LARGE DEVIATIONS AND TRANSITIONS BETWEEN EQUILIBRIA FOR STOCHASTIC LANDAU-LIFSHITZ-GILBERT EQUATION

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Abstract. We study a stochastic Landau-Lifshitz equation on a bounded interval and with finite dimensional noise. We first show that there exists a unique pathwise solution to this equation and that solutions enjoy a global maximal regularity property. Next, we prove a large deviations principle for small noise asymptotic of solutions using the weak convergence method. As a byproduct of the proof we obtain compactness of the solution map for a deterministic Landau-Lifschitz equation, when considered as a transformation of external fields. We then apply this large deviations principle to show that small noise can cause magnetisation reversal and also to show the importance of the shape anisotropy parameter for reducing the disturbance of the solution caused by small noise. The problem is motivated by applications of ferromagnetic nanowires to the fabrication of magnetic memories.

Contents

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1. Introduction

Stochastic PDEs for manifold-valued processes have been introduced by Funaki [24] and recently studied in [9, 12, 13]. In this paper we consider a particular example of such an equation known as the stochastic Landau-Lifshitz-Gilbert (LLG) equation with solutions taking values in the two-dimensional sphere $\mathbb{S}^2$, see [4] or [7]. To introduce this equation, we will need the Sobolev space $H^{1,2}\left(\Lambda,\mathbb{R}^3\right)$ of functions defined on a bounded interval Λ of the real line. To every $\phi\in H^{1,2}\left(\Lambda,\mathbb{R}^3\right)$ we associate the energy functional

$$\mathscr{E}(\phi)=\frac{a}{2}\int\limits_\Lambda|\nabla\phi|^2\,dx+\int\limits_\Lambda f(\phi)\,dx,$$

where

$$f(u)=\frac{\beta}{2}\left(u_2^2+u_3^2\right),\quad u=(u_1,u_2,u_3)\in\mathbb{S}^2\,.$$

Let

$$\mathscr{H}(\phi)=-\nabla\mathscr{E}(\phi)=a\Delta\phi-f'(\phi)$$

denote the the L^2-gradient of the energy functional $\mathscr{E}$. We will consider the following Stratonovitch type stochastic PDE satisfied by a function $M:\Lambda\to\mathbb{R}^3$:

$$\begin{cases} dM=[M\times\mathscr{H}(M)-\alpha M\times(M\times\mathscr{H}(M))]\,dt+\sqrt{\varepsilon}g(M)\circ d\xi, & t\geqslant 0,\\ \left.\dfrac{\partial M}{\partial x}\right|_{\partial\Lambda}=0, & t\geqslant 0,\\ M(0)=M_0\,, \end{cases}\tag{1.1}$$

where ξ is a certain $L^2\left(\Lambda,\mathbb{R}^3\right)$-valued Wiener process and $|M_0(x)|=1$ for all $x\in\Lambda$. The precise definitions of the noise and the function $g:\mathbb{R}^3\to\mathscr{L}\left(\mathbb{R}^3\right)$ are provided in Sections 2 and 3, see (2.1). The definition of solutions to (1.1) is implicitly given in Theorem 3.1. Formal application of the Itô formula easily shows that $|M(t,x)|=1$ for all times and all $x\in\Lambda$ so that (1.1) is indeed an example of a stochastic PDE for an $\mathbb{S}^2$-valued process M.

Equation (1.1) with $\varepsilon=0$ is a relatively simple version of the general Landau-Lifschitz-Gilbert equation that provides a basis for the theory and applications of ferromagnetic materials, and fabrication of magnetic memories in particular, see for example [4, 5, 25, 31]. Let us recall that according to the Landau and Lifschitz theory of ferrormagnetizm [31], modified later by Gilbert [25], the deterministic LLG equation

$$\begin{cases} \dfrac{dm}{dt}=m\times\mathscr{H}(m)-\alpha m\times(m\times\mathscr{H}(m)), & t\geqslant 0,\\ \left.\dfrac{\partial m}{\partial x}\right|_{\partial\Lambda}=0, & t\geqslant 0,\\ m(0)=m_0\,, \end{cases}\tag{1.2}$$

describes the evolution of the magnetisation vector m of a ferromagnet occupying the region Λ. For the derivation of equations (1.1) and (1.2) from physical principles and for the physical motivation to add a stochastic term to equation (1.2), see [4, 5, 8, 29, 31]. Here we mention only that the Landau-Lifschitz theory of ferromagnetizm requires coupling of equations (1.1) and (1.2) with the Maxwell equations in the whole space. They need not be introduced in this paper because in one-dimensional domain the effect of coupling is incorporated in the term $\beta g(M)f(M)$, see [16] for details. Finally, we note that the case of one-dimensional domain while being relatively simple (contrary to the multidimensional case, smooth solutions exist) is important for physics of ferromagnetism and applications of ferromagnetic nanowires, see [16].

To the best of our knowledge (1.1) has not been studied before. The existence of a weak martingale solution is proved for a similar equation in a three-dimensional domain in our earlier work [7]. Kohn, Reznikoff and vanden-Eijnden [29] modelled the magnetisation M in a thin film, assuming that M is constant across the domain for all times and $\beta = 0$. In this case (1.1) reduces to an ordinary stochastic differential equation in $\mathbb{R}^3$. They used the large deviations theory to make a detailed computational and theoretical study of the behaviour of the solution. They also remark that little is known about the behaviour of solutions to the stochastic Landau-Lifshitz equation when M is not constant on the space domain.

In this work we address the question raised in [29]. We show first the existence and uniqueness of smooth pathwise solutions to (1.1). Then we prove the Large Deviations Principle (LDP) for (1.1) and finally, we apply the LDP to the analysis of transitions between equilibria in the limit of vanishing noise.

We will describe now the content and new results obtained in this paper.

We start with Section 2 containing some definitions and auxiliary fact that will are needed later.

In Section 3 we prove the existence of a weak martingale solution stated in Theorem 3.1. The proof combines the ideas of the proof of the existence theorem in [7] with the application of the Girsanov theorem. We only sketch the steps that repeat almost verbatim the arguments from [7] and concentrate on new arguments.

In Section 4 we consider the existence of strong solutions to the stochastic LLG Equation (1.1).

In Theorem 4.2 we state a pathwise uniqueness result for solutions of equation (1.1) with trajectories belonging to the space $S_T = C([0,T];\mathbb{L}^2) \cap L^4\left(0,T;H^{1,2}\left(\Lambda,\mathbb{R}^3\right)\right)$.

In Section 5, we prove maximal regularity of solutions to (1.1). Namely, we show that

$$\mathbb{E}\int_0^T\int_\Lambda |\mathrm{D}M(t,x)|^4\,dx\,dt + \int_0^T\int_\Lambda |\Delta M(t,x)|^2\,dx\,dt < \infty.$$

The proof of this result follows from the maximal regularity and ultracontractivity properties of the heat semigroup generated by the Laplace operator with the Neumann boundary conditions and the estimates for weak solutions of equation (1.1) obtained in Theorem **??**.

The Large Deviations Principle for equation (1.1) is studied in Section 6. We first identify the rate function and prove in Lemma 6.3 that it has compact level sets in the space

$$\mathscr{X}_T = C\left([0,T];H^{1,2}\left(\Lambda;\mathbb{R}^3\right)\right) \cap L^2\left(0,T;H^{2,2}\left(\Lambda;\mathbb{R}^3\right)\right).$$

In particular, we show in Lemma 6.3 certain compactness property of solutions to the deterministic LLG equation. It seems that such a result is new in the deterministic theory and is of independent interest.

The Large Deviations Principle is proved in Theorem 6.1. To prove this theorem, we use

the weak convergence method of Budhiraja and Dupuis [14, Theorem 4.4]. Following their work we show that the two conditions of Budhiraja and Dupuis, see Statements 1 and 2 in Section 6, are satisfied and then Theorem 6.1 easily follows. We note that our proof is simpler than the corresponding proofs in [17] and [21] as we do not need to partition the time interval $[0,T]$ into small subintervals.
In Section 7 we apply the Large Deviations Principle to a simple stochastic model of magnetisation in a needle-shaped domain. we first obtain explicit estimates of the size of domains of attraction of the North and South Pole which are stationary solutions for the deterministic LLG equation. Then we show that in the presence of small noise in equation (1.1) there is a positive probability of transitions transitions between the domains of attraction. Using the Freidlin-Ventzell estimates we obtain explicit estimates for this probability. These estimate show also the importance of the parameter β (interpreted as the measure of shape anisotropy) for reducing the disturbance of the magnetisation caused by small noise. The results we obtain partially answer a question posed in [29] and provide a foundation for the computational study of stability of ferromagnetic nanowires under the influence of small noise.

1.1. **Notations.** The inner product of vectors $x, y \in \mathbb{R}^3$ will be denoted by $x \cdot y$ and $|x|$ will denote the Euclidean norm of x. We will use the standard notation $x \times y$ for the vector product in $\mathbb{R}^3$.
For a domain Λ we will use the notation $\mathbb{L}^p$ for the space $L^p\left(\Lambda;\mathbb{R}^3\right)$, $\mathbb{W}^{1,p}$ for the Sobolev space $W^{1,p}\left(\Lambda;\mathbb{R}^3\right)$ and so on. For $p=2$ we will often write $\mathbb{H}^k$ instead of $\mathbb{W}^{k,2}$. We will always emphasize the norm of the corresponding space writing $|f|_{\mathbb{L}^2}$, $|f|_{\mathbb{H}^1}$ and so on.

We will also need the spaces $L^p(0,T;E)$ and $C([0,T];E)$ of Bochner p-integrable, respectively continuous, functions $f:[0,T]\to E$ with values in a Banach space E. If $E=\mathbb{R}$ then we write simply $L^p(0,T)$ and $C([0,T])$. For a Banach space E we will denote by $\mathscr{L}(E)$ the space of all linear and bounded maps from E to itself.

Throughout the paper C stands for a positive real constant whose actual value may vary from line to line. We include an argument list, $C(a_1,\dots,a_m)$, if we wish to emphasize that the constant depends only on the values of the arguments a_1 to a_m.

2. Preliminaries

We us assume that a $\alpha>0$. Let us denote by g a map $g:\mathbb{R}^3\to\mathscr{L}\left(\mathbb{R}^3\right)$ defined by

$$g:\mathbb{R}^3\ni y\mapsto\left\{\mathbb{R}^3\ni h\mapsto g(y)h:=y\times h-\alpha y\times(y\times h)\in\mathbb{R}^3\right\}\in\mathscr{L}\left(\mathbb{R}^3\right). \tag{2.1}$$

The function g is of class C^∞. In particular, we have

$$\left[g'(y)h\right]z=\mathrm{D}\left[g(y)h\right]z=z\times h-\alpha[z\times(y\times h)+y\times(z\times h)],\quad h,y,z\in\mathbb{R}^3, \tag{2.2}$$

and for every $r>0$

$$\sup_{|y|\leqslant r}\left[|g(y)|_{\mathbb{R}^3}+\left|g'(y)\right|_{\mathscr{L}(\mathbb{R}^3,\mathscr{L}(\mathbb{R}^3))}\right]<\infty. \tag{2.3}$$

Clearly, we can define a map $(u,h)\mapsto(g\circ u)h$, if u,h belong to some function spaces of $\mathbb{R}^3$-valued functions on Λ. For instance, if $u\in\mathbb{L}^\infty$ and $h\in\mathbb{L}^2$ then $(g\circ u)h$ is a well defined element of $\mathbb{L}^2$. We will denote by G a Nemytski type map associated with the function g. To be precise, we will use the notation $G(u)$, if $u\in\mathbb{L}^\infty$, for a linear map defined, for every $q\in[1,\infty]$, by

$$G(u):\mathbb{L}^q\ni h\mapsto u\times h-\alpha u\times(u\times h)\in\mathbb{L}^q. \tag{2.4}$$

For fixed functions $e_i \in \mathbb{L}^2$, $i = 1,2,3$, let $B : \mathbb{R}^3 \to \mathbb{L}^2$ be a linear operator defined by

$$B : \mathbb{R}^3 \ni k \mapsto \sum_{i=1}^{3} k_i e_i \in \mathbb{L}^2. \tag{2.5}$$

In the next lemma we use the notation $e = (e_i)$ and

$$|e|_{\mathbb{L}^\infty} = \max_{1\leqslant i\leqslant 3} |e_i|_{\mathbb{L}^\infty} .$$

Lemma 2.1. *Assume that $q \in [1,\infty]$. Then the map $G : \mathbb{L}^\infty \to \mathscr{L}(\mathbb{L}^q, \mathbb{L}^q)$ is a polynomial map, hence of polynomal growth and locally Lipschitz, i.e. there exists $C_0 > 0$ such that*

$$|G(u)\,h|_{\mathbb{L}^2} \leqslant C_0 |h|_{\mathbb{L}^q}\left[|u|_{\mathbb{L}^\infty} + |u|^2_{\mathbb{L}^\infty}\right], \quad h \in \mathbb{L}^q . \tag{2.6}$$

and, for every $r > 0$ there exists $C_r > 0$ such that for all $u_i \in \mathbb{L}^\infty$, $i = 1,2$ satisfying $|u_i|_{\mathbb{L}^\infty} \leqslant r$, one has

$$|G(u_1)\,h - G(u_2)\,h|_{\mathbb{L}^q} \leqslant C_r |h|_{\mathbb{L}^q}\, |u_1 - u_2|_{\mathbb{L}^\infty} , \quad h \in \mathbb{L}^2 . \tag{2.7}$$

Moreover, there exists $a > 0$ such that

$$|G(u)\,h|_{\mathbb{H}^1} \leqslant a |h|_{\mathbb{H}^1}\left[1 + |u|^2_{\mathbb{H}^1}\right], \quad u, h \in \mathbb{H}^1. \tag{2.8}$$

Proof. The last part of the above Lemma is a consequenc of the fact that $\mathbb{H}^1$ is an algebra. □

Given two vectors $f_2, f_3 \in \mathbb{R}^3$, the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$f(y) = (y \cdot f_2)\, f_2 + (y \cdot f_3)\, f_3, \quad y \in \mathbb{R}^3 . \tag{2.9}$$

As before, we will denote by F the Nemytki map associated with the function f, i.e. for $q \in [1,\infty]$,

$$F : \mathbb{L}^q \ni u \mapsto f \circ u = (u(\cdot) \cdot f_2)\, f_2 + (u(\cdot) \cdot f_3)\, f_3 \in \mathbb{L}^q . \tag{2.10}$$

Note that $F : \mathbb{L}^q \to \mathbb{L}^q$ is a bounded linear map. In conjuction with Lemma 2.1 we get the following result.

Lemma 2.2. *For every $e \in \mathbb{R}^3$, the maps*

$$\begin{aligned} GF: &\quad \mathbb{L}^\infty \ni u \mapsto G(u)F(u) \in \mathbb{L}^\infty \\ G'eGe: &\quad \mathbb{L}^\infty \ni u \mapsto [G'(u)e][G(u)e] \in \mathbb{L}^\infty \end{aligned}$$

Let us recall that $\Lambda \subset \mathbb{R}$ is a bounded interval. We define the Laplacian with the Neumann boundary conditions by $\mathrm{A} : D(\mathrm{A}) \subset \mathbb{L}^2 \to \mathbb{L}^2$ by

$$\left\{ \begin{array}{rcl} D(\mathrm{A}) & := & \{u \in \mathbb{H}^2 : \mathrm{D}u(x) = 0 \text{ for } x \in \partial\Lambda\}, \\ \mathrm{A}u & := & -\Delta u \text{ for } u \in D(\mathrm{A}). \end{array} \right. \tag{2.11}$$

Let us recall that the operator A is self-adjoint and nonnegative and $D\left(A^{1/2}\right)$ when endowed with the graph norm coincides with $\mathbb{H}^1$. Moreover, the operator $(A+I)^{-1}$ is compact.

For any real number $\beta \geqslant 0$, we write $\mathbb{X}^\beta$ for the domain of the fractional power operator $D\left(\mathrm{A}^\beta\right)$ endowed with the norm $|x|_{\mathbb{X}^\beta} = |(I + \mathrm{A})^\beta x|$ and $\mathbb{X}^{-\beta}$ denotes the dual space of $\mathbb{X}^\beta$ so that $\mathbb{X}^\beta \subset \mathbb{H} = \mathbb{H}' \subset \mathbb{X}^{-\beta}$ is a Gelfand triple. Note that for $\beta \in [0, \frac{3}{4})$,

$$\mathbb{X}^\beta = \mathbb{H}^{2\beta}.$$

In what follows we will need the following, well known, interpolation inequality:

$$|u|^2_{\mathbb{L}^\infty} \leqslant k^2 |u|_{\mathbb{H}} |u|_{\mathbb{H}^1} \quad \forall u \in \mathbb{H}^1, \tag{2.12}$$

where the optimal value of the constant k is

$$k = 2\max\left(1, \frac{1}{\sqrt{|\Lambda|}}\right).$$

For $v, w, z \in \mathbb{H}^1$ by the expressions $w \times \Delta v$ and $z \times (w \times \Delta v)$ we understand the unique elements of the dual space $(\mathbb{H}^1)'$ of $\mathbb{H}^1$ such that for any $\phi \in \mathbb{H}^1$

$$_{(\mathbb{H}^1)'}\langle w \times \Delta v,\, \phi\rangle_{\mathbb{H}^1} = -\langle \mathrm{D}(\phi \times w),\, \mathrm{D}v\rangle_{\mathbb{L}^2} \tag{2.13}$$

and

$$_{(\mathbb{H}^1)'}\langle z \times (w \times \Delta v)\,, \phi\rangle_{\mathbb{H}^1} = -\langle \mathrm{D}((\phi \times z) \times w),\, \mathrm{D}v\rangle_{\mathbb{L}^2}, \tag{2.14}$$

respectively. Note that the space $H^1(\Lambda)$ is an algebra, hence for $v, w, z \in \mathbb{H}^1$, linear functionals $\mathbb{H}^1 \ni \phi \mapsto$ RHS of (2.13) (or (2.14)) are continuous. In particular, since $\langle a \times b, a\rangle = 0$ for $a, b \in \mathbb{R}^3$, we obtain

$$_{(\mathbb{H}^1)'}\langle w \times \Delta v,\, v\rangle_{\mathbb{H}^1} = -\langle v \times \mathrm{D}w,\, \mathrm{D}v\rangle_{\mathbb{L}^2} \tag{2.15}$$

$$_{(\mathbb{H}^1)'}\langle z \times (v \times \Delta v)\,, \phi\rangle_{\mathbb{H}^1} = -\langle \mathrm{D}(\phi \times z) \times v,\, \mathrm{D}v\rangle_{\mathbb{L}^2}, \tag{2.16}$$

and since $a \times a = 0$ for $a \in \mathbb{R}^3$, equation (2.13) yields

$$_{(\mathbb{H}^1)'}\langle \phi \times \Delta v,\, \phi\rangle_{\mathbb{H}^1} = -\langle \mathrm{D}(\phi \times \phi),\, \mathrm{D}v\rangle_{\mathbb{L}^2} = 0. \tag{2.17}$$

The maps $\mathbb{H}^1 \ni y \mapsto y \times \Delta y \in (\mathbb{H}^1)'$ and $\mathbb{H}^1 \ni y \mapsto y \times (y \times \Delta y) \in (\mathbb{H}^1)'$ are continuous homogenous polynomials of degree 2, resp. 3 hence they are locally Lipschitz continuous.

3. The Existence of solutions

We will be concerned with the following stochastic integral equation form of problem (1.1)

$$\begin{aligned} M(t) = M_0 &+ \int_0^t [M(s) \times \Delta M(s)) - \alpha M(s) \times (M(s) \times \Delta M(s))]\, ds \\ &+ \sqrt{\varepsilon}\int_0^t G(M(s))B\, dW(s) + \frac{\varepsilon}{2}\sum_{i=1}^3 \int_0^t \left[G'(M(s))e_i\right]\big(G(M(s))e_i\big)\, ds \\ &- \beta\int_0^t G(M(s))F(M(s))\, ds, \qquad t \in [0, T]. \end{aligned} \tag{3.1}$$

where G and F are the Nemytski maps associated with functions g and f defined in the previous section. For instance, since $\mathbb{H}^1$ is an algebra, the map

$$G(\cdot)B : \mathbb{H}^1 \ni M \mapsto \left\{\mathbb{R}^3 \ni k \mapsto \sum_{i=1}^3 G(M)(k_i e_i) = \sum_{i=1}^3 k_i G(M)e_i \in \mathbb{H}^1\right\} \in \mathscr{L}(\mathbb{R}^3, \mathbb{H}^1)$$

is a continuous polynomial function (and hence of C^∞-class and Lipschitz on balls).

Note, that the expresssion

$$\sqrt{\varepsilon}\int_0^t G(M(s))B\,dW(s)+\frac{\varepsilon}{2}\sum_{i=1}^3\int_0^t \left[G'(M(s))e_i\right]\left(G(M(s))e_i\right)ds$$

can be identified with the Stratonovich integral

$$\sqrt{\varepsilon}\int_0^t G(M(s))B\circ dW(s)$$

but we will not use this concept in the paper.
We will now formulate the main result of this Section.

Theorem 3.1 (Existence of a weak martingale solution)**.** *Assume that $e=(e_i)_{i=1}^3\in\left(\mathbb{H}^1\right)^3$, $\|e\|_{\mathbb{H}^1}\leqslant r$ and that function f defined by (2.9) is fixed. Assume also that $M_0\in\mathbb{H}^1$, $\|M_0\|_{\mathbb{H}^1}\leqslant\rho$. Then there exists a system*

$$\Big(\Omega,\mathscr{F},\mathbb{F},\mathbb{P},W,M\Big) \tag{3.2}$$

consisting of a probability space $(\Omega,\mathscr{F},\mathbb{P})$, of a filtration $\mathbb{F}=(\mathscr{F}_t)$, of a canonical $\mathbb{R}^3$-valued $\mathbb{F}$-Wiener process $W=(W(t))$ and of an $\mathbb{F}$-progressively measurable process $M=(M(t))$ such that

(1) *for each $\beta<\frac{1}{2}$ the paths of M are continuous $\mathbb{H}^{2\beta}$-valued functions $\mathbb{P}$-a.s.;*
(2) *For every $p\geqslant 1$ and every $T>0$,*

$$\mathbb{E}\sup_{t\in[0,T]}|M(t)|_{\mathbb{H}^1}^p\leqslant C\left(T,p,\alpha,\rho,r\right); \tag{3.3}$$

(3) *For almost every $t\in[0,\infty)$, $M(t)\times\Delta M(t)\in\mathbb{L}^2$ and every $T>0$ we have*

$$\mathbb{E}\left(\int_0^T|M(s)\times\Delta M(s)|_{\mathbb{L}^2}^2\,ds\right)^p\leqslant C\left(T,p,\alpha,\rho,r\right) \tag{3.4}$$

(4) *$|M(t)(x)|_{\mathbb{R}^3}=1$ for all $x\in\Lambda$ and for all $t\in[0,\infty)$, $\mathbb{P}$-a.s.;*
(5) *For every $t\in[0,\infty)$ equation (3.1) holds $\mathbb{P}$-a.s.*
(6) *for every $\alpha\in\left(0,\frac{1}{2}\right)$, $\mathbb{P}$-a.s.,*

$$u(\cdot)\in C^\alpha\left([0,T],\mathbb{L}^2\right). \tag{3.5}$$

Note that in Theorem 3.1, M is an $\mathbb{H}^1$-valued process, hence the expressions $M(s)\times\Delta M(s)$ and $M(s)\times(M(s)\times\Delta M(s))$ are interpreted in the sense of (2.13) and (2.14) respectively.

