where

$$\kappa = \frac{\alpha\beta}{8\max_{1\leqslant i\leqslant 3}|a^i|^2\,|\Lambda|(1+\alpha^2)}.$$

Proof. We shall use the Freidlin-Wentsell formulation of the upper bound of the large deviation principle (see, for example, [9, Proposition 12.2]) satisfied by $\{\mathscr{L}(Y^\varepsilon):\varepsilon\in(0,1)\}$. Recall that $\mathscr{I}$, defined in (7.31), is the rate function of the large deviation principle. Our main task is to show that

$$\left\{v\in\mathscr{X}_T : I_T(v)\leqslant\kappa r^2\right\}\subset\left\{v\in C([0,T];\mathbb{H}^1) : \sup_{t\in[0,T]}|v(t)+e_1|_{\mathbb{H}^1}\leqslant r\right\}.$$

Take $\psi \in L^2(0,T;\mathbb{R}^3)$ such that

$$\frac{1}{2}\int_0^T |\psi(s)|^2\,ds \leqslant \kappa r^2. \tag{7.36}$$

For simplicity of notation, in this proof we write y in place of y_ψ. By Lemma 7.6 we have for all $t \in [0,T]$,

$$\begin{aligned}
|y(t\wedge\tau_\psi)+e_1|^2_{\mathbb{H}^1} &= 2\alpha \int_0^{t\wedge\tau_\psi}\int_\Lambda |\nabla y|^2\,(y\cdot e_1)\;dx\,ds \\
&\quad + 2\alpha\beta \int_0^{t\wedge\tau_\psi}\int_\Lambda (y\cdot e_1)\,|y\times e_1|^2\,dx\,ds \\
&\quad - 2\alpha\beta \sum_{i=1}^3 \int_0^{t\wedge\tau_\psi} \left\langle \frac{1}{2}(y\times e_1), \frac{2}{\alpha\beta}a^i \right\rangle_{\mathbb{H}} \psi_i\,ds \\
&\quad + 2\alpha\beta \sum_{i=1}^3 \int_0^{t\wedge\tau_\psi} \left\langle \frac{1}{2}(y\times e_1), \frac{2}{\beta}(y\times a^i) \right\rangle_{\mathbb{H}} \psi_i\,ds \\
&\leqslant -\frac{3}{2}\alpha\beta \int_0^{t\wedge\tau_\psi} |y\times e_1|^2_{\mathbb{H}}\,ds + \frac{3}{2}\alpha\beta \int_0^{t\wedge\tau_\psi} |y\times e_1|^2_{\mathbb{H}}\,ds \\
&\quad + \frac{4}{\beta}\left(\frac{1}{\alpha}+\alpha\right)|\Lambda| \sum_{i=1}^3 |a^i|^2 \int_0^{t\wedge\tau_\psi} \psi_i^2\,ds,
\end{aligned} \tag{7.37}$$

where we estimated the integrals on the right hand side of the second equality as follows: the first integral vanished thanks to Lemma 7.6, Lemma 7.1 was used for the integrand of the second integral and the Cauchy-Schwarz inequality and Young's inequality were used for the integrands of the other integrals. Using (7.36) in (7.37), we obtain

$$|y(t\wedge\tau_\psi)+e_1|_{\mathbb{H}^1} \leqslant r < \frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha} \quad \forall t\in[0,T]. \tag{7.38}$$

From (7.38) and the definition of τ_ψ, we conclude that $\tau_\psi > T$. Hence we have

$$\sup_{t\in[0,T]} |y(t)+e_1|_{\mathbb{H}^1} \leqslant r.$$

By the Freidlin-Wentsell formulation of the upper bound of the large deviation principle, since $r<\rho$, given $\xi\in(0,\infty)$, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, inequality (7.35) holds. □

Remark 7.8. Our use of Lemma 7.6 in the proof of Proposition 7.7 means that, in this proposition, we did not need to allow for the spatial variation of magnetization on Λ.