Proof. The proof of Theorem 3.1 is very similar to the proof of Theorem 2.7 in [7]. Here we only sketch the main arguments. Full details can be found in [10]. It is sufficient to prove the theorem for a bounded time interval $[0,T]$. We start with some auxiliary definitions. For each $n\in\mathbb{N}$, let $\mathbb{H}_n$ be the linear span of the first n elements of the orthonormal basis of $\mathbb{L}^2$ composed of eigenvectors of A and let

$$\pi_n:\mathbb{L}^2\to\mathbb{H}_n \tag{3.6}$$

be the corresponding orthogonal projection. Let us define a map $G_n:\mathbb{H}_n\to\mathscr{L}(\mathbb{H}_n)$ by

$$G_n(u)=\pi_nG\left(\pi_nu\right)\pi_n,\quad u\in\mathbb{H}_n.$$

and let $G_n' : \mathbb{H}_n \to \mathscr{L}\big(\mathbb{H}_n, \mathscr{L}(\mathbb{H}_n)\big)$ be the Fréchet derivative of G_n. Since the space $\mathbb{H}_n$ is finite dimensional and contained in $\mathbb{L}^\infty$,
For each $n \in \mathbb{N}$, we define a process $M_n : [0,T] \times \Omega \to \mathbb{H}_n$ to be a solution of the following ordinary stochastic differential equation on $\mathbb{H}_n$:

$$\begin{aligned} M_n(t) &= \pi_n M_0 + \int_0^t \pi_n(M_n \times \Delta M_n)\, ds \\ &- \alpha \int_0^t \pi_n(M_n \times (M_n \times \Delta M_n))\, ds \\ &+ \sqrt{\varepsilon} \int_0^t G_n(M_n)\, B dW(s) + \frac{\varepsilon}{2} \sum_{i=1}^3 \int_0^t \left[G_n'(M_n)\, e_i\right] \left(G_n(M_n)\, e_i\right)\, ds \\ &- \beta \int_0^t G_n(M_n)\, F(M_n)\, ds\,. \end{aligned} \tag{3.7}$$

Since the space $\mathbb{H}_n$ is finite dimensional and contained in $\mathbb{L}^\infty$, by Lemmata 2.1 and 2.2, the maps the maps G_n, $[G_n'(\cdot)\, e_i]\,(G_n(\cdot)\, e_i)$ and $G_n(\cdot)\, F(\cdot)$ are bounded polynomial maps on $\mathbb{H}_n$, hence locally Lipschitz and of polynomal growth. Since the coefficients in (3.7) are of one-sided linear growth, by standard arguments we can prove, see e.g. [2], that for each $n \in \mathbb{N}$, equation (3.7) has a unique strong (in the probabilistic sense) solution. Applying the Itô formula and the Gronwall Lemma to the processes $|M_n(\cdot)|^2_{\mathbb{H}}$ and $|M_n(\cdot)|^2_{\mathbb{H}^1}$, one can obtain the following, uniform in $n \in \mathbb{N}$, estimates.

Lemma 3.2. *Let the assumptions of Theorem 3.1 be satisfyied. Then for each $n \in \mathbb{N}$*

$$|M_n(t)|_{\mathbb{L}^2} = |\pi_n u_0|_{\mathbb{L}^2}, \quad \text{for all} \quad t \in [0,T] \quad \mathbb{P}-a.s.$$

Moreover, for each $p \in [1,\infty)$ there exists a constant $C(T,p,\alpha,\rho,r)$ such that, if $\|M_0\|_{\mathbb{H}^1} \leqslant \rho$ and $\|e\|_{\mathbb{H}^1} \leqslant r$, then for every $n \in \mathbb{N}$

$$\mathbb{E} \sup_{t\in[0,T]} |M_n(t)|^p_{\mathbb{H}^1} \leqslant C(T,p,\alpha,\rho,r)\,, \tag{3.8}$$

$$\mathbb{E}\left(\int_0^T |M_n(s) \times \Delta M_n(s)|^2_{\mathbb{L}^2}\, ds\right)^p \leqslant C(T,p,\alpha,\rho,r)$$

and

$$\mathbb{E}\left(\int_0^T |M_n(s) \times (M_n(s) \times \Delta M_n(s))|^2_{\mathbb{L}^2}\, ds\right)^p \leqslant C(T,p,\alpha,\rho,r)\,.$$

The above *a priori* estimates from Lemma 3.2 on the sequence (M_n) imply, by applying two key results of Flandoli and Gątarek [23, Lemma 2.1 and Theorem 2.2], that the corresponding sequence of laws of pairs (W, M_n) is tight on the space $C\left([0,T],\mathbb{R}^3\right) \times \big[C([0,T];\mathbb{X}^{-1/2}) \cap L^4(0,T;\mathbb{L}^4)\big]$ and hence by the Prokhorod Theorem, modulo extracting a subsequnce, these laws converge weakly to a Borel probability measure $\mathbb{P}^{W,M}$ on $C\left([0,T],\mathbb{R}^3\right) \times \big[C([0,T];\mathbb{X}^{-1/2}) \cap L^4(0,T;\mathbb{L}^4)\big]$. Next we have the following result.

Proposition 3.3. *There exists a probability space* $(\Omega', \mathscr{F}', \mathbb{P}')$ *and there exists a sequence* (W_n', M_n') *of* $C\left([0,T],\mathbb{R}^3\right) \times \left[C([0,T];\mathbb{X}^{-1/2}) \cap L^4(0,T;\mathbb{L}^4)\right]$*-valued random variables defined on* $(\Omega', \mathscr{F}', \mathbb{P}')$ *such that the laws of* (W, M_n) *and* (W_n', M_n') *are equal for each* $n \in \mathbb{N}$ *and* (W_n', M_n') *converges pointwise in* $C\left([0,T],\mathbb{R}^3\right) \times C([0,T];\mathbb{X}^{-\frac{1}{2}})$*,* $\mathbb{P}'$*-a.s., to a limit* (W', M') *whose law is equal to* $\mathbb{P}^{W,M}$*.*

Proof. The proposition follows from the Skorohod theorem (see [27, Theorem 4.30]). □

It remains to show that the pointwise limit (W', M') defined on the probability space $(\Omega', \mathscr{F}', \mathbb{P}')$ satisfies all the claims of Theorem 3.1. For each $n \in \mathbb{N}$, (W_n', M_n') satisfies an equation obtained from (3.7) by replacing W and M_n by W_n' and M_n', respectively. Then the processes M_n' satisfy the estimates of Lemma 3.2. These estimates together with the pointwise convergence of the sequence $((W_n', M_n'))_{n\in\mathbb{N}}$ imply that the (W', M') satisfies equation (3.1). The proof of part (5) of Theorem 3.1 is analogous to the proofs of Lemma 5.1 and Lemma 5.2 in [7].
The proof of part (6) is similar to the proof of inequality (2.17) in Theorem 2.7(c) in [7]. The only difference being the last two terms on the RHS of equation (3.1). However, by part (5), the integrands in these terms are uniformly bounded and hence by Lemmata (2.1) and 2.2 we infer that the expectation of the increments corresponding to these terms is Lipschitz with respect to the time parameters. □

4. The pathwise uniqueness and the existence of a strong solution

The main result in this section is Theorem 4.2, on pathwise uniqueness of solutions of equation (3.1). Although we could have formulated a theorem of Yamada-Watanabe type on the uniqueness in law and the existence of a strong solution to equation (3.1) we have decided to do so at the end of the next section after we had proved some further regularity properties of the solutions.

We start with a simple

Lemma 4.1. *Let* u *be an element of* $\mathbb{H}^1$ *such that*

$$|u(x)| = 1 \quad \text{for all } x \in \Lambda. \tag{4.1}$$

Then, in $(\mathbb{H}^1)'$*, we have*

$$u \times (u \times \Delta u) = -|\mathrm{D}u|^2 u - \Delta u. \tag{4.2}$$

Proof. Let us choose and fix $u, \phi \in \mathbb{H}^1$. Note that by (4.1), $\mathrm{D}|u|^2 = 0$. By equality (2.16) and the product rule we have

$$-{}_{(\mathbb{H}^1)'}\langle u \times (u \times \Delta u)\,, \phi\rangle_{\mathbb{H}^1} = \langle \mathrm{D}(\phi \times u) \times u,\, \mathrm{D}u\rangle_{\mathbb{L}^2} = \langle (\mathrm{D}\phi \times u) \times u,\, \mathrm{D}u\rangle_{\mathbb{L}^2} + \langle (\phi \times \mathrm{D}u) \times u,\, \mathrm{D}u\rangle_{\mathbb{L}^2}.$$

Invoking a well known identity

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c, \quad a, b, c \in \mathbb{R}^3,$$

we obtain

$$\begin{aligned} \langle (\mathrm{D}\phi \times u) \times u,\, \mathrm{D}u\rangle_{\mathbb{L}^2} &= \langle (\mathrm{D}\phi \cdot u)u,\, \mathrm{D}u\rangle_{\mathbb{L}^2} - \langle (u \cdot u)\mathrm{D}\phi,\, \mathrm{D}u\rangle_{\mathbb{L}^2} \\ = \frac{1}{2}\int_\Lambda (\mathrm{D}\phi(x) \cdot u(x))\mathrm{D}|u(x)|^2\,dx - \langle |u|^2 \mathrm{D}\phi,\, \mathrm{D}u\rangle_{\mathbb{L}^2} &= -\langle \mathrm{D}\phi,\, \mathrm{D}u\rangle_{\mathbb{L}^2} \end{aligned}$$

and similarly

$$\begin{aligned}\langle(\phi\times \mathrm{D}u)\times u,\, \mathrm{D}u\rangle_{\mathbb{L}^2} &= \langle(\phi\cdot u)\mathrm{D}u,\, \mathrm{D}u\rangle_{\mathbb{L}^2} - \langle(\mathrm{D}u\cdot u)\phi,\, \mathrm{D}u\rangle_{\mathbb{L}^2}\\ = \langle(\phi\cdot u)\mathrm{D}u,\, \mathrm{D}u\rangle_{\mathbb{L}^2} - \frac{1}{2}\langle(\mathrm{D}|u|^2)\phi,\, \mathrm{D}u\rangle_{\mathbb{L}^2} &= \langle(\phi\cdot u)\mathrm{D}u,\, \mathrm{D}u\rangle_{\mathbb{L}^2}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}-{}_{(\mathbb{H}^1)'}\langle u\times(u\times\Delta u),\phi\rangle_{\mathbb{H}^1} &= -\langle \mathrm{D}\phi,\, \mathrm{D}u\rangle_{\mathbb{L}^2} + \langle(\phi\cdot u)\mathrm{D}u,\, \mathrm{D}u\rangle_{\mathbb{L}^2}\\ &= {}_{(\mathbb{H}^1)'}\left\langle |\mathrm{D}u|^2 u + \Delta u,\phi\right\rangle_{\mathbb{H}^1}\end{aligned}$$

□

The following uniqueness result applies to a more general problem than (3.1). It will be used, in this generality, in the uniqueness part of the proof of Theorem 6.2.

Theorem 4.2 (Pathwise uniqueness)**.** *Assume that $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$, where $\mathbb{F}=(\mathscr{F}_t)_{t\in[0,T]}$, is a filtered probability space and $W=(W(t))_{t\in[0,T]}$ is an $\mathbb{R}^3$-valued $\mathbb{F}$-Wiener process. Assume that $e=(e_i)_{i=1}^3\in\left(\mathbb{H}^1\right)^3$. Let $M_1,\ M_2:[0,T]\times\Omega\to\mathbb{H}$ be $\mathbb{F}$-progressively measurable continuous processes such that, for $i=1,2$, the paths of M_i lie in $L^4\left(0,T;\mathbb{H}^1\right)$, satisfy property (4) from Theorem 3.1 and each M_i satisfies the equation*

$$\begin{aligned}M_i(t) &= M_0 + \int_0^t M_i\times\Delta M_i\,ds - \alpha\int_0^t M_i\times(M_i\times\Delta M_i)\,ds\\ &\quad+\sqrt{\varepsilon}\int_0^t G\left(M_i\right)B dW(s) + \frac{\varepsilon}{2}\sum_{j=1}^3\int_0^t\left[G'\left(M_i\right)e_j\right]G\left(M_i\right)e_j\,ds\\ &\quad-\beta\int_0^t G\left(M_i\right)F\left(M_i\right)\,ds + \int_0^t G\left(M_i\right)Bh(s)\,ds \qquad (4.3)\end{aligned}$$

for all $t\in[0,T]$, $\mathbb{P}$-almost everywhere. Then

$$M_1(\cdot,\omega)=M_2(\cdot,\omega),\quad \text{for}\quad \mathbb{P}-a.e.\ \omega\in\Omega.$$

Proof. Let us fix $h\in\mathscr{P}_T$ and let $R>0$ be such that

$$\int_0^T|h(t)|^2\,dt\leqslant R^2,\quad \mathbb{P}-\text{a.s.}$$

Note that the above implies that

$$\int_0^T|h(t)|\,dt\leqslant R\sqrt{T},\quad \mathbb{P}-\text{a.s.} \qquad (4.4)$$

First, we note that by Lemma 4.1 the following equality holds in $\mathbb{X}^{-1/2}$.

$$M_i(s)\times(M_i(s)\times\Delta M_i(s)) = -|\mathrm{D}M_i(s)|^2M_i(s)-\Delta M_i(s).$$

Let us assume that M_1 and M_2 are two solutions satisfying all assumptions. Because both M_i satisfy (4) from Theorem 3.1, we infer that $|M_i|$ are bounded. Hence, by the local Lipschitz property of maps G, G' and f, as well by the assumptions that each $e_i \in \mathbb{L}^\infty$, there exists a constant $C_1 > 0$, such that for all $t \in [0,T]$,

$$\sum_{i=1}^{3} |G(M_2(t))e_i - G(M_1(t))e_i|^2_{\mathbb{L}^2} \leqslant C_1 |e|^2_{\mathbb{L}^\infty} |M_2(t) - M_1(t)|^2_{\mathbb{L}^2}, \tag{4.5}$$

$$\sum_{i=1}^{3} |G'(M_2(t))e_i G(M_2(t))e_i - G'(M_1(t))e_i G(M_1(t))e_i|^2_{\mathbb{L}^2} \leqslant C_1 |e|^2_{\mathbb{L}^\infty} |M_2(t) - M_1(t)|^2_{L^2}, \tag{4.6}$$

$$\langle \big[\big(G\left(M_2(t)\right) - G\left(M_1(t)\right)\big)\big] Bh(s), M_2(t) - M_1(t)\rangle_{\mathbb{L}^2} \leqslant C|h(t)||e|_{\mathbb{L}^\infty} |M_2(t) - M_1(t)|^2_{\mathbb{L}^2} \tag{4.7}$$

$$\langle G\left(M_2(t)\right) F\left(M_2(t)\right) - G\left(M_1(t)\right) F\left(M_1(t)\right), M_2(t) - M_1(t)\rangle_{\mathbb{L}^2} \leqslant C_1 |M_2(t) - M_1(t)|^2_{\mathbb{L}^2} \tag{4.8}$$

Let $Z = M_2 - M_1$. Then the process Z belongs to $\mathbb{M}^2(0,T;V) \cap L^2(\Omega, C([0,T];\mathbb{H})$ and by Lemma 4.1 is a weak solution of the problem

$$\begin{aligned} dZ(t) &= \alpha AZ\,dt + \Big[\alpha\big(|\mathrm{D}M_2|^2 M_2 - |\mathrm{D}M_1|^2 M_1\big)\Big]\,dt \\ &+ \Big[M_2 \times \Delta M_2 - M_1 \times \Delta M_1\Big[\,dt \\ &+ \sqrt{\varepsilon}\big(G\left(M_2\right) - G\left(M_1\right)\big)\,BdW(s) \\ &+ \frac{\varepsilon}{2}\sum_{j=1}^{3}\Big[G'\left(M_2\right)e_j G\left(M_2\right)e_j - G'\left(M_1\right)e_j G\left(M_1\right)e_j\Big]\,dt \\ &- \beta\Big[G\left(M_2\right)F\left(M_2\right) - G\left(M_1\right)F\left(M_1\right)\Big]\,dt. \\ &+ \Big[\big(G\left(M_2(t)\right) - G\left(M_1(t)\right)\big)\Big]Bh(s)\,dt \end{aligned} \tag{4.9}$$

We can check that all assumptions of the Itô Lemma from [34] are satisfied and therefore

$$\begin{aligned}
\frac{1}{2}d|Z(t)|^2_{\mathrm{H}} &= -\langle AZ, Z\rangle\, dt \\
&+ \alpha\langle |\mathrm{D}M_2(t)|^2 M_2(t)\, dt - |\mathrm{D}M_1(t)|^2 M_1(t), Z\rangle\, dt \\
&+ \alpha\langle \big(\mathrm{D}M_1(t) + \mathrm{D}M_2(t)\big)M_1(t)\mathrm{D}Z, Z\rangle\, dt \\
&+ \Big[\langle M_2(t)\times \Delta Z, Z\rangle - \langle Z\times \Delta M_1(t), Z\rangle\Big]\, dt \\
&+ \frac{\varepsilon}{2}\sum_{j=1}^{3}\langle G'\left(M_2(t)\right)e_j G\left(M_2(t)\right)e_j - G'\left(M_1(t)\right)e_j G\left(M_1(t)\right)e_j, Z\rangle\, dt \\
&\quad - \beta\langle G\left(M_2(t)\right)F\left(M_2(t)\right) - G\left(M_1(t)\right)F\left(M_1(t)\right), Z\rangle\, dt \\
&+ \langle\big[\big(G\left(M_2(t)\right) - G\left(M_1(t)\right)\big)\big]Bh(s), Z\rangle\, dt \\
&+ \frac{1}{2}\varepsilon\sum_{j=1}^{3}|\big(G\left(M_2(t)\right) - G\left(M_1(t)\right)\big)e_j|^2_{\mathrm{H}}dt \\
&\quad + \sqrt{\varepsilon}\sum_{j=1}^{3}\langle G\left(M_2(t)\right) - G\left(M_1(t)\right)\big)e_j, Z\rangle\, dW_j(s) \\
&= \sum_{i=1}^{8} I_i(t)\, dt + \sum_{j=1}^{3} I_{9,j}(t)\, dW_j(t)
\end{aligned} \tag{4.10}$$

We will estimate all the terms in (4.10). In what follows we will often use inequality (2.12) and k is the constant from that inequality. Let us start with the $1^{\rm st}$ term:

$$I_1(t) = -\langle AZ(t), Z(t)\rangle \;=\; -|\mathrm{D}Z(t)|^2.$$

As for the 2nd term we have

$$\begin{aligned}
&\langle |\mathrm{D}M_2|^2 M_2 - |\mathrm{D}M_1|^2 M_1, Z\rangle \\
&\qquad = \langle |\mathrm{D}M_2|^2 Z, Z\rangle + \langle (\mathrm{D}M_1 + \mathrm{D}M_2)M_1\mathrm{D}Z, Z\rangle =: II_0 + \sum_{i=1}^{2} II_i\,.
\end{aligned}$$

Next,

$$\begin{aligned}
II_0 &\leqslant |\mathrm{D}M_2|^2_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^\infty} \\
&\leqslant k^2|\mathrm{D}M_2|^2_{L^2}|Z|_{\mathbb{L}^2}|Z|_{\mathbb{H}^1} \\
&\leqslant k^2|\mathrm{D}M_2|^2_{\mathbb{L}^2}|Z|_{\mathbb{L}^2}\big(|Z|_{\mathbb{L}^2} + |\mathrm{D}Z|_{\mathbb{L}^2}\big) \\
&\leqslant k^2|\mathrm{D}M_2|^2_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2} + k^2|\mathrm{D}M_2|^2_{\mathbb{L}^2}|Z|_{\mathbb{L}^2}|\mathrm{D}Z|_{\mathbb{L}^2} \\
&\leqslant k^2|\mathrm{D}M_2|^2_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2} + \frac{k^4}{2\eta^2}|\mathrm{D}M_2|^4_{\mathbb{L}^2}|Z|^2_{\mathbb{L}^2} + \eta^2|\mathrm{D}Z|^2_{\mathbb{L}^2}\,,
\end{aligned}$$

and, for $i=1,2$,

$$\begin{aligned} II_i &\leqslant |\mathrm{D}M_i|_{\mathbb{L}^2}|M_1|_{\mathbb{L}^\infty}|\mathrm{D}Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^\infty} \leqslant |\mathrm{D}M_i|_{\mathbb{L}^2}|\mathrm{D}Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^\infty} \\ &\leqslant k|\mathrm{D}M_i|_{\mathbb{L}^2}|\mathrm{D}Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^2}^{\frac{1}{2}}\big(|Z|_{\mathbb{L}^2}^{\frac{1}{2}}+|\mathrm{D}Z|_{\mathbb{L}^2}^{\frac{1}{2}}\big) \\ &\leqslant k|\mathrm{D}M_i|_{\mathbb{L}^2}|\mathrm{D}Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^2}+k|\mathrm{D}M_i|_{\mathbb{L}^2}|Z|_{\mathbb{L}^2}^{\frac{1}{2}}|\mathrm{D}Z|_{\mathbb{L}^2}^{\frac{3}{2}} \\ &\leqslant \frac{k^2}{\eta^2}|\mathrm{D}M_i|_{\mathbb{L}^2}^2|Z|_{\mathbb{L}^2}^2+\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2+\frac{k^4}{4\eta^6}|\mathrm{D}M_i|_{\mathbb{L}^2}^4|Z|_{\mathbb{L}^2}^2+\frac{3}{4}\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2 . \end{aligned}$$

Hence,

$$\begin{aligned} I_2(t)=\langle |\mathrm{D}M_2|^2M_2-|\mathrm{D}M_1|^2M_1,Z\rangle &\leqslant k^2\Big[|\mathrm{D}M_2|_{\mathbb{L}^2}^2+\frac{k^2}{2\eta^2}|\mathrm{D}M_2|_{\mathbb{L}^2}^4 \\ &+\sum_{i=1}^{2}\frac{1}{\eta^2}|\mathrm{D}M_i|_{\mathbb{L}^2}^2+\frac{k^2}{4\eta^6}\sum_{i=1}^{2}|\mathrm{D}M_i|_{\mathbb{L}^2}^4\Big]|Z|_{\mathbb{L}^2}^2+\frac{5}{2}\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2 \end{aligned}$$

Let us note now that by (2.17), the 2^{nd} part of the 4^{th} term, i.e. $\langle Z\times\Delta M_1,Z\rangle$ is equal to 0. Next, by definition (2.15), similarly as the estimate of II_i above, we have the following estimates for the 1st part of the 4^{th} term using the bound $|Z|_{L^\infty}\leqslant 2$, we get

$$\begin{aligned} \langle M_2\times\Delta Z,Z\rangle &= -\langle Z\times \mathrm{D}M_2,\mathrm{D}Z\rangle \leqslant |Z|_{\mathbb{L}^\infty}|\mathrm{D}M_2|_{\mathbb{L}^2}|\mathrm{D}Z|_{\mathbb{L}^2} \\ &\leqslant \frac{k^2}{\eta^2}|\mathrm{D}M_i|_{\mathbb{L}^2}^2|Z|_{\mathbb{L}^2}^2+\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2+\frac{k^4}{4\eta^6}|\mathrm{D}M_i|_{\mathbb{L}^2}^4|Z|_{\mathbb{L}^2}^2+\frac{3}{4}\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2 \end{aligned}$$

Therefore, we get the following inequality for the 4^{th} term

$$\begin{aligned} I_4(t)&=\Big[\langle M_2(t)\times\Delta Z,Z\rangle-\langle Z\times\Delta M_1(t),Z\rangle\Big] \\ &\leqslant \frac{k^2}{\eta^2}|\mathrm{D}M_i|_{\mathbb{L}^2}^2|Z|_{\mathbb{L}^2}^2+\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2+\frac{k^4}{4\eta^6}|\mathrm{D}M_i|_{\mathbb{L}^2}^4|Z|_{\mathbb{L}^2}^2+\frac{3}{4}\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2 \end{aligned}$$

Next, we will deal with the 3^{rd} term. Since $|M_1|_{\mathbb{L}^\infty}=1$, the Hölder inequality yields

$$\begin{aligned} \langle \mathrm{D}M_jM_1\mathrm{D}Z,Z\rangle &\leqslant |\mathrm{D}M_j|_{\mathbb{L}^2}|M_1|_{\mathbb{L}^\infty}|\mathrm{D}Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^\infty}\leqslant \mathrm{D}M_j|_{\mathbb{L}^2}|\mathrm{D}Z|_{\mathbb{L}^2}|Z|_{\mathbb{L}^\infty} \\ &\leqslant \frac{k^2}{\eta^2}|\mathrm{D}M_i|_{\mathbb{L}^2}^2|Z|_{\mathbb{L}^2}^2+\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2+\frac{k^4}{4\eta^6}|\mathrm{D}M_i|_{\mathbb{L}^2}^4|Z|_{\mathbb{L}^2}^2+\frac{3}{4}\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2. \end{aligned}$$

Therefore, we get the following inequality for the 3^{rd} term

$$\begin{aligned} I_3(t)=\langle\big(\mathrm{D}M_1+\mathrm{D}M_2\big)M_1\mathrm{D}Z,Z\rangle &= \sum_{j=1}^{2}\langle \mathrm{D}M_jM_1\mathrm{D}Z,Z\rangle\leqslant\frac{k^2}{\eta^2}\Big(\sum_{i=1}^{2}|\mathrm{D}M_i|_{\mathbb{L}^2}^2\Big)|Z|_{\mathbb{L}^2}^2 \\ &+ \eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2+\frac{k^4}{4\eta^6}\Big(\sum_{i=1}^{2}|\mathrm{D}M_i|_{\mathbb{L}^2}^4\Big)|Z|_{\mathbb{L}^2}^2+\frac{3}{2}\eta^2|\mathrm{D}Z|_{\mathbb{L}^2}^2 \end{aligned}$$

By inequalities (4.5), (4.7) and (4.8) we get the following bound for the 5^{th}, 6^{th} and 7^{th} terms

$$\sum_{i=5,6,8} I_i(t)\leqslant C_1|Z(t)|_{\mathbb{L}^2}^2.$$

Finally, for the last term we get by (4.7)

$$I_7(t) \leqslant C_1|F(t)||Z(t)|^2_{\mathbb{L}^2}.$$

Finally, let us define an $\mathbb{R}$-valued process

$$\xi_9(t) := \int_0^t \sum_{j=1}^3 I_{9,j}(s)\, dW_j(s), \quad t \in [0,T].$$

Obviously, ξ_9 is an $\mathbb{L}^2$-valued martingale. Next we add together the terms containing $\eta^2|\mathrm{D}Z|^2_{\mathbb{L}^2}$ to obtain

$$\frac{19}{4}\eta^2|\mathrm{D}Z|^2_{\mathbb{L}^2} \leqslant 5\eta^2|\mathrm{D}Z|^2_{\mathbb{L}^2}\,.$$

Choosing η in such a way that $5\eta^2 = \frac{1}{2}$, for a number $C>0$ we introduce a process

$$\begin{aligned}
\varphi(t) = \varphi_C(t) &= C + k^2\Big[|\mathrm{D}M_2|^2_{\mathbb{L}^2} + \frac{k^2}{2\eta^2}|\mathrm{D}M_2(t)|^4_{\mathbb{L}^2} \\
&+ \sum_{i=1}^2 \frac{1}{\eta^2}|\mathrm{D}M_i(t)|^2_{\mathbb{L}^2} + \frac{k^2}{4\eta^6}\sum_{i=1}^2 |\mathrm{D}M_i(t)|^4_{\mathbb{L}^2}\Big] \\
&+ \frac{k^2}{\eta^2}|\mathrm{D}M_i(t)|^2_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}|\mathrm{D}M_i(t)|^4_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}\sum_{i=1}^2 |\mathrm{D}M_i(t)|^4_{\mathbb{L}^2}, \quad t \in [0,T].
\end{aligned}$$

From all our inequalities we infer that there exist a constant $C>0$ such that

$$|Z(t)|^2_{\mathbb{L}^2} \leqslant \int_0^t \varphi_C(s)\,|Z(s)|^2_{\mathbb{L}^2}\, ds + \xi_9(t), \quad t \in [0,T] \tag{4.11}$$

By the Itô Lemma applied to the following an $\mathbb{R}$-valued process,

$$Y(t) := |Z(t)|^2_{\mathbb{L}^2} e^{-\int_0^t \varphi_C(s)\, ds}, \quad t \in [0,T],$$

see [37] for a similar idea, we infer that

$$\begin{aligned}
Y(t) &\leqslant \int_0^t e^{-\int_0^s \varphi_C(r)\, dr}\, d\xi_9(s) \\
&= \sqrt{\varepsilon}\sum_{j=1}^3 \int_0^t e^{-\int_0^t \varphi_C(s)\, ds} \langle G\left(M_2(s)\right) - G\left(M_1(s)\right)\big)e_j, Z\rangle\, dW_j(s), \quad t \in [0,T].
\end{aligned}$$

Since M_1, M_2 and Z are uniformly bounded and G is locally Lipschitz the process defined by the RHS of the last inequality is an $\mathbb{F}$-martingale.

Thus, we infer that

$$\mathbb{E}Y(t) \leqslant 0, \quad t \in [0,T]\,,$$

and since Y is nonnegative, we deduce that $Y(t) = 0$, $\mathbb{P}$-a.s., for every $t \in [0,T]$. Finally, the definition of Y yields

$$Z(t) = 0 \quad \mathbb{P}-a.s., \quad \text{for every} \quad t \in [0,T]\,.$$

This completes the proof. □

Remark 4.3. Let us note first that the processes M_i, $i=1,2$ in Theorem 4.2 satisfy weaker conditions that those guaranteed by the existence result from Theorem 3.1. Hence our uniqueness result in Theorem 4.2 holds in the following sense.
Suppose that M_1 is a solution satisfying assumptions of Theorem 4.2 and M_2 a solution in the sense of Theorem 3.1, both defined on the same filtered probability space, then $M_1=M_2$.

5. Further regularity

In this section, we assume that a system $\Big(\Omega,\mathscr{F},\mathbb{F},\mathbb{P},W,M\Big)$ is a weak martingale solution to problem (3.1) such that M has paths in the space S_T defined by

$$S_T := C([0,T];\mathbb{H})\cap L^4(0,T;\mathbb{H}^1). \tag{5.1}$$

Some regularity properties of M are listed in Theorem 3.1. The main result of this section is Theorem 5.3, where we prove stronger regularity of the solution. In Proposition 5.5, we use this estimate to show that paths of M lie in $C([0,T];\mathbb{H}^1)$, $\mathbb{P}$-almost everywhere; this improves upon the continuity property in Theorem 3.1.
We start with a lemma that expresses M in a mild-form which allows us to exploit the regularizing properties of the semigroup (e^{-tA}). The proof of this well known fact is omitted, see for instance

Lemma 5.1. *For each $t\in[0,T]$, $\mathbb{P}$-a.s.*

$$\begin{aligned} M(t) = e^{-\alpha tA}M_0 \;&+\; \int_0^t e^{-\alpha(t-s)A}(M(s)\times\Delta M(s))ds + \alpha\int_0^t e^{-\alpha(t-s)A}\left(|\mathrm{D}M(s)|^2M(s)\right)\,ds \\ &+\; \varepsilon^{\frac12}\int_0^t e^{-\alpha(t-s)A}G(M(s))B\,dW(s) \\ &-\; \beta\int_0^t e^{-\alpha(t-s)A}G(M(s))F(M(s))\,ds \\ &+\; \frac{\varepsilon}{2}\sum_{i=1}^3\int_0^t e^{-\alpha(t-s)A}G'(M(s))e_iG(M(s))e_i\,ds. \end{aligned} \tag{5.2}$$

Before we state the main result of this section let us make the following important remark.

Remark 5.2. Suppose that the vector $\Delta M(t,x)\in\mathbb{R}^3$ is a.e. well defined and that

$$|M(t,x)|^2 = 1 \text{ a.e..}$$

Then we infer that

$$M(t,x)\cdot\Delta M(t,x) = -|\mathrm{D}M(t,x)|^2,\quad a.e.$$

and therefore, an elementary identity

$$|a\times b|^2+|a\cdot b|^2 = |a|^2\cdot|b|^2,\quad a,b\in\mathbb{R}^3,$$

yields

$$|M(t,x)\times\Delta M(t,x)|^2+|\mathrm{D}M(t,x)|^4 = |\Delta M(t,x)|^2,\quad t,x-a.e.$$

Theorem 5.3. *Assume that $p \in [1,\infty)$. Then for every $M_0 \in \mathbb{H}^1$ and $e = \big(e_i\big)_{i=1}^3 \in (\mathbb{H}^1)^3$ Then there exists a constant $C_p = C_p\left(\alpha, T, \|M_0\|_{\mathbb{H}^1}, |e|_{\mathbb{H}^1}\right)$ such that the unique solution M of the problem (3.1) satifies*

$$\mathbb{E}\left(\int_0^T |\mathrm{D}M(t)|_{\mathbb{L}^4}^4\,dt + \int_0^T |\Delta M(t)|_{\mathbb{L}^2}^2\,dt\right)^p \leqslant C_p. \tag{5.3}$$

Definition 5.4. *A weak martingale solution*

$$\Big(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, M\Big) \tag{5.4}$$

to problem (3.1) is called a martingale strong solution to to problem (3.1) iff it satisfies condition (5.3) for $p = 1$.

Proof. By the uniqueness it is sufficient to prove the theorem for the solution constructed in Theorem 3.1. Let us describe the structure of the proof. In Step 1 we will show the first part of inequality (5.3) for every $p \in [1,\infty)$. In Step 2 we will show the second part of inequality (5.3) for every $p = 1$. In step 3 we will use Step 2 and Remark 5.2 to deduce the second part of inequality (5.3) for every $p \in [1,\infty)$.

We will use repeatedly the following well known properties of the semigroup $\big(e^{-tA}\big)$. The semigroup $\big(e^{-tA}\big)$, where A is defined in 2.11, is ultracontractive, see, for example, [3], that is, there exists $C > 0$ such that if $1 \leqslant p \leqslant q \leqslant \infty$, then

$$|\mathrm{e}^{-tA} f|_{\mathbb{L}^q} \leqslant \frac{C}{t^{\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}} |f|_{\mathbb{L}^p}, \quad f \in \mathbb{L}^p,\ t > 0. \tag{5.5}$$

It is also well known that A has maximal regularity property, that is, there exists $C > 0$ such that for any $f \in L^2\left(0, T; \mathbb{H}\right)$ and

$$u(t) = \int_0^t e^{-(t-s)A} f(s)ds, \quad t \in [0,T],$$

we have

$$\int_0^T |Au(t)|_{\mathbb{H}}^2\,dt \leqslant C\int_0^T |f(t)|_{\mathbb{H}}^2\,dt. \tag{5.6}$$

Let us fix for the rest of the proof $T > 0$, and an auxiliary number $\delta \in \left(\frac{5}{8}, \frac{3}{4}\right)$. Let us also fix $\rho > 0$ and $r > 0$ such that $\|M_0\|_{\mathbb{H}^1} \leqslant \rho$ and $\|e\|_{\mathbb{H}^1} \leqslant r$.

- **Step 1** Let us additionally choose and fix $p \in [1,\infty)$. By Lemma 5.1 M can be written as a sum of six terms:

$$M(t) = \sum_{i=0}^{5} m_i(t),$$

and we will consider each term separately. In what follows, C stands for a generic constant that depends on p, T, α, ρ and r only. In order to simplify notation, we put,

without loss of generality, $\varepsilon=\alpha=\beta=1$.
We will show first that

$$\mathbb{E}\Big(\int_0^T |M(t)|^4_{\mathbb{W}^{1,4}}\,dt\Big)^p \leqslant C\,(p,T,\alpha,\rho,r)\,. \tag{5.7}$$

To this end we will prove a stronger estimate:

$$\mathbb{E}\Big(\int_0^T \Big|A^\delta M(t)\Big|^4_{\mathbb{L}^2}\,dt\Big)^p \leqslant C\,(p,T,\alpha,\rho,r)\,. \tag{5.8}$$

Then (5.7) will follow from the Sobolev imbedding $\mathbb{X}^\delta \hookrightarrow \mathbb{W}^{1,4}$.
We start with m_0. For each $t\in(0,T]$, we have

$$\Big|A^\delta e^{-tA}M_0\Big|^4_{\mathbb{L}^2} \leqslant \frac{C}{t^{4\delta-2}}\,|M_0|^4_{\mathbb{H}^1}\,,$$

and therefore, since $\delta<\frac{3}{4}$, we infer that

$$\int_0^T \Big|A^\delta m_0(t)\Big|^4_{\mathbb{L}^2}\,dt \leqslant C\,|M_0|^4_{\mathbb{H}^1}\,. \tag{5.9}$$

We will consider m_1. Putting $f=M\times\Delta M$ we have

$$|A^\delta e^{-(t-s)A}f(s)|_{\mathbb{L}^2} \leqslant C(t-s)^{-\delta}|f(s)|_{\mathbb{L}^2}, \quad 0<s<t<T,$$

hence applying the Young inequality we obtain

$$\begin{aligned}\int_0^T |A^\delta m_1(t)|^4_{\mathbb{L}^2}\,dt &\leqslant C\int_0^T\left(\int_0^t (t-s)^{-\delta}|f(s)|_{\mathbb{L}^2}\,ds\right)^4 dt\\ &\leqslant C\left(\int_0^T s^{-\frac{4\delta}{3}}\,ds\right)^3\left(\int_0^T |f(s)|^2_{\mathbb{L}^2}\,,ds\right)^2.\end{aligned}$$

Thereby, since $\frac{4\delta}{3}<1$, part (3) of Theorem 3.1 yields

$$\mathbb{E}\Big(\int_0^T |A^\delta m_1(t)|^4_{\mathbb{L}^2}\,dt\Big)^p \leqslant C(2p,T,\alpha,\rho,r). \tag{5.10}$$

Since for every $t\in[0,T]$, $|M(t,x)|=1$ almost everywhere, and $e_i\in\mathbb{H}^1$, $i=1,2,3$, the estimate (2.3) implies that there exists deterministic $c>0$ such that

$$\sum_{i=1}^3 |G(M)e_i|_{\mathbb{L}^2} + \sum_{i=1}^3 |G'(M)e_iG(M)e_i|_{\mathbb{L}^2} \leqslant c.$$

Therefore, the same arguments as for m_1 yield

$$\mathbb{E}\Big(\int_0^T |A^\delta m_5(t)|^4_{\mathbb{L}^2}\,dt\Big)^p \leqslant C(p,T,\alpha,\rho,r). \tag{5.11}$$

We will now consider the term m_2 using the fact that $f = |\mathrm{D}M|^2 M \in \mathbb{L}^1(0,T;\mathbb{L}^2)$. Invoking the semigroup property of e^{-tA} and the ultracontractive estimate (5.5) with $p=1$ and $q=2$ we find that there exists $C>0$ such that $\mathbb{P}$-a.s.

$$|A^\delta e^{-(t-s)A} f(s)|_{\mathbb{L}^2} \leqslant \frac{C}{(t-s)^{\delta+\frac14}} \sup_{r\in[0,T]} |M(r)|^2_{\mathbb{H}^1}, \quad 0<s<t\in[0,T].$$

Therefore,

$$\int_0^T \left| \int_0^t A^\delta e^{-(t-s)A} f(s)\,ds \right|^4_{\mathbb{L}^2} dt \leqslant C \sup_{r\in[0,T]} |M(r)|^8_{\mathbb{H}^1} \int_0^T \left(\int_0^t \frac{ds}{(t-s)^{\delta+\frac14}}\,ds \right)^4 dt$$

hence (since $\delta+\frac14<1$) Theorem 3.1 yields

$$\mathbb{E}\int_0^T \left|A^\delta m_2(t)\right|^4_{\mathbb{L}^2} dt \leqslant C(T,\rho,r). \tag{5.12}$$

In order to estimate m_3 we recall that there exist $a_r>0$ such that

$$\|G(M)e_i\|_{\mathbb{H}^1} \leqslant a_r(1+\|M\|^2_{\mathbb{H}^1}), \quad i=1,2,3. \tag{5.13}$$

Invoking Lemma 7.2 in [20] we find that for $i=1,2,3$ and any $t\in[0,T]$

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t A^\delta e^{-(t-s)A} G(M(s))e_i\,dW(s)\right|^4_{\mathbb{L}^2} \\
&\leqslant C(T)\mathbb{E}\left(\int_0^t |A^\delta e^{-\alpha(t-s)A} G(M(s))e_i|^2_{\mathbb{L}^2}\,ds\right)^2 \\
&= C(T)\mathbb{E}\left(\int_0^t |A^{\delta-\frac12} e^{-(t-s)A} A^{\frac12} G(M(s))e_i|^2_{\mathbb{L}^2}\,ds\right)^2 \\
&\leqslant C(T)\mathbb{E}\left(\int_0^t \frac{|G(M(s))e_i|^2_{\mathbb{H}^1}}{(t-s)^{2\delta-1}}\,ds\right)^2 \\
&\leqslant C(T)\mathbb{E}\sup_{r\in[0,T]}\left[1+|M(r)|^8_{\mathbb{H}^1}\right].
\end{aligned}$$

Thus, Theorem 3.1 now yields

$$\mathbb{E}\int_0^T |A^\delta m_3(t)|^4_{\mathbb{L}^2}\,dt \leqslant C(T,\rho,r). \tag{5.14}$$

Because by inequality (2.8),

$$|G(M)F(M)|_{\mathbb{H}^1} \leqslant aC(1+\|M\|^2_{\mathbb{H}^1})\|M\|^2_{\mathbb{H}^1}$$

the case of m_4 can be treated very easily.

Finally, combining estimates (5.9) to (5.14) we obtain (5.8) and (5.7) follows.

- **Step 2** We will prove that

$$\mathbb{E}\int_0^T |AM(t)|^2_{\mathbb{L}^2}\,dt \leqslant C(T,\rho,r). \tag{5.15}$$

To this end we note first that using the maximal inequality (5.15) and the first part of the proof it is easy to see that

$$\mathbb{E}\int_0^T |Am_i(t)|^2_{\mathbb{L}^2}\;dt \leqslant C(T,\rho,r), \quad i=1,2,4. \tag{5.16}$$

The estimate

$$\int_0^T |Am_0(t)|^2_{\mathbb{L}^2}\;dt \leqslant C\left(T,\rho\right), \tag{5.17}$$

is an immediate consequence of the fact that $M_0 \in \mathbb{H}^1 = D\left(A^{1/2}\right)$.
We will consider now the stochastic term m_3. Using (5.13), a result of Pardoux in [34] and part 1 of Theorem 3.1 we find that

$$\mathbb{E}\int_0^T |Am_3(t)|^2_{\mathbb{L}^2}dt \leqslant C\mathbb{E}\int_0^T \left(|M(t)|^2_{\mathbb{H}^1}+1\right)\,dt \leqslant C(T,\rho,r). \tag{5.18}$$

Combining (5.16), (5.17) and (5.18) we obtain (5.15).
- **Step 3** Take $p \geqslant 1$. By Step 2 and Remark we infer that

$$\mathbb{E}\left(\int_0^T |\Delta M(t)|^2_{\mathbb{L}^2}\;dt\right)^p = \mathbb{E}\left(\int_0^T\int_\Lambda |\Delta M(t.x)|^2\;dx\,dt\right)^p \tag{5.19}$$

$$= \quad \mathbb{E}\left(\int_0^T\int_\Lambda |M(t,x)\times\Delta M(t,x)|^2\,dx\,dt + \int_0^T\int_\Lambda |\mathrm{D}M(t,x)|^4\,dx\,dt\right)^p.$$

Hence the second part of inequality (5.3) in Theorem 5.3 folllows from the first part (proved above in Step 2) and inequality (3.4) from the Theorem 3.1 about the existence of weak solutions.