Appendix A. Uniqueness of strong solutions

The aim of this paper is to extend the celebrated result of Pardoux [17] and Krylov-Rosovski [15] to stochastic parabolic equations driven by Poisson type noise. To put our results into right framework let us recall the result of Pardoux and Krylov-Rosovski. We use the formulation of the former author. Suppose

$$V \subset H = H' \subset V'$$

is a Gelfand triple of Hilbert spaces. We will study the following equation

$$\begin{aligned} &du(t) + A(t)u(t)\,dt - C(t)u(t)dW(t) = f(t)\,dt + g(t)\,dW(t), \quad t \geqslant 0,\\ &u(0) = u_0. \end{aligned} \tag{A.1}$$

We suppose that K is a real separable Hilbert space, $A(t)$, $C(t)$, $t \in [0,T]$ are two families of linear operators satisfying the following assumptions

$$A \in L^\infty(0,T;\mathscr{L}(V,V')), \tag{A.2}$$

$$B \in L^\infty(0,T;R(K,H)), \tag{A.3}$$

where $R(H,H)$ is the space of all γ-radonifying, i.e. Hilbert-Schmidt, operators form K to H, and the following coercivity assumption. There exists $\nu > 0$ and $\lambda \in \mathbb{R}$ such that for a.a. $t \in [0,T]$,

$$\langle A(t)u, u\rangle + \lambda |u|^2 \geqslant \nu \|u\|^2 + \frac{1}{2}\,\|C(t)u\|^2_{R(K,H)}, \quad u \in V. \tag{A.4}$$

In the above, $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_{R(K,H)}$ denote respectively the norm in V, H and $R(K,H)$. By $\langle\cdot,\cdot\rangle$ we denote the duality between V' and V, while the inner products in V and H will be denoted respectively by $(\cdot,\cdot)_V$ and respectively $(\cdot,\cdot)_H$.

Moreover, $W(t)$, $t \geqslant 0$, is a canonical K-cylindrical Wiener process defined on some fixed complete filtered probability space. Moreover, $f(t)$, $t \in [0,T]$ and $g(t)$, $t \in [0,T]$ are progressively measurable V' and resp. $R(K,H)$-valued processes such that

$$\mathbb{E}\int_0^T |f(t)|^2_{V'}\,dt < \infty, \tag{A.5}$$

$$\mathbb{E}\int_0^T \|g(t)\|^2_{R(K,H)}\,dt < \infty. \tag{A.6}$$

Suppose finally, that u_0 belongs to $L^2(\Omega,\mathscr{F}_0;H)$. Under the above assumptions Pardoux [?], see Theorem 1.3,

Theorem A.1. *There exists a unique progressively measurable process $u(t)$ such that u is a solution to problem (A.1) and moreover*

$$\mathbb{E}\int_0^T \|u(t)\|^2\,dt < \infty \tag{A.7}$$

$$u \in L^2(\Omega; C(0,T;H)) \tag{A.8}$$

$$|u(t)|^2 + 2\int_0^t \langle A(s)u(s), u(s)\rangle\,ds = |u_0|^2 + 2\int_0^t (g(s) + C(s)u(s), u(s))dW(s) \tag{A.9}$$

$$+ \int_0^t (u(s), f(s)\,ds + \int_0^t \|C(s)u(s) + g(s)\|^2_{R(K,H)}\,ds, \quad a.s..$$

Our aim is this paper is to generalise this result in the following sense.

Suppose that τ is a accessible stopping time and let τ_u be a certain increasing sequence of stopping times $\mathbb{P}$-a.s. convergent to τ. Assume that $f(t)$, $t \in [0,\tau)$ and $g(t)$, $t \in [0,\tau)$ are progressively measurable V' and resp. $R(K,H)$-valued processes such that for each $n \in \mathbb{N}$,

$$\mathbb{E}\int_0^{\tau_n} |f(t)|^2_{V'}\,dt < \infty, \tag{A.10}$$

$$\mathbb{E}\int_0^{\tau_n} \|g(t)\|^2_{R(K,H)}\,dt < \infty. \tag{A.11}$$

First we shall generalise [17, Theorem 1.2].

Theorem A.2. *In addition to the above assumption let us assume that $u(t)$, $t \in [0,\tau)$ is a progressively measurable V-valued process such that for each $n \in \mathbb{N}$,*

$$\mathbb{E}\int_0^{\tau_n} |u(t)|^2_V\,dt < \infty, \tag{A.12}$$

$$\mathbb{E}\sup_{t\in[0,\tau_n]} |u(t)|^2_H\,dt < \infty. \tag{A.13}$$

Suppose also that $\psi : H \to \mathbb{R}$ is a twice Fréchet differentiable function such that

(i) *ψ, ψ' and ψ'' are bounded on balls,*

(iii) *for each operator $Q \in \mathscr{T}_1(H)$, the function $H \ni x \mapsto \operatorname{tr}_H(Q \circ \psi''(x)) \in \mathbb{R}$ is continuous,*