The proof is complete. □

Proposition 5.5. *$\mathbb{P}$ almost surely, the paths of M lie in the space $C([0,T];\mathbb{H}^1)$.*

Proof. The proposition follows easily from the results in [34]. □

Corollary 5.6. *Let $e_i \in H^1$, $i=1,2,3$. Let W be an $\mathbb{F}$ Wiener process defined on the probability space $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$. Then, for every $M_0 \in \mathbb{H}^1$ and $\varepsilon > 0$, there exists a unique* *pathwise solution* $M^\varepsilon \in C\left([0,T];\mathbb{H}^1\right) \cap L^2\left(0,T;D(A)\right)$ *of the problem* (3.1)*, i.e.*

$$\begin{aligned} M(t) = M_0 &+ \alpha \int_0^t \Delta M(s)\,ds + \alpha \int_0^t |\mathrm{D}M(s)|^2 M(s)\,ds + \int_0^t M(s)\times \Delta M(s)\,ds \\ &+ \frac{\varepsilon}{2}\sum_{i=1}^3 \int_0^t G'(M(s))e_i G(M(s))e_i\,ds + \sqrt{\varepsilon}\int_0^t G(M(s))B\,dW(s) \\ &- \beta \int_0^t G(M(s))F(M(s))\,ds, \end{aligned} \tag{5.20}$$

where all the integrals are the Bochner or the Itô integrals in $\mathbb{L}^2$*.*

In what follows we will denote by $\mathscr{X}_T$ the Banach space

$$\mathscr{X}_T = C\left([0,T];\mathbb{H}^1\right) \cap L^2(0,T;D(A)). \tag{5.21}$$

By an infinite-dimensional version of the Yamada and Watanabe Theorem, see [33, Theorems 12.1 (part 3) and 13.2], the pathwise uniqueness and the existence of weak solutions implies uniqueness in law and the existence of a strong solution. In Theorem 5.7 below, we state such a result for equation (3.1).

Using the additional regularity results proven in this section, we have the following result.

Theorem 5.7. *Let assumptions of Theorem 4.2 be satisfied. Then uniqueness in law and the existence of a strong solution holds for equation (3.1) in the following sense:*

(1) *if* $\left(\Omega,\mathscr{F},\mathbb{F},\mathbb{P},W,M\right)$ *and* $\left(\Omega',\mathscr{F}',\mathbb{F}',\mathbb{P}',W',M'\right)$ *are two martingale strong solutions to problem (3.1) such that both* M *and* M' *are* $\mathscr{X}_T$*-valued random variables, then* M *and* M' *have the same laws on* $\mathscr{X}_T$*;*

(2) *for every* $\varepsilon > 0$ *there exists a Borel measurable function*

$$J^\varepsilon : {}_0C\left([0,T];\mathbb{R}^3\right) := \left\{\omega \in C\left([0,T];\mathbb{R}^3\right) : \omega(0) = 0\right\} \to \mathscr{X}_T \tag{5.22}$$

such that for any filtered probability space $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$*, where the filtration* $\mathbb{F} = (\mathscr{F}_t)$ *is such that* $\mathscr{F}_0$ *contains all* $\mathbb{P}$*-null sets from* $\mathscr{F}$*, and for any* $\mathbb{R}^3$*-valued* $\mathbb{F}$*-Wiener process* $W = (W(t))_{t\in[0,T]}$*, the system* $\left(\Omega,\mathscr{F},\mathbb{F},\mathbb{P},W,M^\varepsilon\right)$*, where* $M^\varepsilon = J^\varepsilon \circ W$*, i.e.*

$$M^\varepsilon : \Omega \ni \omega \mapsto J^\varepsilon\big(W(\omega)\big) \in \mathscr{X}_T,$$

is a strong martingale solution[1] *to problem (3.1).*

6. The Large Deviations Principle

In this section we will prove the large deviation principle for the family of laws of the solutions M^ε of equation (3.1) with the parameter $\varepsilon \in (0,1]$ approaching zero and fixed $M_0 \in \mathbb{H}^1$.

In what follows we will denote by M^ε the unique strong martingale solution to the problem (3.1).

The main result in this section is as follows.

[1] In particular, M is $\mathbb{F}$-progressively measurable.

Theorem 6.1. *The family of laws* $\{\mathscr{L}(M^\varepsilon) : \varepsilon \in (0,1]\}$ *on* $\mathscr{X}_T$ *satisfies the large deviation principle with rate function* I *defined below in equation* (6.11).

Before we embark on the proof of the above result we will present the necessary background. In particular we will formulate crucial Lemmata 6.3 and 6.4. Then we will present the proof of Theorem 6.1. This will be followed by the proof of Lemma 6.3. The proof of Lemma 6.4 will be given at the very end of this section.

In order to prove the above result we will present some results due to Buhhiraja and Dupuis [14]. Following that paper we will formulate some two general claims. These claims will be consequence of Lemmata 6.3 and 6.4 which we first only formulate. This preliminary material will be followed by the proof of Theorem 6.1. This will then be followed by the proof of Lemma 6.3. The proof of Lemma 6.4 will be given at the very end of this section.

6.1. **Large Deviations Principle according to Buhhiraja and Dupuis.**

In order to prove the Large Deviations Principle formulated in Theorem 6.1 holds we need to consider an equation slightly more general than equation (3.1).

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, be the classical Wiener space, i.e.

$$\begin{aligned}
\Omega &= {}_0C\left([0,T];\mathbb{R}^3\right), \\
\mathbb{P} & \quad \text{is the Wiener measure on } \Omega, \\
W &= \big(W(t)=W_t\big)_{t\in[0,T]} \text{ is the canonical } \mathbb{R}^3\text{-valued Wiener process on } (\Omega,\mathbb{P}), \\
\mathbb{F} &= (\mathscr{F}_t)_{t\in[0,T]} \text{ is the } \mathbb{P}\text{-completion of the natural filtration } \mathbb{F}^0 = \big(\mathscr{F}^0_t\big)_{t\in[0,T]} \text{ generated by } W.
\end{aligned}$$

Note that filtration $\mathbb{F} = (\mathscr{F}_t)$ is such that $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets from $\mathscr{F}$.

By Theorem 5.7 for every $\varepsilon > 0$ there exists a Borel map

$$J^\varepsilon : {}_0C\left([0,T];\mathbb{R}^3\right) \to \mathscr{X}_T \tag{6.1}$$

the system $\Big(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, M^\varepsilon\Big)$, where

$$M^\varepsilon : \Omega \ni \omega \mapsto J^\varepsilon\big(W(\omega)\big) \in \mathscr{X}_T,$$

is a martingale strong solution[2] to problem (3.1).
By $\mathbb{E}$ we will denote the integration with respect to the measure $\mathbb{P}$.

Suppose that X is a separable Banach space. We say that an X-valued $\mathbb{F}$-predictable process $h : [0,T] \times {}_0C\left([0,T];\mathbb{R}^3\right) \to \mathrm{X}$ belongs to $\mathscr{P}_T(\mathrm{X})$ iff

$$\|h\|_T^2 = \operatorname*{ess\,sup}_{\omega\in\Omega} \int_0^T |h(t,\omega)|^2_{\mathrm{X}}\, dt < \infty. \tag{6.2}$$

[2]In particular, M is $\mathbb{F}$-progressively measurable.

Given $h \in \mathscr{P}_T(\mathbb{R}^3)$ we can consider an equation

$$\begin{aligned} M(t) = M_0 &+ \int_0^t [M \times \Delta M - \alpha M \times (M \times \Delta M)]\, ds \\ &+ \sqrt{\varepsilon} \int_0^t G(M)B\, dW(s) + \frac{\varepsilon}{2} \sum_{i=1}^3 \int_0^t \left[G'(M)e_i\right]\big(G(M)e_i\big)\, ds \\ &- \beta \int_0^t G(M)f(M)\, ds + \int_0^t G(M)Bh(s)\, ds, \quad t \in [0,T]. \end{aligned} \tag{6.3}$$

Theorem 6.2. *Assume that $h \in \mathscr{P}_T(\mathbb{R}^3)$ and $\varepsilon \in (0,1]$. Then there exists a process $\widetilde{M} = M^{h,\varepsilon}$ such that the system*

$$\left(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, \widetilde{M}\right)$$

is a strong martingale solution of problem (6.3) such that for every $p \geqslant 1$,

$$\mathbb{E} \sup_{t\in[0,T]} |\widetilde{M}(t)|^p_{\mathbb{H}^1} < \infty, \tag{6.4}$$

$$\mathbb{E}\left(\int_0^T |\mathrm{D}\widetilde{M}(t)|^4_{\mathbb{L}^4}\, dt + \int_0^T \left|\Delta \widetilde{M}(t)\right|^2_{\mathbb{L}^2}\, dt\right)^p < \infty. \tag{6.5}$$

New proof. **Part II**. The existence. Let us fix $\varepsilon > 0$. For any $h \in \mathscr{P}_T(\mathbb{R}^3)$ let us put

$$\tilde{\rho}_h = \exp\left(-\frac{1}{\sqrt{\varepsilon}} \int_0^T h(s)\, dW(s) + \frac{1}{2\varepsilon} \int_0^T |h(s)|^2\, ds\right). \tag{6.6}$$

and

$$\widetilde{W}_h(t) = W(t) + \frac{1}{\sqrt{\varepsilon}} \int_0^t h(s)\, ds, \quad t \in [0,T]. \tag{6.7}$$

Since $h \in \mathscr{P}_T(\mathbb{R}^3)$ we infer that

$$\mathbb{E}\big(\tilde{\rho}_h\big)^2 < \infty$$

and therefore there exists a probability measure $\tilde{\mathbb{P}}_h$ on $\mathscr{F}_T$ such that

$$\frac{d\mathbb{P}_h}{d\mathbb{P}} = \tilde{\rho}_h.$$

Invoking the Girsanov Theorem we find that the process $\widetilde{W}_h$ is a Wiener process on probability space $(\Omega, \mathbb{P}_h)$. Note that now $\Omega = {}_0C\left([0,T];\mathbb{R}^3\right)$. Therefore, by part (ii) of Theorem 5.7, if the process $\widetilde{M}$ is defined by

$$\widetilde{M} : \Omega \ni \omega \mapsto J^\varepsilon\big(\widetilde{W}_h(\omega)\big) \in \mathscr{X}_T$$

then the system

$$\left(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}_h, \widetilde{W}_h, \widetilde{M}\right)$$

is a strong martingale solution of problem (3.1). In particular, by Theorems 3.1 and 5.3,

$$\widetilde{\mathbb{E}} \sup_{t\in[0,T]} |\widetilde{M}(t)|_{\mathbb{H}^1}^{2p} < \infty, \;\; p \geqslant 1,$$

$$\widetilde{\mathbb{E}} \left(\int_0^T |\mathrm{D}\widetilde{M}(t)|_{\mathbb{L}^4}^4 \, dt + \int_0^T \left| \Delta \widetilde{M}(t) \right|_{\mathbb{L}^2}^2 \, dt \right)^{2p} < \infty, \;\; p \geqslant 1.$$

On the other hand, since $h \in \mathscr{P}_T(\mathbb{R}^3)$ we infer that

$$\widetilde{\mathbb{E}}\big(\tilde{\rho}_h\big)^{-2} < \infty \tag{6.8}$$

and therefore $\mathbb{P}$ is absolutely continuous w.r.t. $\mathbb{P}_h$ and

$$\frac{d\mathbb{P}}{d\mathbb{P}_h} = \tilde{\rho}_h^{-1}.$$

Therefore, by applying the Hölder inequality we infer that for any $p \geqslant 1$.

$$\mathbb{E} \sup_{t\in[0,T]} |\widetilde{M}(t)|_{\mathbb{H}^1}^{p} < \infty,$$

$$\mathbb{E} \left(\int_0^T |\mathrm{D}\widetilde{M}(t)|_{\mathbb{L}^4}^4 \, dt + \int_0^T \left| \Delta \widetilde{M}(t) \right|_{\mathbb{L}^2}^2 \, dt \right)^{p} < \infty.$$

Therefore, by a standard argument, we infer that the system

$$\left(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, \widetilde{M}\right)$$

is a strong martingale solution of problem (6.3), see e.g. Appendix A in [19]. This completes the existence proof.

Part II. Uniqueness follows from Theorem 4.2.

□

Let us note that that we have used the Girsanov Theorem only to prove the existence of a solution to problem (6.3). Having this done, we return to our fixed probability space with a fixed Wiener process.

Let now define a Borel map

$$J^0 : {}_0C([0,T];\mathbb{R}^3) \to \mathscr{X}_T.$$

If $x \in {}_0C([0,T];\mathbb{R}^3) \setminus {}_0H^{1,2}([0,T];\mathbb{R}^3)$, then we put $J^0(x) = 0$. If $x' = h$ for some $h \in L^2\left(0,T;\mathbb{R}^3\right)$, then by Corollary 5.6 there exist a unique function $y_h \in \mathscr{X}_T$ that is the unique solution of the equation

$$\begin{aligned} y_h(t) = M_0 \;\; &+ \int_0^t [y_h(s) \times \Delta y_h(s) - \alpha y_h(s) \times (y_h(s) \times \Delta y_h(s))] \, ds \\ &- \;\; \beta \int_0^t G\left(y_h\right) f\left(y_h\right) ds + \int_0^t G\left(y_h\right) Bh(s) \, ds, \end{aligned} \tag{6.9}$$

where the mapping G has been defined in (2.10). We put

$$J^0(x) := y_h, \quad x = \int_0^\cdot h(s)\,ds, h \in L^2\left(0,T;\mathbb{R}^3\right).$$

Let us note that in view of Lemma , problem can be written in the following equivalent form

$$\begin{aligned} y_h(t) = M_0 \quad &+ \quad \alpha \int\limits_0^t \Delta y_h\,ds + \alpha \int\limits_0^t |\mathrm{D} y_h|^2\, y_h\,ds + \int\limits_0^t y_h \times \Delta y_h\,ds \\ &- \quad \beta \int\limits_0^t G(y_h)\, f(y_h)\,ds + \int\limits_0^t G(y_h)\, Bh(s)\,ds, \end{aligned} \tag{6.10}$$

We can easily prove that the map $J^0 : {}_0H^{1,2}([0,T];\mathbb{R}^3) \to \mathscr{X}_T$ is continuous. Since ${}_0H^{1,2}([0,T];\mathbb{R}^3)$ is a Borel subset of ${}_0C([0,T];\mathbb{R}^3)$, we infer that the map $J^0 : {}_0C([0,T];\mathbb{R}^3) \to \mathscr{X}_T$ is Borel measurable. We define now the rate function $I : \mathscr{X}_T \to [0,\infty]$ by the formula

$$I(u) := \inf\left\{ \frac{1}{2}\int\limits_0^T |h(s)|^2\,ds : \; h \in L^2\left(0,T;\mathbb{R}^3\right) \text{ and } u = J^0\Big(\int_0^\cdot h(s)\,ds\Big) \right\}, \tag{6.11}$$

where $\inf \varnothing = \infty$.

In order to prove Theorem 6.1, i.e. that the family of laws $\{\mathscr{L}(J_0^\varepsilon(W)) : \varepsilon \in (0,1]\}$ satisfies the large deviation principle on $\mathscr{X}_T$ with the rate function I we will follow the weak convergence method of Budhiraja and Dupuis [14], see also Duan and Millet [21] and Chueshov and Millet [17]. To this end we need to show that the following two statements are true.

Statement 1. *For each $R > 0$, the set $\{y_h : h \in B_R\}$ is a compact subset of $\mathscr{X}_T$.*

In the above, for $R > 0$ we denote by B_R the closed call of radius R in the Hilbert space $L^2\left(0,T;\mathbb{R}^3\right)$ endowed with the weak topology.

Statement 2. *Assume that $R > 0$, that (ε_n) is an $(0,1]$-valued sequence convergent to 0, that (h_n) is a sequence of $\mathbb{R}^3$-valued $\mathbb{F}$-predictable $\mathbb{R}^3$-valued processes, indexed by $[0,T]$, such that $\|h\|_T \leqslant R$ on Ω and the laws $\mathscr{L}(h_n)$ converge weakly on B_R to the law $\mathscr{L}(h)$. Then the processes*

$${}_0C([0,T],\mathbb{R}^3) \ni \omega \mapsto J^{\varepsilon_n}\Big(\omega + \frac{1}{\sqrt{\varepsilon_n}}\int_0^\cdot h_{\varepsilon_m}(s)\,ds\Big) \in \mathscr{X}_T$$

converge in law on $\mathscr{X}_T$ to $J^0(\int_0^\cdot h(s)\,ds)$.

The remaining part of this section is devoted to the proof of these two statements.

Lemma 6.3. *Suppose that $(h_n) \subset L^2\left(0,T;\mathbb{R}^3\right)$ is a sequence converging weakly to h. Then the sequence y_{h_n} converges strongly to y_h in $\mathscr{X}_T$. In particular, for every $R > 0$, the mapping*

$$B_R \ni h \mapsto J^0\Big(\int_0^\cdot h(s)\,ds\Big) \in \mathscr{X}_T$$

is Borel.
In particular, if $R > 0$ and h and $\tilde{h}$ are two B_R-valued random variables, possibly defined on different probability spaces, with the same laws, then the laws of $\mathscr{X}_T$-valued random variables $\Omega \ni \omega \mapsto J^0\big(\int_0^\cdot h(s,\omega)\,ds\big) \in \mathscr{X}_T$ and $\tilde{\Omega} \ni \tilde{\omega} \mapsto J^0\big(\int_0^\cdot \tilde{h}(s,\tilde{\omega})\,ds\big) \in \mathscr{X}_T$ are also equal.

Lemma 6.4. *Assume that $R>0$ and that an $(0,1]$-valued sequence (ε_n) converges to 0 and (h_n) is an $\mathscr{P}_T(\mathbb{R}^3)$-valued sequence such that*

$$\sup_{n\in\mathbb{N}}\int_0^T |h_n(t)|^2\,dt \leqslant R, \quad \text{for every } \omega\in\Omega, \tag{6.12}$$

and $\mathscr{L}(h_n)$ converges to $\mathscr{L}(h)$ weakly on B_R. Then the sequence of $\mathscr{X}_T$-valued random variables

$${}_0C([0,T],\mathbb{R}^3)\ni\omega\mapsto J^{\varepsilon_n}(\omega+\frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h_n(s)\,ds)-J^0(\int_0^\cdot h_n(s)\,ds)\in\mathscr{X}_T$$

converges in probability to 0.

It seems that it will be useful to introduce some temporary notation. The process (of function) $J^0(\int_0^\cdot h(s)\,ds)$ will be denoted by $\Phi^0(h)$ and the process ${}_0C([0,T],\mathbb{R}^3)\ni\omega\mapsto J^\varepsilon(\omega+\frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h(s)\,ds)$ will be denoted by $\Phi^\varepsilon(h)$.

6.2. **Proof of the main result from this section.**

Proof of Theorem 6.1. Obviously Statement 1 follows from Lemma 6.3.

The proof we propose here seem to based on a new idea of using deterministic result from Statement 1 and the Skorokhod embedding theorem on a separable metric space B_R.
Now we will occupy ourselves with a proof of Statement 2. For this aim let us choose and fix that $R>0$. Consider also an $(0,1]$-valued sequence (ε_n) that is convergent to 0 and a sequence (h_n) of $(\mathbb{F})$-predictable processes satisfying condition (6.12) that converges to h in law on B_R. Then, the following claims hold true.

(a) the $\mathscr{X}_T$-valued random variables $\Phi^{\varepsilon_n}(h_n)-\Phi^0(h_n)$ of converges in probability to 0,
(b) $\Phi^0(h_n)$ converges in law on $\mathscr{X}_T$ to $\Phi^0(h)$.

Claim **(a)** follows from by Lemma 6.4.
To prove Claim **(b)** let us first recall that B_R is a separable metric space. By the assumptions, the laws on B_R of the sequence of laws $(\mathscr{L}(h_n))$ converges weakly to the law $\mathscr{L}(h)$. Hence, by the Skorodhod Theorem, see for example, [27, Theorem 4.30], there exists a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$, and, on that probability space, there exist B_R-valued random variables $\tilde{h}_n$ and $\tilde{h}$, with the same laws as h_n and h, such that $\tilde{h}_n\to\tilde{h}$ in B_R, pointwise on $\tilde{\Omega}$. By the main part of Lemma 6.3 this implies that

$$\Phi^0(\tilde{h}_n)\to\Phi^0(\tilde{h}) \text{ in } \mathscr{X}_T \text{ pointwise on } \tilde{\Omega}.$$

Moreover, by the second part of Lemma 6.3, the laws of $\Phi^0(\tilde{h}_n)$ and $\Phi^0(\tilde{h})$ are equal, respectively, to the laws of $\Phi^0(h_n)$ and $\Phi^0(h)$.

Note that we can choose a subsequence, without introducing a new notation such that

(a') the sequence $\Phi^{\varepsilon_n}(\tilde{h}_n)-\Phi^0(\tilde{h}_n)$ of $\mathscr{X}_T$-valued random variables converges to 0, $\tilde{\mathbb{P}}$- almost surely.

These two convergence results imply that $(\Phi^{\varepsilon_n}(h_n))$ converges in law on $\mathscr{X}_T$ to $\Phi^0(h)$. Indeed, for any globally Lipschitz continuous and bounded function $f:\mathscr{X}_T\to\mathbb{R}$, see Dudley

[22, Theorem 11.3.3] we have

$$\left|\int_{\mathscr{X}_T} f(x)\, d\mathscr{L}(\Phi^{\varepsilon_n}(h_n)) - \int_{\mathscr{X}_T} f(x)\, d\mathscr{L}(\Phi^0(h))\right| = \left|\int_{\mathscr{X}_T} f(x)\, d\mathscr{L}(\Phi^{\varepsilon_n}(\tilde{h}_n)) - \int_{\mathscr{X}_T} f(x)\, d\mathscr{L}(\Phi^0(\tilde{h}))\right|$$
$$= \left|\int_{\tilde{\Omega}} f\left(\Phi^{\varepsilon_n}(\tilde{h}_n)\right)\, d\tilde{\mathbb{P}} - \int_{\Omega} f(\Phi^0(\tilde{h}))\, d\tilde{\mathbb{P}}\right|$$
$$\leqslant \int_{\tilde{\Omega}} |f(\Phi^{\varepsilon_n}(\tilde{h}_n)) - f(\Phi^0(\tilde{h}_n))|\, d\tilde{\mathbb{P}} + \left|\int_{\tilde{\Omega}} f(\Phi^0(\tilde{h}_n))\, d\mathbb{P} - \int_{\tilde{\Omega}} f(\Phi^0(\tilde{h}))\, d\tilde{\mathbb{P}}\right|$$

Now, we observe that because $\Phi^0(\tilde{h}_n) \to \Phi^0(\tilde{h})$ a.s. and f is a bounded and continuous function, we infer that the 2nd term on the RHS converges to 0. The first term converges to 0 because it is bounded by $|f|_{Lip} \int_{\tilde{\Omega}} |\Phi^{\varepsilon_n}(h_n) - \Phi^0(h_n)|\, d\tilde{\mathbb{P}}$ and the sequence $\Phi^{\varepsilon_n}(h_n) - \Phi^0(h_n)$ is $\tilde{\mathbb{P}}$-a.s. convergent.

Therefore, Statement 2 is true as well and thus we conclude the proof of Theorem 6.1. □

6.3. **Proof of the auxiliary results.**

Proof of Lemma 6.3. Let us assume that $h_n \to h$ weakly in $L^2(0,T;\mathbb{R}^3)$. To simplify notation, we write y_n for y_{h_n}, y for y_h and set $u_n = y_n - y$. We have to show that $u_n \to 0$ in $\mathscr{X}_T$.

Let us put

$$R^2 = \sup_{n\in\mathbb{N}} \int_0^T |h_n|^2(s)\, ds. \tag{6.13}$$

By Theorems[3] 3.1 and 5.3 and the uniqueness of solutions, there exists a finite constant $C = C(T,\alpha,R,\rho)$, such that if $|M_0|_{\mathbb{H}^1} \leqslant \rho$, then

$$\sup_{t\in[0,T]} |y_n(t)|_{\mathbb{H}^1} \leqslant C, \quad n \in \overline{\mathbb{N}}, \tag{6.14}$$

$$\int_0^T \left(|\Delta y_n(s)|^2_{\mathbb{L}^2} + |\mathrm{D}y_n|^4_{\mathbb{L}^4}\right) ds \leqslant C, \quad n \in \overline{\mathbb{N}} \tag{6.15}$$

and

$$\int_0^T |y_n'(s)|^2_{\mathbb{L}^2}\, ds \leqslant C, \quad n \in \overline{\mathbb{N}}. \tag{6.16}$$

Let us also recall that

$$|y_n(t)(x)| = 1, \quad x \in \Lambda,\ t \in [0,T],\ n \in \overline{\mathbb{N}}. \tag{6.17}$$

The same properties hold for y. Hence, in particular,

$$|u_n(t)(x)| \leqslant 2, \quad x \in \Lambda,\ t \in [0,T],\ n \in \overline{\mathbb{N}}. \tag{6.18}$$

[3] In fact, the corresponding result for equations (6.9), or (6.10), could be proven directly without invoking the Girsanov Theorem.

Step 1. The above estimates, together with standard compactness results, see e.g. [39], imply that the sequence y_n has a subsequence, for which we do not introduce a separate notation, which converges weakly in $L^2(0,T;\mathbb{H}^2)$, strongly in $L^q(0,Y;\mathbb{H}^1)\cap C([0,T];\mathbb{L}^2)$, for any $q<\infty$ and in $C_{\mathrm{w}}([0,T];\mathbb{H}^1)$ to some $\bar{y}\in L^2(0,T;\mathbb{H}^2)\cap C([0,T];\mathbb{H}^1)$ such that $\bar{y}'\in L^2(0,T;\mathbb{L}^2)$. Standard argument, see e.g. section 7 of [11] imply that $\bar{y}$ is a unique solution of the problem (6.10). A deterministic version of our uniqueness result Theorem 4.2 implies then that, recall that $y=y_h$, $\bar{y}=y$. Using the subsequence argument, we deduce that the whole sequence y_n converges to y weakly in $L^2(0,T;\mathbb{H}^2)$, strongly in $L^q(0,Y;\mathbb{H}^1)\cap C([0,T];\mathbb{L}^2)$, for any $q<\infty$ and in $C_{\mathrm{w}}([0,T];\mathbb{H}^1)$.

Step 2. Let $q\in L^2(0,T;\mathbb{L}^2)$. We claim that

$$\lim_{n\to\infty}\left(\sup_{t\in[0,T]}\left|\int_0^t\langle q(s),u_n(s)\rangle_{\mathbb{L}^2}(h_n(s)-h(s))\,ds\right|\right)=0. \tag{6.19}$$

By *Step 1* we can assume that there exists an element $u_\infty\in C([0,T];\mathbb{L}^2)$ such that $u_n\to u_\infty$ in $C([0,T];\mathbb{L}^2)$. For $n\in\mathbb{N}\cup\{\infty\}$ we define an operator $\mathscr{K}_n:L^2\left(0,T;\mathbb{R}^3\right)\to{}_0C\left([0,T];\mathbb{R}^3\right)$ by the following formula

$$\mathscr{K}_n v(t)=\int_0^t\langle q(s),u_n(s)\rangle_{\mathbb{L}^2}\,v(s)\,ds,\quad t\in[0,T],\ \ v\in L^2\left(0,T;\mathbb{R}^3\right).$$

Each operator $\mathscr{K}_n$, is compact because the function $\langle q(\cdot),u_n(\cdot)\rangle_{\mathbb{L}^2}$ belongs to $L^2\left(0,T;\mathbb{R}\right)$. Moreover, since the sequence $\langle q(\cdot),u_n(\cdot)\rangle_{\mathbb{L}^2}$ converges strongly in $L^2\left(0,T;\mathbb{R}\right)$ to a function $\langle q(\cdot),u_\infty(\cdot)\rangle_{\mathbb{L}^2}$ we infer that

$$\lim_{n\to\infty}\|\mathscr{K}_n-\mathscr{K}_\infty\|=0.$$

Since

$$|\mathscr{K}_n\left(h_n-h\right)|_{{}_0C\left([0,T];\mathbb{R}^3\right))}\leqslant\|\mathscr{K}_n-\mathscr{K}_\infty\|\cdot|h_n-h|_{L^2([0,T];\mathbb{R}^3)}+|\mathscr{K}_\infty\left(h_n-h\right)|_{{}_0C\left([0,T];\mathbb{R}^3\right)},$$

the claim (6.19) follows immediately by the compactness of K_∞ because $h_n\to h$ weakly in $L^2\left([0,T];\mathbb{R}^3\right)$.

Step 3. We will show that

$$\lim_{n\to\infty}\Big[\sup_{t\in[0,T]}|u_n(t)|^2_{\mathbb{H}^1}+\alpha\int_0^T|\Delta u_n|^2_{\mathbb{L}^2}\,ds\Big]=0. \tag{6.20}$$

Without loss of generality we may assume that $e_2=e_3=0$ and we put $e=e_1\in\mathbb{H}^1$. In particular, we can assume that all functions h and h_n are $\mathbb{R}$-valued, i.e. $h,h_n\in L^2(0,T;\mathbb{R})$. Note that in this case, the last term in (6.10) reads

$$\int_0^t G\left(y_h\right)Bh(s)\,ds=\int_0^t h(s)G(y_h(s))e\,ds. \tag{6.21}$$

Let us recall, see (3.6), that for $N\in\mathbb{N}$, $\pi_N:\mathbb{L}^2\to\mathbb{H}_n$ is the orthogonal projection onto the finite dimensional subspace of $\mathbb{L}^2$ spanned by the first N eigenvectors of the Neumann Laplacian.

For the aim of proving (6.20) we will show that there exist $C>0$ such that for every $N\in\mathbb{N}$,

$$\limsup_{n\to\infty}\left(\sup_{t\in[0,T]}|\mathrm{D}u_n(t)|^2_{\mathbb{L}^2}+\alpha\int_0^T|\Delta u_n|^2_{\mathbb{L}^2}\,ds\right)\leqslant C\,|e-\pi_N e|^2_{\mathbb{L}^2}\,. \tag{6.22}$$

By (6.10), for each $n\in\mathbb{N}$, we have

$$\begin{aligned} u_n(t) &= \alpha\int_0^t\Delta u_n\,ds+\alpha\int_0^t\Big\{|\mathrm{D}y_n|^2\,y_n-|\mathrm{D}y|^2\,y\Big\}\,ds \\ &+ \int_0^t\Big\{y_n\times\Delta y_n-y\times\Delta y\Big\}\,ds-\beta\int_0^t\Big\{G\left(y_n\right)F\left(y_n\right)-G\left(y\right)F\left(y\right)\Big\}\,ds \\ &+ \int_0^t\Big\{G\left(y_n\right)Bh_n(s)-G\left(y\right)Bh(s)\Big\}\,ds, \end{aligned}\tag{6.23}$$

Therefore, by some simple algebraic manipulations, formula (6.21) and the linearity[4] of the function f, we infer that

$$\begin{aligned} u_n(t) &= \alpha\int_0^t\Delta u_n\,ds \\ &+ \alpha\int_0^t(|\mathrm{D}y_n|-|\mathrm{D}y|)(|\mathrm{D}y_n|+|\mathrm{D}y|)y_n\,ds+\alpha\int_0^t|\mathrm{D}y|^2u_n\,ds \\ &+ \int_0^t u_n\times\Delta y_n\,ds+\int_0^t y\times\Delta u_n\,ds \\ &- \beta\int_0^t G\left(y_n(s)\right)F\left(u_n(s)\right)\,ds-\beta\int_0^t\left(G\left(y_n(s)\right)-G(y(s))\right)f(y(s))\,ds \\ &+ \int_0^t h_n(s)\big[G\left(y_n(s)\right)e-G\left(y(s)\right)e\big]\,ds+\int_0^t\left[h_n(s)-h(s)\right]G(y(s))e\,ds. \end{aligned}\tag{6.24}$$

In order to prove (6.20) we could follow a standard method of getting a'priori bounds by invoking the Gronwall Lemma. This would work easily but the last term on the RHS of (6.24). In order to be able to deal with that term we could use Step 2. However this would work had the function e were more regular, at least from $\mathbb{H}^2$. For this purpose, we will introduce an approximation of e by a sequence of more regular functions, see below, and then prove, instead of (6.20), (6.22).

Since u_n is a strong solution of the above equation, by Lions-Magenes [32], we infer that

[4] In fact, Lipschitz property of f would be sufficient.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\mathrm{D}u_n(t)|^2 &= -\alpha|\Delta u_n(t)|^2 - \alpha\langle(|\mathrm{D}y_n(t)| - |\mathrm{D}y(t)|)(|\mathrm{D}y_n(t)| + |\mathrm{D}y(t)|)y_n(t), \Delta u_n(t)\rangle \\ &- \alpha\langle|\mathrm{D}y(t)|^2 u_n(t), \Delta u_n(t)\rangle - \langle u_n(t)\times\Delta y_n(t), \Delta u_n(t)\rangle \qquad (6.25)\\ &+ \beta\langle G(y_n(t))\,F(u_n(t)), \Delta u_n\rangle + \beta\langle (G(y_n(t)) - G(y(t)))\,f(y(t)), \Delta u_n\rangle \\ &- \langle h_n(t)\big[G(y_n(t))\,e - G(y(t))\,e\big], \Delta u_n\rangle - \langle [h_n(t)-h(t)]\,G(y(t))e, \Delta u_n\rangle\end{aligned}$$

Let us now fix an auxiliary natural number N. Subtracting and adding $\pi_N e$ in the last term of the above equality and using integration by parts we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\mathrm{D}u_n(t)|^2 &= -\alpha|\Delta u_n(t)|^2 - \alpha\langle(|\mathrm{D}y_n(t)| - |\mathrm{D}y(t)|)(|\mathrm{D}y_n(t)| + |\mathrm{D}y(t)|)y_n(t), \Delta u_n(t)\rangle \\ &- \alpha\langle|\mathrm{D}y(t)|^2 u_n(t), \Delta u_n(t)\rangle - \langle u_n(t)\times\Delta y_n(t), \Delta u_n(t)\rangle \qquad (6.26)\\ &+ \beta\langle G(y_n(t))\,F(u_n(t)), \Delta u_n(t)\rangle + \beta\langle (G(y_n(t)) - G(y(t)))\,f(y(t)), \Delta u_n(t)\rangle \\ &- \langle h_n(t)\big[G(y_n(t))\,e - G(y(t))\,e\big], \Delta u_n\rangle \\ &- \langle [h_n(t)-h(t)]\big(G(y(t))e - G(y(t))\pi_N e\big), \Delta u_n\rangle \\ &- \langle [h_n(t)-h(t)]\,\Delta\big(G(y(t))\pi_N e\big), u_n\rangle\end{aligned}$$

Let us show how we estimate each of the terms on the RHS above. All norm below, unless otherwise stated, are those in $\mathbb{L}^2$. We fix $\varepsilon > 0$. We begin with the 1st term. By the Young inequality and inequality (6.17) we have, where C is the constant from the GNI below,

$$-\alpha\langle(|\mathrm{D}y_n| - |\mathrm{D}y|)|\mathrm{D}y|y_n, \Delta u_n\rangle \leqslant \frac{\varepsilon}{2}|\Delta u_n|^2 + \frac{1}{2\varepsilon}|\mathrm{D}u_n|^2_{\mathbb{L}^4}|\mathrm{D}y|^2_{\mathbb{L}^4}$$

Note that by the Gagliardo-Nirenberg inequality (and again and inequality (6.17))

$$|\mathrm{D}u_n|^2_{\mathbb{L}^4} \leqslant C\big[|u_n|^2 + |\Delta u_n|^2\big]^{1/2}|u_n|_{\mathbb{L}^\infty} \leqslant C\big[|u_n| + |\Delta u_n|\big]|u_n|_{\mathbb{H}^1}$$

Hence

$$|\mathrm{D}u_n|^2_{\mathbb{L}^4}|\mathrm{D}y|^2_{\mathbb{L}^4} \leqslant \frac{\varepsilon^2}{2}|\Delta u_n|^2 + (C + \frac{1}{2\varepsilon^2})|\mathrm{D}y|^4_{\mathbb{L}^4}|u_n|^2_{\mathbb{H}^1}$$

Therefore,

$$-\alpha\langle(|\mathrm{D}y_n| - |\mathrm{D}y|)|\mathrm{D}y|y_n, \Delta u_n\rangle \leqslant \frac{3\varepsilon}{4}|\Delta u_n|^2 + (\frac{C}{2\varepsilon} + \frac{1}{4\varepsilon^3})|\mathrm{D}y|^4_{\mathbb{L}^4}|u_n|^2_{\mathbb{H}^1}. \qquad (6.27)$$

Similarly, we also get

$$-\alpha\langle(|\mathrm{D}y_n| - |\mathrm{D}y|)|\mathrm{D}y_n|y_n, \Delta u_n\rangle \leqslant \frac{3\varepsilon}{4}|\Delta u_n|^2 + \frac{1}{4\varepsilon^3})|\mathrm{D}y_n|^4_{\mathbb{L}^4}|u_n|^2_{\mathbb{H}^1}. \qquad (6.28)$$

In an almost identical way, where instead of inequality (6.17) we use (6.18), we get

$$-\alpha\langle|\mathrm{D}y|^2 u_n, \Delta u_n\rangle \leqslant \frac{\varepsilon}{2}|\Delta u_n|^2 + \frac{2\alpha^2}{\varepsilon}|\mathrm{D}y|^4_{\mathbb{L}^4}|u_n|^2. \qquad (6.29)$$

Next, with $C > 0$ such that $|u|^2_{\mathbb{L}^\infty} \leqslant |u|^2_{\mathbb{H}^1}$ we have

$$\begin{aligned}-\langle u_n\times\Delta y_n, \Delta u_n\rangle &\leqslant \frac{\varepsilon}{2}|\Delta u_n|^2 + \frac{1}{2\varepsilon}|u_n|^2_{\mathbb{L}^\infty}|\Delta y_n|^2 \\ &\leqslant \frac{\varepsilon}{2}|\Delta u_n|^2 + \frac{C}{2\varepsilon}|\Delta y_n|^2|u_n|^2_{\mathbb{H}^1} \qquad (6.30)\end{aligned}$$

The next two terms are easy. By inequalities (6.17) and (6.18), and the Lipschitz continuity of functions f and g on balls we infer that

$$\beta\langle G\left(y_n(t)\right)F\left(u_n(t)\right), \Delta u_n\rangle \leqslant \frac{\varepsilon}{2}|\Delta u_n|^2 + \frac{C\beta^2}{2\varepsilon}|u_n|^2 \tag{6.31}$$

and

$$\beta\langle (G\left(y_n(t)\right) - G(y(t)))\, f(y(t)), \Delta u_n\rangle \leqslant \frac{\varepsilon}{2}|\Delta u_n|^2 + \frac{C\beta^2}{2\varepsilon}|u_n|^2 \tag{6.32}$$

Next, by inequality (6.17) and the Lipschitz continuity of function g on balls we infer that

$$\langle h_n(t)\big[G\left(y_n(t)\right)e - G\left(y(t)\right)e\big], \Delta u_n(t)\rangle \leqslant \frac{\varepsilon}{2}|\Delta u_n|^2 + \frac{C}{2\varepsilon}|h_n(t)|^2|u_n(t)|^2 \tag{6.33}$$

$$\langle [h_n(t) - h(t)]\big(G(y(t))e - G(y(t))\pi_N e\big), \Delta u_n\rangle \leqslant \frac{\varepsilon}{2}|\Delta u_n|^2 + \frac{C}{2\varepsilon}|h_n(t) - h(t)|^2|e - \pi_N e|^2 \tag{6.34}$$

We leave the last term unchanged. From all the inequalities above we deduce that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\mathrm{D}u_n(t)|^2 \quad &+ \quad \alpha|\Delta u_n(t)|^2 \\
&\leqslant \quad \frac{3\varepsilon}{4}|\Delta u_n(t)|^2 + (\frac{C}{2\varepsilon} + \frac{1}{4\varepsilon^3})|\mathrm{D}y(t)|^4_{\mathbb{L}^4}|u_n(t)|^2_{\mathbb{H}^1} \\
&+ \quad \frac{3\varepsilon}{4}|\Delta u_n(t)|^2 + (\frac{C}{2\varepsilon} + \frac{1}{4\varepsilon^3})|\mathrm{D}y_n(t)|^4_{\mathbb{L}^4}|u_n(t)|^2_{\mathbb{H}^1} \\
&+ \quad \frac{\varepsilon}{2}|\Delta u_n(t)|^2 + \frac{2\alpha^2}{\varepsilon}|\mathrm{D}y(t)|^4_{\mathbb{L}^4}|u_n(t)|^2 \\
&+ \quad \frac{\varepsilon}{2}|\Delta u_n(t)|^2 + \frac{C}{2\varepsilon}|\Delta y_n|^2|u_n(t)|^2_{\mathbb{H}^1} \\
&+ \quad \frac{\varepsilon}{2}|\Delta u_n(t)|^2 + \frac{C\beta^2}{2\varepsilon}|u_n(t)|^2 \\
&+ \quad \frac{\varepsilon}{2}|\Delta u_n(t)|^2 + \frac{C\beta^2}{2\varepsilon}|u_n(t)|^2 \\
&+ \quad \frac{\varepsilon}{2}|\Delta u_n(t)|^2 + \frac{C}{2\varepsilon}|h_n(t)|^2|u_n(t)|^2 \\
&+ \quad \frac{\varepsilon}{2}|\Delta u_n(t)|^2 + \frac{C}{2\varepsilon}|h_n(t) - h(t)|^2|e - \pi_N e|^2 \\
&- \quad \langle [h_n(t) - h(t)]\,\Delta\big(G(y(t))\pi_N e\big), u_n(t)\rangle
\end{aligned}$$

Let us now choose $\varepsilon = \frac{\alpha}{9} > 0$, i.e. such that

$$\frac{\alpha}{2} = \big(2\times\frac{3}{4} + 6\times\frac{1}{2}\big)\varepsilon = \frac{9}{2}\varepsilon.$$

Then we get

$$\frac{d}{dt}|\mathrm{D}u_n(t)|^2 \quad + \quad \alpha|\Delta u_n(t)|^2 \tag{6.35}$$

$$\leqslant \quad \psi_n(t)|u_n(t)|^2_{\mathbb{H}^1} + \chi_n(t)|u_n(t)|^2_{\mathbb{L}^2} + \frac{9C}{2\alpha}|h_n(t) - h(t)|^2|e - \pi_N e|^2 \tag{6.36}$$

$$- \quad \langle [h_n(t) - h(t)]\,\Delta\big(G(y(t))\pi_N e\big), u_n(t)\rangle$$

where

$$\psi_n(t) = \left(\frac{9C}{2\alpha}+\frac{9^3}{4\alpha^3}\right)\left(|\mathrm{D}y(t)|_{\mathbb{L}^4}^4+|\mathrm{D}y_n(t)|_{\mathbb{L}^4}^4\right)+\frac{9C}{2\alpha}|\Delta y_n|^2 \tag{6.37}$$

$$\chi_n(t) = \frac{18\alpha^2}{\alpha}|\mathrm{D}y(t)|_{\mathbb{L}^4}^4+\frac{9C\beta^2}{\alpha}+\frac{9C}{2\alpha}|h_n(t)|^2 \tag{6.38}$$

Therefore, with

$$b_{n,N} := \sup_{t\in[0,T]} |\langle [h_n(t)-h(t)]\,\Delta\big(G(y(t))\pi_N e\big), u_n(t)\rangle|$$

we infer that for any $t\in[0,T]$,

$$|\mathrm{D}u_n(t)|^2 + \alpha\int_0^t |\Delta u_n(s)|^2\,ds \tag{6.39}$$

$$\leqslant \int_0^t \psi_n(s)|u_n(s)|_{\mathbb{H}^1}^2\,ds+\int_0^T \chi_n(s)|u_n(s)|_{\mathbb{L}^2}^2\,ds \tag{6.40}$$

$$+ \frac{9C}{\alpha}|e-\pi_N e|^2\int_0^T\left(|h_n(s)|^2+|h(s)|^2\right)ds+b_{n,N}.$$

Therefore, by the Gronwall Lemma and our assumption (6.13) we get

$$|\mathrm{D}u_n(t)|^2 \leqslant \Big[\frac{18CR^2}{\alpha}|e-\pi_N e|^2+b_{n,N}+\int_0^T \chi_n(s)|u_n(s)|_{\mathbb{L}^2}^2\,ds\Big]\,e^{\int_0^t\psi_n(s)},\quad t\in[0,T].$$

By estimates (6.14) and (6.15)

$$\gamma := \sup_{n\in\mathbb{N}}\int_0^T \psi_n(s)\,ds<\infty$$

and γ depends on α, T, R, ρ and r only. Therefore, we infer that there exists a constant $C_T>0$ such that

$$\sup_{t\in[0,T]}|u_n(t)|_{\mathbb{H}^1}^2+\alpha\int_0^T|\Delta u_n|_{\mathbb{L}^2}^2\,ds\leqslant C_T e^{\gamma T}\Big[\frac{18CR^2}{\alpha}|e-\pi_N e|^2+b_{n,N}+\int_0^T\chi_n(s)|u_n(s)|_{\mathbb{L}^2}^2\,ds\Big]. \tag{6.41}$$

Therefore, since by Claim(6.19) $b_{n,N}\to 0$ as $n\to\infty$, and, by Step 1, $\int_0^T \chi_n(s)|u_n(s)|_{\mathbb{L}^2}^2\,ds$ converges to 0, we conclude the proof of (6.22) and so of (6.20) as well.