(iv) *for each $x \in V$, the restriction of $d_x\psi = \psi'(x)$ to the space V is continuous and, if $\nabla_V\psi(x)$ denotes the unique element in V such that*

$$(d_x\psi)(y) = (\nabla_V\psi(x), y)_V,\ y \in V,$$

then the map $V \ni x \mapsto \nabla_V\psi(x) \in V$ is (V,s)-(V,w) continuous, where (V,s), resp. (V,w) denotes the space V endowed with the strong, resp. weak, topology;

(v) *the map $V \ni x \mapsto \nabla_V\psi \in V$ is of linear growth, i.e. there exists $k > 0$ such that $\|\nabla_V\psi(x)\| \leqslant k(1 + \|x\|)$, $x \in V$.*

Suppose that that u is a local solution of the problem (A.1)*, i.e. for each $n\in\mathbb{N}$, for all $t\geqslant 0$, $\mathbb{P}$-a.s.,*

$$u(t\wedge\tau_n)) = u(0) + \int_0^{t\wedge\tau_n} \big[C(s)u(s)+g(s)\big]\,dW(s) + \int_0^{t\wedge\tau_n} \big[-Au(s)+f(s)\big]\,ds. \tag{A.14}$$

Then, for each $n\in\mathbb{N}$, for all $t\geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\psi(u(t\wedge\tau_n)) &= \psi(u(0)) + 2\int_0^{t\wedge\tau_n} (\nabla_H\psi(u(s)),\big[C(s)u(s)+g(s)\big]dW(s))_H \qquad \text{(A.15)}\\ &+ \int_0^{t\wedge\tau_n} \langle -A(s)u(s)+f(s),\nabla_V(u(s))\rangle\,ds + \int_0^{t\wedge\tau_n} \operatorname{tr}_{g(s)}\psi''(u(s))\,ds.\end{aligned}$$

Remark A.3. $\nabla_H\psi(x)$ denotes the unique element in H such that

$$(d_x\psi)(y) = (\nabla_H\psi(x),y)_H,\ y\in H$$

$$\psi'': H\to\mathscr{L}(H,\mathscr{L}(H,\mathbb{R}))\cong\mathscr{L}(H,H;\mathbb{R})=\mathbb{L}(H,H;\mathbb{R}).$$

Proof. The proof of the above result is a combination of the proof of [**?**, Theorems 1.2 and 1.3] and the approximation argument from □

In particular, with function $\psi(x)=|x|^2$, $x\in H$, we have the following result.

Corollary A.4. *Suppose that τ is a accessible stopping time and let τ_u be a certain increasing sequence of stopping times $\mathbb{P}$-a.s. convergent to τ. Assume that $f(t)$, $t\in[0,\tau)$ and $g(t)$, $t\in[0,\tau)$ are progressively measurable V' and resp. $R(K,H)$-valued processes such that for each $n\in\mathbb{N}$, they satisfy the conditions* (A.10) *and* (A.11)*. Suppose that $u(t)$, $t\in[0,\tau)$ is a progressively measurable V-valued process such that for each $n\in\mathbb{N}$, it satisfies the conditions* (A.12) *and* (A.13)*. Suppose finally that u is a local solution of the problem* (A.1)*. Then, for each $n\in\mathbb{N}$, for all $t\geqslant 0$, $\mathbb{P}$-a.s.,*

$$\begin{aligned}|u(t\wedge\tau_n)|^2 &+ 2\int_0^{t\wedge\tau_n}\langle A(s)u(s),u(s)\rangle\,ds \qquad \text{(A.16)}\\ &= |u(0)|^2 + 2\int_0^{t\wedge\tau_n}(u(s),(g(s)+C(s)u(s))dW(s)) \qquad \text{(A.17)}\\ &+ \int_0^{t\wedge\tau_n}\langle f(s),u(s)\rangle\,ds + \int_0^{t\wedge\tau_n}\|C(s)u(s)+g(s)\|^2_{R(K,H)}\,ds.\end{aligned}$$

Assume now that we have also a nonnegative progressively measurable process $a(t)$, $0\leqslant t<\tau$ and we define another nonnegative process y by

$$y(t) := e^{-a(t)},\ 0\leqslant t<\tau.$$

Then $dy(t) = -a(t)y(t)\,dt$ and by the chain rule we infer that the process $y(t)|u(t)|^2$, $0 \leqslant t < \tau$ satisfies the following

$$\begin{aligned} d\big[y(t)|u(t)|^2\big] &= y(t)\,d[|u(t)|^2] + |u(t)|^2\,dy(t) \\ &= y(t)(u(t),(g(t)+C(t)u(t))\,dW(t)) + y(t)\big[-2\langle A(t)u(t),u(t)\rangle \\ &+ \langle f(t),u(t)\rangle + \|C(t)u(t)+g(t)\|^2_{R(K,H)} - a(t)|u(t)|^2\big]\,dt \end{aligned}$$