Step 4. We complete the proof of Lemma 6.3 by taking the limit as $N\to\infty$. □

Note, that Statement 1 follows Lemma 6.3.

Now we will occupy ourselves with the proof of that Statement 2. For this purpose let us chhose and fix the following processes:

$$Y_n=\Phi^{\varepsilon_n}(h_n)\text{ and }y_n=\Phi^0(h_n).$$

Let $N>|M_0|_{\mathbb{H}^1}$ be fixed. For each $n\in\mathbb{N}$ we define an $(\mathscr{F}_t)$-stopping time

$$\tau_n=\inf\{t>0:|Y_n(t)|_{\mathbb{H}^1}\geqslant N\}\wedge T. \tag{6.42}$$

Lemma 6.5. *For τ_n as defined in (6.42) we have*

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{t\in[0,T]}\left|Y_n\left(t\wedge\tau_n\right)-y_n\left(t\wedge\tau_n\right)\right|^2_{\mathbb{L}^2}+\int_0^{\tau_N}\left|Y_n-y_n\right|^2_{\mathbb{H}^1}\,ds\right)=0.$$

Proof. Let $X_n=Y_n-y_n$. We assume without loss of generality that $\beta=0$, $e_2=e_3=0$ and $e_1=h$. Then for any $n\in\mathbb{N}$ we have

$$\begin{aligned}dX_n=\alpha\Delta X_n dt&\\&+\alpha\left(\mathrm{D}X_n\right)\cdot\left(\mathrm{D}\left(Y_n+y_n\right)\right)Y_n dt+\alpha\left|\mathrm{D}y_n\right|^2X_n dt\\&+X_n\times\Delta Y_n dt+y_n\times\Delta X_n dt\\&+\left(G\left(Y_n\right)-G\left(y_n\right)\right)hh_n dt\\&+\sqrt{\varepsilon_n}G\left(Y_n\right)hdW+\frac{\varepsilon_n}{2}G'\left(Y_n\right)G\left(Y_n\right)hdt\end{aligned}\tag{6.43}$$

Using a version of the Itô formula given in [34] and integration by parts we obtain

$$\begin{aligned}\frac{1}{2}d\left|X_n\right|^2_{\mathbb{L}^2}=-\alpha\left|X_n\right|^2_{\mathbb{H}^1}dt+\alpha\left|\left|\mathrm{D}y_n\right|X_n\right|^2_{\mathbb{L}^2}dt&\\+\alpha\left\langle X_n,\left(\mathrm{D}X_n\right)\cdot\left(\mathrm{D}\left(Y_n+y_n\right)\right)Y_n\right\rangle_{\mathbb{L}^2}dt&\\-\left\langle \mathrm{D}X_n,X_n\times\mathrm{D}y_n\right\rangle_{\mathbb{L}^2}dt&\\+\left\langle\left(G\left(Y_n\right)-G\left(y_n\right)\right)h,X_n\right\rangle_{\mathbb{L}^2}h_n dt&\\+\frac{\varepsilon_n}{2}z_n dt+\sqrt{\varepsilon_n}\left\langle G\left(Y_n\right)h,X_n\right\rangle_{\mathbb{L}^2}dW&\end{aligned}$$

where z_n is a process defined by

$$z_n=\left\langle G'\left(Y_n\right)G\left(Y_n\right)h,X_n\right\rangle_{\mathbb{L}^2}+\left|G\left(Y_n\right)h\right|^2_{\mathbb{L}^2}.$$

Therefore

$$\begin{aligned}\left|X_n(t)\right|^2_{\mathbb{L}^2}+2\alpha\int_0^t\left|X_n\right|^2_{\mathbb{H}^1}ds\leqslant C\int_0^t\left|X_n\right|_{\mathbb{L}^2}\left|X_n\right|_{\mathbb{H}^1}\left|y_n\right|^2_{\mathbb{H}^1}ds&\\+C\int_0^t\left|X_n\right|^{3/2}_{\mathbb{H}^1}\left|X_n\right|_{\mathbb{L}^2}\left(\left|y_n\right|_{\mathbb{H}^1}+\left|y_n\right|_{\mathbb{H}^1}\right)ds&\\+C\int_0^t\left|X_n\right|^{3/2}_{\mathbb{H}^1}\left|X_n\right|_{\mathbb{L}^2}\left|y_n\right|_{\mathbb{H}^1}ds&\\+C\int_0^t\left|X_n\right|^2_{\mathbb{L}^2}\left|h_n\right|ds&\\+C\varepsilon_n+\sqrt{\varepsilon_n}\left|\int_0^t\left\langle G\left(Y_n\right)h,X_n\right\rangle_{\mathbb{L}^2}dW\right|.&\end{aligned}$$

By (6.14) we have $\sup_n |y_n|_{\mathbb{H}^1} < \infty$ and therefore, using repeatedly the Young inequality we find that there exists $C > 0$ such that for all $t \in [0,T]$

$$|X_n(t)|^2_{\mathbb{L}^2} + \alpha \int_0^t |X_n|^2_{\mathbb{H}^1}\, ds \leqslant C \int_0^t |X_n|^2_{\mathbb{L}^2} \left(1 + |h_n| + \beta\, |y_n|^4_{\mathbb{H}^1}\right) ds$$
$$+ C\varepsilon_n + \sqrt{\varepsilon_n} \left| \int_0^t \langle G\left(Y_n\right) h, X_n \rangle_{\mathbb{L}^2}\, dW \right|.$$

Denoting the left hand side of the above inequality by L_t and using the definition of τ_n we have

$$\begin{aligned} L_{t\wedge\tau_n} &\leqslant C \int_0^{t\wedge\tau_n} |X_n|^2_{\mathbb{L}^2} \left(1 + |h_n| + \beta\, |y_n|^4_{\mathbb{H}^1}\right) ds \\ &\quad + C\varepsilon_n + \sqrt{\varepsilon_n} \left| \int_0^{t\wedge\tau_n} \langle G\left(Y_n\right) h, X_n \rangle_{\mathbb{L}^2}\, dW \right| \\ &\leqslant \int_0^{t\wedge\tau_n} |X_n|^2_{\mathbb{L}^2}\, \psi_{n,N} ds + C\varepsilon_n \\ &\quad + \sqrt{\varepsilon_n} \left| \int_0^{t\wedge\tau_n} \langle G\left(Y_n\right) h, X_n \rangle_{\mathbb{L}^2}\, dW \right|, \end{aligned}$$

where

$$\psi_{n,N}(s) = 1 + |h_n(s)| + \beta N^4, \quad s \leqslant T.$$

Since

$$\sup_n \sup_{t\in[0,T]} \langle G\left(Y_n(t)\right), X_n(t) \rangle^2 \leqslant C, \quad \mathbb{P} - a.s.,$$

the Burkoholder-Davis-Gundy inequality yields

$$\mathbb{E} \sup_{s\leqslant t} L_{s\wedge\tau_n} \leqslant C\sqrt{\varepsilon_n} + \int_0^t \mathbb{E} \sup_{r\leqslant s} |X_n\left(r\wedge\tau_n\right)|^2_{\mathbb{L}^2}\, \psi_{n,N} ds, \tag{6.44}$$

and therefore

$$\mathbb{E} \sup_{r\leqslant t} |X_n(r\wedge\tau_n)|^2_{\mathbb{L}^2} \leqslant C\sqrt{\varepsilon_n} + \int_0^t \mathbb{E} \sup_{r\leqslant s} |X_n\left(r\wedge\tau_n\right)|^2_{\mathbb{L}^2}\, \psi_{n,N} ds.$$

Clearly,

$$\sup_{n\in\mathbb{N}} \int_0^T \psi_{n,N} ds < \infty,$$

hence the Gronwall Lemma implies

$$\mathbb{E}\sup_{r\in[0,T]}|X_n(r\wedge\tau_n)|^2_{\mathbb{L}^2}\leqslant C\sqrt{\varepsilon_n}\mathrm{e}^{\int_0^T\psi_{n,N}ds}\to 0 \text{ as } n\to\infty.$$

Returning now to (6.44), we also have

$$\mathbb{E}\int_0^{\tau_n}|X_n(s)|^2_{\mathbb{H}^1}\,ds\leqslant C\sqrt{\varepsilon_n}\mathrm{e}^{\int_0^T\psi_{n,N}ds}\to 0 \text{ as } n\to\infty.$$

This completes the proof of Lemma 6.5. □

Lemma 6.6. *For the stopping time τ_n defined in (6.42) we have*

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{t\in[0,T]}|\mathrm{D}(Y_n(t\wedge\tau_n)-y_n(t\wedge\tau_n))|^2_{\mathbb{L}^2}+\int_0^{\tau_n}|\Delta(Y_n-y_n)|^2_{\mathbb{L}^2}\,ds\right)=0.$$

Proof. By a version of the Itô formula, see [34],

$$\frac{1}{2}d|\mathrm{D}(Y_n(t)-y_n(t))|^2_{\mathbb{L}^2}=-\left\langle\Delta\left(Y_n-y_n\right),d\left(Y_n-y_n\right)\right\rangle_{\mathbb{L}^2}+\varepsilon_n\left|\mathrm{D}G\left(Y_n\right)h\right|^2_{\mathbb{L}^2}dt.$$

Therefore, putting $X_n=Y_n-y_n$ and invoking equality (6.43) we obtain for any $\eta>0$

$$\begin{aligned}\frac{1}{2}d\left|\mathrm{D}X_n(t)\right|^2_{\mathbb{L}^2}=&-\alpha\left|\Delta X_n\right|^2_{\mathbb{L}^2}\\&-\alpha\left\langle\Delta X_n,\mathrm{D}X_n\cdot\left(\mathrm{D}Y_n+\mathrm{D}y_n\right)Y_n\right\rangle_{\mathbb{L}^2}dt\\&-\alpha\left\langle\Delta X_n,\left|\mathrm{D}y_n\right|^2X_n\right\rangle_{\mathbb{L}^2}dt\\&-\left\langle X_n\times\Delta y_n,\Delta X_n\right\rangle_{\mathbb{L}^2}dt\\&-\left\langle\left(G\left(Y_n\right)-G\left(y_n\right)\right)h,\Delta X_n\right\rangle_{\mathbb{L}^2}h_ndt\\&-\sqrt{\varepsilon_n}\left\langle\mathrm{D}G\left(Y_n\right)h,\Delta X_n\right\rangle_{\mathbb{L}^2}dW\\&-\frac{\varepsilon_n}{2}\left\langle G'\left(Y_n\right)G\left(Y_n\right)h,\Delta X_n\right\rangle_{\mathbb{L}^2}dt\\&+\varepsilon_n\left|\mathrm{D}G\left(Y_n\right)h\right|^2_{\mathbb{L}^2}dt.\end{aligned}\tag{6.45}$$

We will estimate the terms in (6.45). First, noting that

$$\left\langle X_n\times\Delta Y_n,\Delta X_n\right\rangle_{\mathbb{L}^2}=\left\langle X_n\times\Delta u_n,\Delta X_n\right\rangle_{\mathbb{L}^2}$$

we find that

$$\left|\left\langle X_n\times\Delta Y_n,\Delta X_n\right\rangle_{\mathbb{L}^2}\right|\leqslant C\eta^2\left|\Delta X_n\right|^2_{\mathbb{L}^2}+\frac{C}{\eta^2}\left|X_n\right|_{\mathbb{L}^2}\left|X_n\right|_{\mathbb{H}^1}.\tag{6.46}$$

Next, by the Young inequality and the interpolation inequality (2.12)

$$\begin{aligned}
|\langle \Delta X_n, \mathrm{D}X_n \cdot (\mathrm{D}Y_n + \mathrm{D}y_n)\, Y_n\rangle_{\mathbb{L}^2}| &\leqslant C\eta^2 \left|\Delta X_n\right|^2_{\mathbb{L}^2} \\
&\quad + \frac{C}{\eta^2} \int\limits_\Lambda \left|\mathrm{D}X_n\right|^2 \left(\left|\mathrm{D}Y_n\right|^2 + \left|\mathrm{D}y_n\right|^2\right) dx \\
&\leqslant C\eta^2 \left|\Delta X_n\right|^2_{\mathbb{L}^2} \\
&\quad + \frac{C}{\eta^2} \left|\mathrm{D}X_n\right|^2_\infty \int\limits_\Lambda \left(\left|\mathrm{D}Y_n\right|^2 + \left|\mathrm{D}y_n\right|^2\right) dx \\
&\leqslant C\eta^2 \left|\Delta X_n\right|^2_{\mathbb{L}^2} \\
&\quad + \frac{C}{\eta^2} \left|X_n\right|_{\mathbb{H}^1} \left(\left|X_n\right|_{\mathbb{H}^1} + \left|\Delta X_n\right|_{\mathbb{L}^2}\right) \left(\left|Y_n\right|^2_{\mathbb{H}^1} + \left|y_n\right|^2_{\mathbb{H}^1}\right)
\end{aligned},$$

and thereby

$$\begin{aligned}
|\langle \Delta X_n, \mathrm{D}X_n \cdot (\mathrm{D}Y_n + \mathrm{D}y_n)\, Y_n\rangle_{\mathbb{L}^2}| &\leqslant C\eta^2 \left|\Delta X_n\right|^2_{\mathbb{L}^2} \\
&\quad + \frac{C}{\eta^2} \left|X_n\right|^2_{\mathbb{H}^1} \left(\left|Y_n\right|^2_{\mathbb{H}^1} + \left|y_n\right|^2_{\mathbb{H}^1}\right) \\
&\quad + \frac{C}{\eta^6} \left|X_n\right|^2_{\mathbb{H}^1} \left(\left|Y_n\right|^4_{\mathbb{H}^1} + \left|y_n\right|^4_{\mathbb{H}^1}\right) \\
&\leqslant C\eta^2 \left|\Delta X_n\right|^2_{\mathbb{L}^2} + C_\eta \left|X_n\right|^2_{\mathbb{H}^1} \left(1 + \left|Y_n\right|^4_{\mathbb{H}^1}\right).
\end{aligned} \tag{6.47}$$

Finally, using (2.12) we obtain

$$\begin{aligned}
|\left\langle \Delta X_n, \left|\mathrm{D}y_n\right|^2 X_n \right\rangle_{\mathbb{L}^2}| &\leqslant \left|\Delta X_n\right|_{\mathbb{L}^2} \left|\mathrm{D}y_n\right|_{\mathbb{L}^\infty} \left|\mathrm{D}y_n\right|_{\mathbb{L}^2} \left|X_n\right|_{\mathbb{L}^\infty} \\
&\leqslant C\eta^2 \left|\Delta X_n\right|^2_{\mathbb{L}^2} \\
&\quad + \frac{C}{\eta^2} \left|y_n\right|_{\mathbb{H}^1} \left(\left|y_n\right|_{\mathbb{H}^1} + \left|\Delta y_n\right|_{\mathbb{L}^2}\right) \left|y_n\right|^2_{\mathbb{H}^1} \left|X_n\right|_{\mathbb{L}^2} \left|X_n\right|_{\mathbb{H}^1}.
\end{aligned} \tag{6.48}$$

Taking into account (6.46), (6.47) and (6.48) we obtain from (6.45)

$$
\begin{aligned}
|\mathrm{D}X_n(t)|^2_{\mathbb{L}^2} + 2\alpha \int_0^t |\Delta X_n|^2_{\mathbb{L}^2}\,ds \leqslant{}& C\eta^2 \int_0^t |\Delta X_n|^2_{\mathbb{L}^2}\,ds \\
&+ C_\eta \sup_{r\leqslant t}\left(1+|Y_n|^4_{\mathbb{H}^1}\right)\int_0^t |X_n|^2_{\mathbb{H}^1}\,ds \\
&+ C\eta^2 \int_0^t |\Delta X_n|^2_{\mathbb{L}^2}\,ds + C_\eta\left(\sup_{r\leqslant t}|X_n(r)|_{\mathbb{L}^2}\right)\left(\sup_{r\leqslant t}|X_n(r)|_{\mathbb{H}^1}\right) \\
&+ C\eta^2 \int_0^t |\Delta X_n|^2_{\mathbb{L}^2}\,ds + C_\eta\left(\sup_{r\leqslant t}|X_n(r)|_{\mathbb{L}^2}\right)\left(\sup_{r\leqslant t}|X_n(r)|_{\mathbb{H}^1}\right) \\
&+ C\eta^2 \int_0^t |\Delta X_n|^2_{\mathbb{L}^2}\,ds + C_\eta \sup_{r\leqslant t}|X_n(r)|^2_{\mathbb{L}^2} \\
&+ \sqrt{\varepsilon_n}\left|\int_0^t \langle \mathrm{D}G\left(Y_n\right)h, \Delta X_n\rangle_{\mathbb{L}^2}\,dW\right| \\
&+ C\varepsilon_n \int_0^t \left(1+|\Delta X_n|^2_{\mathbb{L}^2}\right)ds.
\end{aligned}
\tag{6.49}
$$

Choosing η in such a way that $4C\eta^2 = \alpha$ we obtain

$$
\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]}|\mathrm{D}X_n\left(t\wedge\tau_n\right)|^2_{\mathbb{L}^2} + \alpha\int_0^{t\wedge\tau_n}|\Delta X_n|^2_{\mathbb{L}^2}\,ds\right) \leqslant{}& C_\eta\left(1+N^4\right)\mathbb{E}\int_0^{\tau_n}|X_n|^2_{\mathbb{H}^1}\,ds \\
&+ C_\eta\left(1+N\right)\mathbb{E}\sup_{t\in[0,T]}|X_n\left(t\wedge\tau_n\right)|_{\mathbb{L}^2} \\
&+ \sqrt{\varepsilon_n}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^{t\wedge\tau_n}\langle \mathrm{D}G\left(Y_n\right)h, \Delta X_n\rangle_{\mathbb{L}^2}\,dW\right| \\
&+ C\varepsilon_n\mathbb{E}\int_0^T\left(1+|\Delta X_n|^2_{\mathbb{L}^2}\right)ds.
\end{aligned}
$$

By Theorem 5.3 there exists a finite constant C, depending on T, α, R, M_0 and h only, such that for each $n\in\mathbb{N}$

$$
\mathbb{E}\int_0^T |\Delta Y_n(s)|^2_{\mathbb{L}^2}\,ds \leqslant C(T,\alpha,M,u_0,h),
$$

hence invoking the Burkholder-Davis-Gundy inequality we find that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\mathrm{D}X_n\left(t\wedge\tau_n\right)\right|^2_{\mathbb{L}^2}+\alpha\int\limits_0^{t\wedge\tau_n}\left|\Delta X_n\right|^2_{\mathbb{L}^2}ds\right)\leqslant C_\eta\left(1+N^4\right)\mathbb{E}\int\limits_0^{\tau_n}\left|X_n\right|^2_{\mathbb{H}^1}\,ds$$
$$+C_\eta\left(1+N\right)\mathbb{E}\sup_{t\in[0,T]}\left|X_n\left(t\wedge\tau_n\right)\right|_{\mathbb{L}^2}$$
$$+C(1+N)\sqrt{\varepsilon_n}.$$

Finally, Lemma 6.6 follows from Lemma 6.5. □

We will conclude this section with the promised proof of Lemma 6.4.

Proof of Lemma 6.4. We will use the same notation as in the proof of Lemma (6.6). Let $\delta>0$ and $\nu>0$. Invoking part (2) of Theorem (3.1) we can find $N>|M_0|_{\mathbb{H}^1}$ such that

$$\frac{1}{N}\sup_{n\in\mathbb{N}}\mathbb{E}\sup_{t\in[0,T]}|Y_n(t)|_{\mathbb{H}^1}<\frac{\nu}{2}.$$

Then invoking Lemma 6.6 we find that for all n sufficiently large

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]}|Y_n(t)-u_n(t)|^2_{\mathbb{H}^1}+\int\limits_0^T|Y_n-u_n|^2_{D(A)}\,ds\geqslant\delta\right)\\
\leqslant\;&\mathbb{P}\left(\sup_{t\in[0,T]}|Y_n(t\wedge\tau_n)-u_n(t\wedge\tau_n)|^2_{\mathbb{H}^1}+\int\limits_0^{\tau_n}|Y_n-u_n|^2_{D(A)}\,ds\geqslant\delta,\;\tau_n=T\right)\\
&+\mathbb{P}\left(\sup_{t\in[0,T]}|Y_n(t)|_{\mathbb{H}^1}\geqslant N\right)\\
\leqslant\;&\frac{1}{\delta}\mathbb{E}\left(\sup_{t\in[0,T]}|Y_n(t\wedge\tau_n)-u_n(t\wedge\tau_n)|^2_{\mathbb{H}^1}+\int\limits_0^{\tau_n}|Y_n-u_n|^2_{D(A)}\,ds\right)\\
&+\frac{1}{N}\mathbb{E}\sup_{t\in[0,T]}|Y_n(t)|_{\mathbb{H}^1}\\
<\;&\nu.
\end{aligned}$$

□

7. Application to a model of a ferromagnetic needle

In this section we will use the large deviation principle established in the previous section to investigate the dynamics of a stochastic Landau-Lifshitz model of magnetization in a needle-shaped particle. Here the shape anisotropy energy is crucial. When there is no applied field and no noise in the field, the shape anisotropy energy gives rise to two locally stable stationary states of opposite magnetization. We add a small noise term to the field and use the large deviation principle to show that noise induced magnetization reversal occurs and to quantify the effect of material parameters on sensitivity to noise.