Let us now assume that the processes f and g are of special form. To be precise, let us assume that

There exist $\alpha,\beta > 0$ such that $\alpha+\beta < 1$, there exist $C_1, C_2 > 0$ and there exists a nonnegative progressively measurable process $\varphi(t)$, $0 \leqslant t < \tau$, such that

$$|\langle f(t),u(t)\rangle \leqslant \alpha\nu\|u(t)\|^2 + \frac{C_1}{\nu}\varphi(t)|u(t)|^2, \tag{A.18}$$

$$\|C(t)u(t)+g(t)\|^2_{R(K,H)} \leqslant \beta\nu\|u(t)\|^2 + \frac{C_2}{\nu}\varphi(t)|u(t)|^2, \tag{A.19}$$

Then, we have

$$\begin{aligned} d\big[y(t)|u(t)|^2\big] &+ 2y(t)\langle A(t)u(t),u(t)\rangle - y(t)(u(t),(g(t)+C(t)u(t))\,dW(t)) \\ &= y(t)\big[\langle f(t),u(t)\rangle + \|C(t)u(t)+g(t)\|^2_{R(K,H)} - a(t)|u(t)|^2\big]\,dt \\ &\leqslant (\alpha+\beta)\nu y(t)\|u(t)\|^2 + \big[\frac{C_1+C_2}{\nu}\varphi(t) - a(t)\big]y(t)|u(t)|^2, \end{aligned}$$

Applying next assumption (A.4) we infer that

$$\begin{aligned} d\big[y(t)|u(t)|^2\big] &+ \nu y(t)\|u(t)\|^2 + \frac{1}{2}\,y(t)\|C(t)u(t)\|^2_{R(K,H)} - y(t)(u(t),(g(t)+C(t)u(t))\,dW(t)) \\ &\leqslant (\alpha+\beta)\nu y(t)\|u(t)\|^2 + [\frac{C_1+C_2}{\nu}\varphi(t) + \lambda - a(t)]y(t)|u(t)|^2, \end{aligned}$$

Therefore, with $\delta = \nu(1-\alpha-\beta)$ and $C_3 = \frac{C_1+C_2}{\nu}$ we infer that for all $n \in \mathbb{N}$,

$$\begin{aligned} d\big[y(t)|u(t)|^2\big] &+ \delta y(t)\|u(t)\|^2 - y(t)(u(t),(g(t)+C(t)u(t))\,dW(t)) \\ &\leqslant [C_3\varphi(t) + \lambda - a(t)]y(t)\varphi(t)|u(t)|^2, \end{aligned}$$

Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned} y(t\wedge\tau_n)|u(t\wedge\tau_n)|^2 &+ \delta\int_0^{t\wedge\tau_n} y(s)\|u(s)\|^2\,ds - \int_0^{t\wedge\tau_n} y(s)(u(s),(g(s)+C(s)u(s))\,dW(s)) \\ &\leqslant y(0)|u(0)|^2 + \int_0^{t\wedge\tau_n} [C_3\varphi(t) + \lambda - a(t)]y(t)\varphi(s)|u(s)|^2\,ds \end{aligned}$$

Since the process $\int\limits_0^{t\wedge\tau_n} y(s)(u(s),(g(s)+C(s)u(s))\,dW(s))$ is a martingale, we get, by taking the expectation, that

$$\begin{aligned}\mathbb{E}\big[y(t\wedge\tau_n)|u(t\wedge\tau_n)|^2\big] \quad &+ \quad \delta\mathbb{E}\int\limits_0^{t\wedge\tau_n} y(s)\|u(s)\|^2\,ds \leqslant \mathbb{E}\big[y(0)|u(0)|^2\big] \\ &+ \quad \mathbb{E}\int\limits_0^{t\wedge\tau_n}[C_3\varphi(t)+\lambda-a(t)]y(t)\varphi(s)|u(s)|^2\,ds\end{aligned}$$

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Department of Mathematics, The University of York, Heslington, York YO10 5DD, UK
E-mail address: zb500@york.ac.uk

School of Mathematics and Statistics, The University of Sydney, Sydney 2006, Australia
E-mail address: Beniamin.Goldys@sydney.edu.au

School of Mathematics and Statistics, UNSW, Sydney 2052, Australia