The axis of the needle is represented by the interval Λ and at each $x \in \Lambda$ the magnetization $u(x) \in \mathbb{S}^2$ is assumed to be constant over the cross-section of the needle. We define the total magnetic energy of magnetization $u \in \mathbb{H}^1$ of the needle by

$$E_t(u) = \frac{1}{2} \int_\Lambda |\mathrm{D}u(x)|^2 \, dx + \beta \int_\Lambda \Phi(u(x)) \, dx - \int_\Lambda \mathscr{K}(t,x) \cdot u(x) \, dx, \tag{7.1}$$

where

$$\Phi(u) = \Phi(u_1, u_2, u_3) = \frac{1}{2}\left(u_2^2 + u_3^2\right),$$

β is the positive real shape anisotropy parameter and $\mathscr{K}$ is the externally applied magnetic field, such that $\mathscr{K}(t) \in \mathbb{H}$ for each t.

With this magnetic energy, the deterministic Landau-Lifshitz equation becomes:

$$\frac{\partial y}{\partial t}(t) = y \times \Delta y - \alpha y \times (y \times \Delta y) + G(y)\left(-\beta f(y) + \mathscr{K}(t)\right) \tag{7.2}$$

where $f(y) = \mathrm{D}\Phi(y)$, $y \in \mathbb{R}^3$. We assume, as before, that the initial state $u_0 \in \mathbb{H}^1$ and $|u_0(x)|_{\mathbb{R}^3} = 1$ for all $x \in \Lambda$. We also assume that the applied field $\mathscr{K}(t) : \Lambda \to \mathbb{R}^3$ is constant on Λ at each time t. Equation (7.2) has nice features: the dynamics of the solution can be studied using elementary techniques and, when the externally applied field $\mathscr{K}$ is zero, the equation has two stable stationary states, $\zeta_+ = (1,0,0)$ and $\zeta_- = (-1,0,0) = -\zeta_+$. In what follow we will abuse the notation as by $\zeta_\pm$. We will also denote a constant function $\Lambda \ni x \mapsto \zeta_\pm \in \mathbb{S}^2$ which obviously belongs to $\mathbb{H}^1$.

We now outline the structure of this example. In Proposition 7.2, we show that if the applied field $\mathscr{K}$ is zero and the initial state y_0 satisfies

$$|y_0 - \zeta_\pm|_{\mathbb{H}^1} < \frac{1}{2k^2\sqrt{|\Lambda|}} \frac{\alpha}{1+2\alpha},$$

then the solution $y(t)$ of (7.2) converges to $\zeta_\pm$ in $\mathbb{H}^1$ as t goes to ∞. In Lemma 7.3, we show that if λ exceeds a certain value (depending on α and β) and the applied field is $\mathscr{K} = \lambda \mathbf{m} + \beta f(\mathbf{m})$ and $|y_0 - \mathbf{m}|_{\mathbb{H}^1} < \frac{1}{k}$, then $y(t)$ converges in $\mathbb{H}^1$ to $\mathbf{m}$ as t goes to ∞. Lemma 7.3 is used to show that, given $\delta \in (0,\infty)$ and $T \in (0,\infty)$, there is a piecewise constant (in time) externally applied field, $\mathscr{K}$, which drives the magnetization from the initial state ζ_- to the $\mathbb{H}^1$-ball centred at ζ_+ and of radius δ by time T; in short, in the deterministic system, this applied field causes magnetization reversal by time T(see Definition 7.4). What we are really interested in is the effect of adding a small noise term to the field. We will show that if $\mathscr{K}$ is zero but a noise term multiplied by $\sqrt{\varepsilon}$ is added to the field, then the solution of the resulting stochastic equation exhibits magnetization reversal by time T with positive probability for all sufficiently small positive ε. This result, in Proposition 7.5, is obtained using the lower bound of the large deviation principle. Finally, in Proposition 7.7, the upper bound of the large deviation principle is used: we obtain an exponential estimate of the probability that, in time interval $[0,T]$, the stochastic magnetization leaves a given $\mathbb{H}^1$-ball centred at the initial state ζ_- and of radius less than or equal to $\frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}$. This estimate emphasizes the importance of a large value of β for reducing the disturbance in the magnetization caused by noise in the field.

7.1. **Stable stationary states of the deterministic equation.** In this subsection, we identify stable stationary states of the deterministic equation (7.2) when the applied field $\mathscr{K}$ does not vary with time.

Let $\zeta \in \mathbb{S}^2$. Since the time derivative $\frac{dy}{dt}$ of the solution y to (7.2), belongs to $L^2(0,T;\mathbb{H})$ and y belongs to $L^2(0,T;D(A))$, we have for all $t \geqslant 0$:

$$\begin{aligned} |y(t)-\zeta|^2_{\mathbb{H}} &= |y_0-\zeta|^2_{\mathbb{H}} \\ &+ 2\int_0^t \langle y-\zeta, y\times\Delta y - \alpha y\times(y\times\Delta y) \\ &\qquad +G(y)(-\beta f(y)+\mathscr{K}\rangle_{\mathbb{H}}\, ds \\ &= |y_0-\zeta|^2_{\mathbb{H}} + 2\int_0^t \langle -\zeta, y\times\Delta y - \alpha y\times(y\times\Delta y) \\ &\qquad +G(y)(-\beta f(y)+\mathscr{K}\rangle_{\mathbb{H}}\, ds \end{aligned} \tag{7.3}$$

$$|\mathrm{D}y(t)|^2_{\mathbb{H}} y(s) + y(s)\times(-\beta f(y(s))+\mathscr{K})$$
$$f(y(s))+\mathscr{K}))\rangle_{\mathbb{H}}\, ds$$

$$\begin{aligned} &= |\mathrm{D}y_0|^2_{\mathbb{H}} - 2\int_0^t \langle \Delta y, G(y)\,(-\beta f(y)+\mathscr{K}) \\ &\qquad - \alpha y\times(y\times\Delta y)\rangle_{\mathbb{L}^2}\, ds \end{aligned} \tag{7.4}$$

Lemma 7.1. *Let $u \in \mathbb{H}^1$ be such that $u(x)\in\mathbb{S}^2$ and*

$$|u-\zeta_\pm|_{\mathbb{H}^1} \leqslant \frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}.$$

Then for all $x\in\Lambda$

(1) $\frac{1-u_1^2(x)}{u_1^2(x)} + \alpha\frac{(1-u_1^2(x))^2}{u_1^2(x)} - \alpha u_1^2(x) \leqslant 0$ *for all $x\in\Lambda$,*
(2) $\langle u(x),\zeta_\pm\rangle \geqslant \frac{3}{4}$ *and*
(3) $\frac{7}{8}|u(x)\zeta_\pm|^2 \leqslant |u(x)\times\zeta_\pm|^2$.

Proof. By (2.12)

$$\begin{aligned} \sup_{x\in\Lambda}|u(x)-\zeta_\pm|^2 &\leqslant k^2|u-\zeta_\pm|_{\mathbb{H}}|u-\zeta_\pm|_{\mathbb{H}^1}, \\ &\leqslant k^2\,2\sqrt{|\Lambda|}\,\frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha} = \frac{\alpha}{1+2\alpha}. \end{aligned} \tag{7.5}$$

Invoking (7.5), we find that

$$u_1^2(x) = 1-(u_2^2(x)+u_3^2(x)) \geqslant 1-|u(x)-\zeta_\pm|^2 \geqslant \frac{1+\alpha}{1+2\alpha}, \quad x\in\Lambda. \tag{7.6}$$

Hence one can use (7.6) and straightforward algebraic manipulations to verify that

$$\frac{1-u_1^2(x)}{u_1^2(x)} + \alpha\frac{(1-u_1^2(x))^2}{u_1^2(x)} - \alpha u_1^2(x) \leqslant 0.$$

Statements 2 and 3 of Lemma 7.1 follow easily from (7.5). □

Proposition 7.2. *Let the applied field $\mathscr{K}$ be zero and let $y_0 \in \mathbb{H}^1$ satisfy*

$$|y_0 - \zeta_\pm|_{\mathbb{H}^1} < \frac{1}{2k^2\sqrt{|\Lambda|}} \frac{\alpha}{1+2\alpha}. \tag{7.7}$$

Let the process y be the solution to (7.2). Then $y(t)$ converges to $\zeta_\pm$ in $\mathbb{H}^1$ as $t \to \infty$.

Proof. Using some algebraic manipulation and the fact that $\langle \mathrm{D}y(s), y(s)\rangle = 0$ a.e. on Λ for each $s \geqslant 0$, one may simplify equations (7.4) and (7.4).
We obtain from (7.4):

$$\begin{aligned} |y(t) - \zeta_\pm|^2_{\mathbb{H}} &= |y_0 - \zeta_\pm|^2_{\mathbb{H}} - 2\alpha \int_0^t \int_\Lambda |\mathrm{D}y(s)|^2 \langle y(s), -\zeta_\pm\rangle \, dx \, ds \\ &\quad - 2\alpha\beta \int_0^t \int_\Lambda \langle y(s), -\zeta_\pm\rangle |y(s) \times \zeta_+|^2 \, dx \, ds \quad \forall t \geqslant 0, \end{aligned} \tag{7.8}$$

and

$$|\mathrm{D}y(t)|^2_{\mathbb{H}} = |\mathrm{D}y_0|^2_{\mathbb{H}} - 2\alpha \int_0^t |y(s) \times \Delta y(s)|^2_{\mathbb{H}} \, ds + 2\beta \int_0^t \int_\Lambda R(s) \, dx \, ds \quad \forall t \geqslant 0, \tag{7.9}$$

where

$$\begin{aligned} R &= \mathrm{D}y_1(y_3\mathrm{D}y_2 - y_2\mathrm{D}y_3) + \alpha(\mathrm{D}y_1)^2 - \alpha y_1^2 |\mathrm{D}y|^2 \\ &= \frac{-y_2\mathrm{D}y_2 - y_3\mathrm{D}y_3}{y_1}(y_3\mathrm{D}y_2 - y_2\mathrm{D}y_3) \\ &\quad + \alpha(1-y_1^2)\left(\frac{y_2\mathrm{D}y_2 + y_3\mathrm{D}y_3}{y_1}\right)^2 - \alpha y_1^2((\mathrm{D}y_2)^2 + (\mathrm{D}y_3)^2. \end{aligned} \tag{7.10}$$

Define

$$\tau = \inf\left\{ t \geqslant 0 : \, |y(t) - \zeta_\pm|_{\mathbb{H}^1} \geqslant \frac{1}{2k^2\sqrt{|\Lambda|}} \frac{\alpha}{1+2\alpha} \right\}.$$

Then, by our choice of y_0, $\tau > 0$. For each $s \in [0, \tau)$, $y(s)$ satisfies the hypotheses of Lemma 7.1, hence and

$$y(s)(x) \cdot (-\zeta_\pm) \geqslant \frac{3}{4}, \quad x \in \Lambda,$$

$$|y(s)(x) \times (-\zeta_\pm)\,|^2 \geqslant \frac{7}{8}|y(s)(x) - \zeta_\pm|^2, \quad x \in \Lambda.$$

and, invoking the Cauchy-Schwartz inequality

$$R \leqslant \left(\frac{1-y_1^2}{y_1^2} + \alpha\frac{(1-y_1^2)^2}{y_1^2} - \alpha y_1^2\right)((\mathrm{D}y_2)^2 + (\mathrm{D}y_3)^2) \leqslant 0, \quad x \in \Lambda. \tag{7.11}$$

Consequently, from (7.8) and (7.9) we deduce that the functions $|y(\cdot) - \zeta_\pm|^2_{\mathbb{H}}$ and $|\mathrm{D}y(\cdot)|^2_{\mathbb{H}}$ are nonincreasing on $[0, \tau)$. Furthermore, we have

$$|y(t) - \zeta_\pm|^2_{\mathbb{H}} \leqslant |y_0 - \zeta_\pm|^2_{\mathbb{H}} - \frac{3}{2}\alpha \int_0^t |\mathrm{D}y(s)|^2_{\mathbb{H}} \, ds - \frac{21}{16}\alpha\beta \int_0^t |y(s) - \zeta_\pm|^2_{\mathbb{H}} \, ds, \quad t < \tau, \tag{7.12}$$

and

$$|\mathrm{D}y(t)|_{\mathbb{H}}^2 \leqslant |\mathrm{D}y_0|_{\mathbb{H}}^2, \quad t<\tau. \tag{7.13}$$

Suppose, to get a contradiction, that $\tau<\infty$. Then, from (7.12) and (7.13), we have

$$|y(\tau)-\zeta_{\pm}|_{\mathbb{H}^1} \leqslant |y_0-\zeta_{\pm}|_{\mathbb{H}^1} < \frac{1}{2k^2\sqrt{l(\Lambda)}}\frac{\alpha}{1+2\alpha},$$

which contradicts the definition of τ. Therefore, $\tau=\infty$. Since (7.12) holds for all $t\geqslant 0$, we have

$$\int_0^\infty |\mathrm{D}y(s)|_{\mathbb{H}}^2\,ds + \int_0^\infty |y(s)-\zeta_{\pm}|_{\mathbb{H}}^2\,ds < \infty.$$

Since both integrands are nonincreasing

$$\lim_{t\to\infty}\left(|\mathrm{D}y(t)|_{\mathbb{H}} + |y(t)-\zeta_{\pm}|_{\mathbb{H}}\right) = 0.$$

□

other uniform stationary states of equation (**??**) are points of the form $(0,y_2,y_3)\in\mathbb{R}^3$, where $y_2^2+y_3^2=1$; however, such a point, $(0,y_2,y_3)$, is not a stable stationary state because any given $\mathbb{H}^1$-ball centred at the point contains another uniform state $(\underline{y}_1,\underline{y}_2,\underline{y}_3)\in\mathbb{R}^3$ with lower energy (that is, $\underline{y}_2^2+\underline{y}_3^2<1$) and, by (**??**), energy is nonincreasing. We will show next, that if the applied field has sufficiently large magnitude, then there exists a stable stationary state that is roughly in the direction of the applied field.

Lemma 7.3. *Assume that* $\mathbf{m}\in\mathbb{S}^2$ *and a real number* λ *satisfies*

$$\lambda > \left(\frac{4\beta+4\alpha\beta}{3\alpha} \vee \frac{2\beta+4\alpha\beta-\alpha}{\alpha}\right). \tag{7.14}$$

Let the applied field be[5]

$$\mathscr{K} := \lambda\mathbf{m} + \beta f(\mathbf{m}).$$

Let y *be a solution to the problem* (7.2) *with initial data* y_0 *satisfying* $|y_0-\mathbf{m}|_{\mathbb{H}^1}<\frac{1}{k}$. *Then*

$$|y(t)-\mathbf{m}|_{\mathbb{H}^1} \leqslant |y_0-\mathbf{m}|_{\mathbb{H}^1}\,\mathrm{e}^{-\frac{1}{2}\gamma t} \quad \forall t\geqslant 0, \tag{7.15}$$

where

$$\gamma := (\alpha\lambda+\alpha-2\beta-4\alpha\beta)\wedge\left(\frac{3}{2}\alpha\lambda-2\beta-2\alpha\beta\right) > 0$$

is positive, by condition (7.14).

[5]Note that a constant function $\mathbf{m}$ is a stationary solution to the problem (7.2).

Proof. We have, from (7.3) and (7.4) with ζ replaced by $\mathbf{m}$:

$$
\begin{aligned}
&|y(t)-\mathbf{m}|_{\mathbb{H}}^2 \qquad (7.16)\\
&= |y_0-\mathbf{m}|_{\mathbb{H}}^2 + 2\int_0^t \langle y(s)-\mathbf{m}, y(s)\times(\mathbf{m}-\beta f(y-\mathbf{m}))\rangle_{\mathbb{H}}\,ds\\
&\qquad + 2\alpha\int_0^t \langle \Delta y(s), y(s)\times(y(s)\times\mathbf{m})\rangle_{\mathbb{H}}\,ds\\
&\qquad - 2\alpha\int_0^t \langle y(s)\times\mathbf{m}, y(s)\times(\mathbf{m}-\beta f(y-\mathbf{m}))\rangle_{\mathbb{H}}\,ds\\
&= |y_0-\mathbf{m}|_{\mathbb{H}}^2 - 2\beta\int_0^t \langle y-\mathbf{m}, y\times f(y-\mathbf{m})\rangle_{\mathbb{H}}\,ds\\
&\qquad - 2\alpha\int_0^t\int_\Lambda |\mathrm{D}y|^2(y\cdot\mathbf{m})\,dx\,ds\\
&\qquad - 2\alpha\int_0^t |y\times\mathbf{m}|_{\mathbb{H}}^2\,ds\\
&\qquad + 2\alpha\beta\int_0^t \langle y\times\mathbf{m}, y\times f(y-\mathbf{m})\rangle_{\mathbb{H}}\,ds \quad \forall t\geqslant 0.
\end{aligned}
$$

From (7.4) we have:

$$\begin{aligned}|\mathrm{D}y(t)|^2_{\mathbb{H}} &= |\mathrm{D}y_0|^2_{\mathbb{H}} - 2\int_0^t \langle \Delta y(s), y(s)\times(\mathbf{m}-\beta f(y-\mathbf{m})) \\ &\qquad\qquad - \alpha y(s)\times(y(s)\times\Delta y(s)) \\ &\qquad\qquad - \alpha y(s)\times(y(s)\times(\mathbf{m}-\beta f(y-\mathbf{m})))\rangle_{\mathbb{H}}\,ds \\ &= |\mathrm{D}y_0|^2_{\mathbb{H}} + 2\beta\int_0^t \langle \Delta y, y\times f(y-\mathbf{m})\rangle_{\mathbb{H}}\,ds \\ &\quad - 2\alpha\int_0^t |y\times\Delta y|^2_{\mathbb{H}}\,ds \\ &\quad - 2\alpha\int_0^t\int_\Lambda |\mathrm{D}y|^2(y\cdot\mathbf{m})\,dx\,ds \\ &\quad - 2\alpha\beta\int_0^t \langle \Delta y, y\times(y\times f(y-\mathbf{m}))\rangle_{\mathbb{H}}\,ds \quad \forall t\geqslant 0.\end{aligned} \tag{7.17}$$

Define

$$\tau_1 := \inf\{t\geqslant 0 : |y(t)-\mathbf{m}|_{\mathbb{H}^1}\geqslant \tfrac{1}{k}\}. \tag{7.18}$$

By our choice of y_0, τ_1 is greater than zero. Observe that

$$\sup_{x\in\Lambda}|y(t)(x)-\mathbf{m}|_{\mathbb{R}^3} < 1 \quad \text{for all } t<\tau_1. \tag{7.19}$$

It is easy to check that for every $t<\tau_1$

$$\frac{3}{4}|y(t)-\mathbf{m}|^2_{\mathbb{H}} \leqslant |y(t)\times\mathbf{m}|^2_{\mathbb{H}} \leqslant |y(t)-\mathbf{m}|^2_{\mathbb{H}}, \tag{7.20}$$

and

$$y(t,x)\cdot\mathbf{m}\geqslant \tfrac{1}{2}, \quad x\in\Lambda. \tag{7.21}$$

Adding equalities (7.16) and (7.17) we obtain for $t > 0$

$$\begin{aligned}
&|y(t) - \mathbf{m}|^2_{\mathbb{H}^1} \\
= \quad & |y_0 - \mathbf{m}|^2_{\mathbb{H}^1} - 4\alpha \int_0^t \int_\Lambda |\mathrm{D}y|^2 \, (y \cdot \mathbf{m}) \, dx \, ds \\
& + 2\beta \int_0^t \langle \Delta y, y \times f(y - \mathbf{m}) \rangle_{\mathbb{H}} \, ds \\
& - 2\alpha\beta \int_0^t \langle \Delta y, y \times (y \times f(y - \mathbf{m})) \rangle_{\mathbb{H}} \, ds \\
& - 2\alpha \int_0^t |y \times \mathbf{m}|^2_{\mathbb{H}} \, ds \\
& - 2\beta \int_0^t \langle y - \mathbf{m}, y \times f(y - \mathbf{m}) \rangle_{\mathbb{H}} \, ds \\
& + 2\alpha\beta \int_0^t \langle y \times \mathbf{m}, y \times f(y - \mathbf{m}) \rangle_{\mathbb{H}} \, ds \\
& - 2\alpha \int_0^t |y \times \Delta y|^2_{\mathbb{H}} \, ds \, .
\end{aligned} \tag{7.22}$$

Therefore for every $t < \tau_1$

$$\begin{aligned}
|y(t) - \mathbf{m}|^2_{\mathbb{H}^1} \quad \leqslant \quad & |y_0 - \mathbf{m}|^2_{\mathbb{H}^1} - (2\alpha - 2\beta - 4\alpha\beta) \int_0^t |\mathrm{D}y|^2_{\mathbb{H}} \, ds \\
& - (\tfrac{3}{2}\alpha - 2\beta - 2\alpha\beta) \int_0^t |y - \mathbf{m}|^2_{\mathbb{H}} \, ds \\
& - 2\alpha \int_0^t |y \times \Delta y|^2_{\mathbb{H}} \, ds \, ,
\end{aligned} \tag{7.23}$$

where we used (7.19), (7.20) and (7.21). Because of hypothesis (7.14), the two expressions $(2\alpha - 2\beta - 4\alpha\beta)$ and $(\frac{3}{2}\alpha - 2\beta - 2\alpha\beta)$ on the right hand side of (7.23) are positive numbers. Suppose, to get a contradiction, that $\tau_1 < \infty$. Then, from (7.23), we have

$$|y(\tau_1) - \mathbf{m}|_{\mathbb{H}^1} \leqslant |y_0 - \mathbf{m}|_{\mathbb{H}^1} < \tfrac{1}{k},$$

which contradicts the definition of τ_1 in (7.18). Hence $\tau_1 = \infty$. It now follows from (7.23) that

$$\int_0^\infty |\mathrm{D}y(s)|^2_{\mathbb{H}}\,ds \;<\; \infty, \tag{7.24}$$

$$\int_0^\infty |y(s)-\mathbf{m}|^2_{\mathbb{H}}\,ds \;<\; \infty \tag{7.25}$$

$$\text{and}\quad \int_0^\infty |y(s)\times \Delta y(s)|^2_{\mathbb{H}}\,ds \;<\; \infty. \tag{7.26}$$

From (7.22) and these three inequalities, the function $t\in[0,\infty)\mapsto |y(t)-\mathbf{m}|^2_{\mathbb{H}^1}$ is absolutely continuous and, for almost every $t\geqslant 0$, its derivative is:

$$\begin{aligned}
\frac{d}{dt}|y-\mathbf{m}|^2_{\mathbb{H}^1}(t) \;&=\; -4\alpha\int_\Lambda |\mathrm{D}y(t)|^2(y(t)\cdot\mathbf{m})\,dx \\
&\quad + 2\beta\langle \Delta y(t), y(t)\times f(y-\mathbf{m})\rangle_{\mathbb{H}} \\
&\quad - 2\alpha\beta\langle \Delta y(t), y(t)\times(y(t)\times f(y-\mathbf{m}))\rangle_{\mathbb{H}} \\
&\quad - 2\alpha|y(t)\times\mathbf{m}|^2_{\mathbb{H}} \\
&\quad - 2\beta\big\langle y(t)-\mathbf{m}, y(t)\times f(y-\mathbf{m})\big\rangle_{\mathbb{H}} \\
&\quad + 2\alpha\beta\langle y(t)\times\mathbf{m}, y(t)\times f(y-\mathbf{m})\rangle_{\mathbb{H}} \\
&\quad - 2\alpha|y(t)\times\Delta y(t)|^2_{\mathbb{H}} \\
&\leqslant\; -(2\alpha-2\beta-4\alpha\beta)|\mathrm{D}y(t)|^2_{\mathbb{H}} \\
&\quad - (\tfrac{3}{2}\alpha-2\beta-2\alpha\beta)|y(t)-\mathbf{m}|^2_{\mathbb{H}} \\
&\quad - 2\alpha|y(t)\times\Delta y(t)|^2_{\mathbb{H}} \\
&\leqslant\; -\gamma|y(t)-\mathbf{m}|^2_{\mathbb{H}^1},
\end{aligned} \tag{7.27}$$

where

$$\gamma := (\alpha+\alpha\lambda-2\beta-4\alpha\beta)\wedge\left(\frac{3}{2}\alpha\lambda-2\beta-2\alpha\beta\right) > 0.$$

Now the lemma follows by a standard argument. □

7.2. **Noise induced instability and magnetization reversal.** In Proposition 7.2 we showed that the states ζ_+ and ζ_- are stable stationary states of the deterministic Landau-Lifshitz equation (7.2) when the externally applied field $\mathscr{K}$ is zero. In this section we show that a small noise term in the field may drive the magnetization from the initial state ζ_- to any given $\mathbb{H}^1$-ball centred at ζ_+ in any given time interval $[0,T]$. We also find an exponential upper bound for the probability that small noise in the field drives the magnetization outside a given $\mathbb{H}^1$-ball centred at the initial state ζ_- in time interval $[0,T]$. Firstly we need a definition.

Definition 7.4. *Let δ be a given small positive real number. Suppose that the initial magnetization is ζ_- and that at some time T the magnetization lies in the open $\mathbb{H}^1$-ball centred at ζ_+ and of radius δ. Then we say that* magnetization reversal *has occurred by time T.*

We consider a stochastic equation for the magnetization, obtained by setting $\mathscr{K}$ to zero and adding a three dimensional noise term to the field. Denoting the magnetization by Y, the equation is:

$$\left.\begin{aligned} dY &= (Y \times \Delta Y - \alpha Y \times (Y \times \Delta Y) + \beta G(Y) f(Y))\, dt \\ &\quad + \sqrt{\varepsilon} G(Y) B \circ dW(t) \\ Y(0) &= \zeta_-. \end{aligned}\right\} \tag{7.28}$$

In (7.28), we assume that the vectors $e_1, e_2, e_3 \in \mathbb{R}^3$ are linearly independent. The parameter $\varepsilon > 0$ corresponds to the 'dimensionless temperature' parameter appearing in the following stochastic differential equation (7.29) of Kohn, Reznikoff and Vanden-Eijnden [29]:

$$\dot{m} = m \times (g + \varepsilon^{\frac{1}{2}} \sqrt{\tfrac{2\alpha}{1+\alpha^2}} \dot{W}) - \alpha m \times (m \times (g + \varepsilon^{\frac{1}{2}} \sqrt{\tfrac{2\alpha}{1+\alpha^2}} \dot{W})), \tag{7.29}$$

Fix $T > 0$. There is no deterministic applied field in (7.28) but, as we will see, the lower bound of the large deviation principle satisfied by the solutions Y^ε ($\varepsilon \in (0,1)$) of (7.28) implies that, for all sufficiently small positive ε, the probability of magnetization reversal by time T is positive.

Firstly, we shall use Lemma 7.3 to construct a piecewise constant (in time) deterministic applied field, $\mathscr{K}$, such that the solution y of (7.2), with initial state $\zeta_- = (-1, 0, 0)$, undergoes magnetization reversal by time T.

Take points $u^i \in \mathbb{S}^2$, $i = 0, 1, \ldots, N$, such that $u^0 = \zeta_-$ and $u^N = \zeta_+$ and

$$|u^i - u^{i+1}|_{\mathbb{H}^1} = |u^i - u^{i+1}|_{\mathbb{R}^3} \sqrt{|\Lambda|} < \frac{1}{k} \quad \text{for } i = 0, 1, \ldots, N-1.$$

Let

$$\eta := \min \left\{ \frac{1}{k} - |u^i - u^{i+1}|_{\mathbb{H}^1} : i = 1, \ldots, N-1 \right\} \wedge \frac{\delta}{2}.$$

Using Lemma 7.3, we can take the applied field to be

$$\mathscr{K}(t) := \sum_{i=0}^{N-1} 1_{(i\frac{T}{N}, (i+1)\frac{T}{N}]}(t) \left(R u^{i+1} + \beta f(u^{i+1}) \right), \quad t \geqslant 0, \tag{7.30}$$

with the positive real number R chosen to ensure that, as t varies from $i\frac{T}{N}$ to $(i+1)\frac{T}{N}$, $y(t)$ starts at a distance of less than η from u^i (i.e. $|y(i\frac{T}{N}) - u^i|_{\mathbb{H}^1} < \eta$) and moves to a distance of less than η from u^{i+1} (i.e. $|y((i+1)\frac{T}{N}) - u^{i+1}|_{\mathbb{H}^1} < \eta$). Specifically, we take $R \in (0, \infty)$ such that

$$\frac{1}{k} e^{-\frac{1}{2}[(\alpha R + \alpha - 2\beta - 4\alpha\beta) \wedge (\frac{3}{2}\alpha R - 2\beta - 2\alpha\beta)] \frac{T}{N}} < \eta.$$

For each $i = 0, 1, \ldots, N-1$, let $\phi^{i+1} = (\phi_1^{i+1}, \phi_2^{i+1}, \phi_3^{i+1}) \in \mathbb{R}^3$ be the vector of scalar coefficients satisfying the equality

$$\phi_1^{i+1} a^1 + \phi_2^{i+1} a^2 + \phi_3^{i+1} a^3 = R u^{i+1} + \beta f(u^{i+1}),$$

and define

$$\phi(t) := \sum_{i=0}^{N-1} 1_{(i\frac{T}{N}, (i+1)\frac{T}{N}]}(t)\, \phi^{i+1}, \quad t \in [0, T]. \tag{7.31}$$

We remark that the function ϕ depends on the chosen values of δ and T, the material parameters Λ, α and β and the noise parameters a^1, a^2 and a^3.

Recall that Y^ε denotes the solution of (7.28). By an argument very much like that leading to Theorem 6.1, the family of laws $\{\mathscr{L}(Y^\varepsilon) : \varepsilon \in (0,1)\}$ on $\mathscr{X}_T$ satisfies a large deviation principle. In order to define the rate function, we introduce an equation

$$\begin{aligned} y_\psi(t) &= \zeta_- + \int_0^t y_\psi \times \Delta y_\psi \, ds - \alpha \int_0^t y_\psi \times (y_\psi \times \Delta y_\psi) \, ds \\ &\quad - \beta \int_0^t G(y_\psi) f(y_\psi) \, ds + \int_0^t G(y_\psi) B\psi \, ds. \end{aligned} \tag{7.32}$$

By Corollary 5.6 this equation has unique solution $y_\psi \in \mathscr{X}_T$ for every $\psi \in L^2(0,T;\mathbb{R}^3)$. The rate function $I : \mathscr{X}_T \to [0,\infty]$, is defined by:

$$I_T(v) := \inf\left\{ \frac{1}{2}\int_0^T |\psi(s)|^2 \, ds : \psi \in L^2(0,T;\mathbb{R}^3) \text{ and } v = y_\psi \right\}, \tag{7.33}$$

where the infimum of the empty set is taken to be ∞.
Let y be the solution of equation (7.2) with $y_0 = \zeta_-$ and $\mathscr{K}$ as defined in (7.30). Using the notation in (7.32), we have $y = y_\phi$, for ϕ defined in (7.31). Therefore

$$I_T(y) \leqslant \frac{1}{2}\int_0^T |\phi(s)|^2 \, ds < \infty.$$

Since y undergoes magnetization reversal by time T, paths of Y^ε which lie close to y also undergo magnetization reversal by time T. In particular, by the Freidlin-Wentzell formulation of the lower bound of the large deviation principle (see, for example, [20, Proposition 12.2]), given $\xi > 0$, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in[0,T]} |Y^\varepsilon(t) - y(t)|_{\mathbb{H}^1} + \left(\int_0^T |Y^\varepsilon(s) - y(s)|^2_{D(A)} \, ds \right)^{\frac{1}{2}} < \tfrac{\delta}{2} \right) \\ &\geqslant \exp\left(\tfrac{-I_T(y) - \xi}{\varepsilon} \right) \\ &\geqslant \exp\left(\frac{-\frac{1}{2}\int_0^T |\phi(s)|^2 \, ds - \xi}{\varepsilon} \right). \end{aligned} \tag{7.34}$$

Since we have $|y(T) - \zeta_+|_{\mathbb{H}^1} < \frac{\delta}{2}$, the right hand side of (7.34) provides a lower bound for the probability that Y^ε undergoes magnetization reversal by time T. We summarize our conclusions in the following proposition.

Proposition 7.5. *For all sufficiently small $\varepsilon > 0$, the probability that the solution Y^ε of (7.28) undergoes magnetization reversal by time T is bounded below by the expression on the right hand side of (7.34); in particular, it is positive.*

We shall now use the upper bound of the large deviation principle satisfied by $\{\mathscr{L}(Y^\varepsilon) : \varepsilon \in (0,1)\}$ to find an exponential upper bound for the probability that small noise in the field drives the magnetization outside a given $\mathbb{H}^1$-ball centred at the initial state ζ_- in time interval

$[0,T]$. This is done in Proposition 7.7 below; the proof of the proposition uses Lemma 7.6. In Lemma 7.6 and Proposition 7.7, for h an arbitrary element of $L^2(0,T;\mathbb{R}^3)$, y_h denotes the function in $\mathscr{X}_T$ which satisfies equality (7.32) and τ_h is defined by

$$\tau_h := \inf\left\{ t\in[0,T] : |y_h(t)+\zeta_+|_{\mathbb{H}^1} \geqslant \frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}\right\}.$$

Lemma 7.6. *For each $h\in L^2(0,T;\mathbb{R}^3)$, we have $|\mathrm{D}y_h(t)|_{\mathbb{H}}=0$ for all $t\in[0,\tau_h\wedge T)$.*

Proof. Let $h\in L^2(0,T;\mathbb{R}^3)$. To simplify notation in this proof, we write y instead of y_h. Proceeding as in the derivation of (7.9), we obtain

$$\begin{aligned}|\mathrm{D}y(t)|^2_{\mathbb{H}} &= -2\alpha\int_0^t |y\times\Delta y|^2_{\mathbb{H}}\,ds + 2\beta\int_0^t\int_\Lambda R\,dx\,ds \\ &\quad - 2\alpha\sum_{i=1}^3\int_0^t \langle \mathrm{D}y, y\times(\mathrm{D}y\times a^i)\rangle_{\mathbb{H}} h_i\,ds,\quad t\in[0,T],\end{aligned}\tag{7.35}$$

where $R(s)$ defined in (7.10) satisfies inequality (7.11). For each $s\in[0,\tau_h\wedge T)$, $y(s)$ satisfies the hypotheses of Lemma 7.1, thus we have $R(s)(x)\leqslant 0$ for all $x\in\Lambda$. It follows from (7.35) that for all $t\in[0,\tau_h\wedge T)$:

$$|\mathrm{D}y(t)|^2_{\mathbb{H}} \leqslant 2\alpha\int_0^t |\mathrm{D}y|^2_{\mathbb{H}}\sum_{i=1}^3 |a^i|\cdot|h_i|\,ds. \tag{7.36}$$

By the Gronwall lemma applied to (7.36), $|\mathrm{D}y(t)|^2_{\mathbb{H}}=0$ for all $t\in[0,\tau_h\wedge T)$. □

Proposition 7.7. *Let*

$$0<r<\rho\leqslant\frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha}.$$

The for any $\xi>0$, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$:

$$\mathbb{P}\left(\sup_{t\in[0,T]}|Y^\varepsilon(t)+\zeta_+|_{\mathbb{H}^1}\geqslant\rho\right)\leqslant\exp\left(\frac{-\kappa r^2+\xi}{\varepsilon}\right),\tag{7.37}$$

where

$$\kappa=\frac{\alpha\beta}{8\max_{1\leqslant i\leqslant 3}|a^i|^2\,|\Lambda|(1+\alpha^2)}.$$

Proof. We shall use the Freidlin-Wentzell formulation of the upper bound of the large deviation principle (see, for example, [20, Proposition 12.2]) satisfied by $\{\mathscr{L}(Y^\varepsilon):\varepsilon\in(0,1)\}$. Recall that $\mathscr{I}$, defined in (7.33), is the rate function of the large deviation principle. Our main task is to show that

$$\left\{v\in\mathscr{X}_T:\ I_T(v)\leqslant\kappa r^2\right\}\subset\left\{v\in C([0,T];\mathbb{H}^1):\sup_{t\in[0,T]}|v(t)+\zeta_+|_{\mathbb{H}^1}\leqslant r\right\}.$$

Take $h \in L^2(0,T;\mathbb{R}^3)$ such that

$$\frac{1}{2}\int_0^T |h(s)|^2\,ds \leqslant \kappa r^2. \tag{7.38}$$

For simplicity of notation, in this proof we write y in place of y_h. By Lemma 7.6 we have for all $t \in [0,T]$,

$$\begin{aligned}
|y(t\wedge\tau_h)+\zeta_+|^2_{\mathbb{H}^1} &= 2\alpha \int_0^{t\wedge\tau_h}\int_\Lambda |\mathrm{D}y|^2\,(y\cdot\zeta_+)\,dx\,ds \\
&\quad + 2\alpha\beta\int_0^{t\wedge\tau_h}\int_\Lambda (y\cdot\zeta_+)\,|y\times\zeta_+|^2\,dx\,ds \\
&\quad - 2\alpha\beta\sum_{i=1}^3\int_0^{t\wedge\tau_h}\left\langle \frac{1}{2}(y\times\zeta_+), \frac{2}{\alpha\beta}a^i\right\rangle_{\mathbb{H}} h_i\,ds \\
&\quad + 2\alpha\beta\sum_{i=1}^3\int_0^{t\wedge\tau_h}\left\langle \frac{1}{2}(y\times\zeta_+), \frac{2}{\beta}(y\times a^i)\right\rangle_{\mathbb{H}} h_i\,ds \\
&\leqslant -\frac{3}{2}\alpha\beta\int_0^{t\wedge\tau_h}|y\times\zeta_+|^2_{\mathbb{H}}\,ds + \frac{3}{2}\alpha\beta\int_0^{t\wedge\tau_h}|y\times\zeta_+|^2_{\mathbb{H}}\,ds \\
&\quad + \frac{4}{\beta}\left(\frac{1}{\alpha}+\alpha\right)|\Lambda|\sum_{i=1}^3|a^i|^2\int_0^{t\wedge\tau_h}h_i^2\,ds,
\end{aligned} \tag{7.39}$$

where we estimated the integrals on the right hand side of the second equality as follows: the first integral vanished thanks to Lemma 7.6, Lemma 7.1 was used for the integrand of the second integral and the Cauchy-Schwarz inequality and Young's inequality were used for the integrands of the other integrals. Using (7.38) in (7.39), we obtain

$$|y(t\wedge\tau_h)+\zeta_+|_{\mathbb{H}^1} \leqslant r < \frac{1}{2k^2\sqrt{|\Lambda|}}\frac{\alpha}{1+2\alpha} \qquad \forall t\in[0,T]. \tag{7.40}$$

From (7.40) and the definition of τ_h, we conclude that $\tau_h > T$. Hence we have

$$\sup_{t\in[0,T]}|y(t)+\zeta_+|_{\mathbb{H}^1} \leqslant r.$$

By the Freidlin-Wentzell formulation of the upper bound of the large deviation principle, since $r<\rho$, given $\xi\in(0,\infty)$, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, inequality (7.37) holds. ☐

Remark 7.8. Our use of Lemma 7.6 in the proof of Proposition 7.7 means that, in this proposition, we did not need to allow for the spatial variation of magnetization on Λ.

Appendix A. Budhiraja-Dupuis result

Let us recall Theorem 3.6 from [14].

Theorem A.1. *Assume that* K, H *be separable Hilbert spaces such that the embedding*

$$\mathrm{K} \hookrightarrow \mathrm{H} \tag{A.1}$$

is γ*-radonifying, and*

$$f : {}_0C([0,T];\mathrm{H}) \to \mathbb{R}$$

be a bounded (or bounded from below?) Borel measurable function. Then

$$-\log \mathbb{E} e^{-f(W)} = \inf_{h \in \mathscr{A}} \mathbb{E}\Big(\frac{1}{2}\int_0^T |h(s)|^2_{\mathrm{K}} + f\big(W + \int_0^{\cdot} h(s)\,ds\big)\Big), \tag{A.2}$$

where $\mathscr{A}$ *consist of all* K*-valued predictable processes* h *such that*

$$\mathbb{P}\Big\{\int_0^T |h(s)|^2_{\mathrm{K}} < \infty\Big\} = 1. \tag{A.3}$$

For $R > 0$ we denote by $\mathscr{A}_R$ the subset of $\mathscr{A}$ consisting of of all K-valued predictable processes h satisfying

$$\mathbb{P}\Big\{\int_0^T |h(s)|^2_{\mathrm{K}} \leqslant R^2\Big\} = 1. \tag{A.4}$$

Note that $\bigcup_{R>0} \mathscr{A}_R$ is a proper subset of $\mathscr{A}$. Let us also denote by B_R the closed ball of radius R in the set $L^2(0,T;\mathrm{K})$, i.e.

$$B_R := \big\{h \in L^2(0,T;\mathrm{K}) : \int_0^T |h(s)|^2_{\mathrm{K}} \leqslant R^2\big\}. \tag{A.5}$$

We endow B_R with the weak topology induced by $L^2(0,T;\mathrm{K})$.

Let now E be a Polish space and consider a family, indexed by $\varepsilon \in (0,1]$, of Borel measurable maps

$$J^\varepsilon : {}_0C([0,T];\mathrm{H}) \to E.$$

On the space ${}_0C([0,T];\mathrm{H})$ we consider a Wiener measure $\mathbb{P}$ corresponding to the embedding (A.1) (and the integration w.r.t. $\mathbb{P}$ we denote by $\mathbb{E}$). Note that the RKHS of μ is not the space $L^2(0,T;\mathrm{K})$ but the space ${}_0H^{1,2}(0,T;\mathrm{K})$, where

$${}_0H^{1,2}(0,T;\mathrm{K}) = \big\{\omega \in {}_0C([0,T];\mathrm{K}) : \omega' \in L^2(0,T;\mathrm{K})\big\}.$$

Note that the map

$$L^2(0,T;\mathrm{K}) \ni h \mapsto \int_0^{\cdot} h(s)\,ds \in {}_0H^{1,2}(0,T;\mathrm{K})$$

is an isometric isomorphism.

We denote by μ^ε the "image" measure on E of $\mathbb{P}$ by J^ε, i.e.

$$\mu^\varepsilon = J^\varepsilon(\mathbb{P}), \text{ i.e. } \mu^\varepsilon(A) = \mathbb{P}\big((J^\varepsilon)^{-1}(A)\big), \quad A \in \mathscr{B}(E). \tag{A.6}$$

Assume

Assumption 1. *There exists a Borel measurable map*

$$J^0 : {}_0C([0,T];\mathrm{H}) \to E$$

such that

(BD1) *if $R>0$ and a family $h_\varepsilon \subset \mathscr{A}_R$ converges in law on B_R to $h \in \mathscr{A}_R$, then the processes*

$$ {}_0C([0,T],\mathrm{H}) \ni \omega \mapsto J^\varepsilon(\omega + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot h_\varepsilon(s)\,ds) \in E $$

converge in law, as $\varepsilon \searrow 0$, to the process $J^0(\int_0^\cdot h(s)\,ds)$,
and

(BD2) *the set*

$$ \Big\{ J^0(\int_0^\cdot h(s)\,ds) : h \in B_R \Big\} $$

is compact in E.

We have the following result.

Theorem A.2. [14, Theorem 4.4] *If the assumptions listed above, in particular Assumption 1, are satisfies, then the family of measures μ_ε satisfies the LDP with the rate function I defined by*

$$ I(u) := \inf\big\{ \frac{1}{2} \int_0^T |h(s)|^2_{\mathrm{K}} : h \in L^2(0,T;\mathrm{K}) \text{ and } u = J^0(\int_0^\cdot h(s)\,ds), \big\}. \tag{A.7} $$

Note that

$$ I(u) := \inf\big\{ \frac{1}{2} \int_0^T |y'(s)|^2_{\mathrm{K}} : y \in {}_0H^{1,2}(0,T;\mathrm{K}) \text{ and } u = J^0(y), \big\}. \tag{A.8} $$

Obviously, we put, as always, $\inf \varnothing = \infty$.

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