p-JONES-WENZL IDEMPOTENTS

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ABSTRACT. For a prime number p and any natural number n we introduce, by giving an explicit recursive formula, the p-Jones-Wenzl projector ${}^p\mathrm{JW}_n$, an element of the Temperley-Lieb algebra $TL_n(2)$ with coefficients in $\mathbf{F}_p$. We prove that these projectors give the indecomposable objects in the $\tilde{A}_1$-Hecke category over $\mathbf{F}_p$, or equivalently, they give the projector in $\mathrm{End}_{\mathrm{SL}_2(\overline{\mathbf{F}_p})}((\mathbf{F}_p^2)^{\otimes n})$ to the top tilting module. The way in which we find these projectors is by categorifying the fractal appearing in the expression of the p-canonical basis in terms of the Kazhdan-Lusztig basis for $\tilde{A}_1$.

1. INTRODUCTION

1.1. **A new paradigm.** In recent years a new paradigm has emerged in modular representation theory. The central role that the canonical basis of the Hecke algebra (and its associated Kazhdan-Lusztig polynomials) was believed to play is now known to be played by the p-canonical basis (and its associated p-Kazhdan-Lusztig polynomials). The most groundbreaking papers in this direction are (in our opinion) the paper by Williamson [Wil17] commonly known as "Torsion explosion" (that broke down the old paradigm), the paper by Riche and Williamson [RW18] known as the "Tilting manifesto" (that crystallized the emerging philosophy) and the recent paper by Achar, Makisumi, Riche, and Williamson [AMRW19] (that proved the conjecture in the tilting manifesto).

But although this brought a new scenario into place, there was a widespread feeling that the p-canonical basis was impossible to calculate (if it is not by complicated categorical manipulations). But this belief was again annihilated by the beautiful conjecture by Lusztig and Williamson known as the "billiards conjecture" [LW18], where they conjecture a way in which the p-canonical basis in type $\tilde{A}_2$ can be calculated for some finite (but big) family of elements. It is with the intention of continuing on this path that this paper comes into existence.

1.2. **The SL_2 case.** Let us consider type $\tilde{A}_1$ (the infinite dihedral group). In this case it is easy (and known since the dawn of the theory) to obtain an explicit formula for the canonical basis. In the paper [Eli16], Elias lifted the canonical basis to a categorical level in the $\tilde{A}_1$-Hecke category over a field of characteristic zero. He obtained that the Jones-Wenzl projectors give the indecomposable objects. More precisely, there is a functor from the Temperley-Lieb category to the diagrammatic Hecke category such that the images of the Jones-Wenzl projectors give idempotents in the Bott-Samelson objects projecting to the indecomposable objects.

The main result of this paper is an analogous result, but for fields of positive characteristic. The p-canonical basis of $\tilde{A}_1$ was known since the year 2002 by the work of Erdmann and Henke [EH02] (the group SL_2 is the only semi-simple group for which all tilting characters are known). When one expresses this basis in terms of the canonical basis one obtains a fractal-like structure (see Section 4.4). We lift this construction to a categorical level and obtain what we call the p-Jones-Wenzl projectors with recursive formulas as explicit as in the usual Jones-Wenzl projectors.

We would like to remark that the formulas for the projectors in the characteristic zero case were not so surprising as they already appear in the Temperley-Lieb algebra. The formulas found in this paper are completely new. The most challenging and time-consuming part of the present work was to find the correct definition of the p-Jones-Wenzl projectors.

1.3. **Perspectives.** There are at least four possible applications of our construction, the first one being our main motivation for this work.

(1) Using Elias Quantum Satake [Eli17] and Elias triple clasp expansion [Eli15], together with the main result of Williamson's thesis [Wil11] one is not far from completely understanding the projectors giving the indecomposable objects in type $\tilde{A}_2$ over a field of characteristic zero. The recursive formula for the Jones-Wenzl projector is built-in to the recursive formula for $\mathfrak{sl}_3$ (see Formulas (1.7) and (1.8) of [Eli15]). So, as we have a p-analogue of this part of the formula, we would just need a p-analogue of the other part. If that was achieved, one would probably have the p-canonical basis for the whole $\tilde{A}_2$ (at least conjecturally). Of course, this might go far beyond $\tilde{A}_2$, but as the rank grows, the amount of information obtained via Quantum Satake diminishes gradually. In any case, if this approach works, it would give a good chunk of information in any rank.
(2) The Jones-Wenzl projector JW_n is an endomorphism of the n-fold tensor product $V^{\otimes n}$ of the natural representation of the quantum group $U_q(\mathfrak{sl}_2)$ projecting into the maximal simple module. One would like to obtain a projector satisfying the same property, but when q is a root of unity. Our p-Jones-Wenzl projector is certainly not the answer to this question, but might be an important ingredient.
(3) In the same vein as the last point, it would be desirable to get the underlying quiver for $\mathrm{Tilt}_0(\mathrm{SL}_2)$ in prime characteristic, following the approach of [RW18] using the methods in [AT17] (the latter calculate the quiver in the root of unity case using the Jones-Wenzl projector).
(4) The Jones-Wenzl projectors play a key role in the definition of Reshetikhin-Turaev 3-manifold invariants. It is appealing to replace in that definition the Jones-Wenzl projector by the p-Jones-Wenzl projector and see if one obtains an invariant of some kind of object. For example, could it be that if one does this process to the colored Jones polynomial one obtains an invariant of framed links (necessarily more refined than the usual colored Jones polynomial)?

1.4. **Acknowledgements.** The second author presented these results at the conference "Categorification and higher representation theory" in July 2018 at the Mittag Leffler institute. We would like to thank the organisers for the opportunity to discuss this construction and the participants (especially Jonathan Brundan, Daniel Tubbenhauer and Marko Stosic) for useful comments. We would also like to thank Geordie Williamson for interesting discussions.

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2. DEFINITION OF THE p-JONES-WENZL IDEMPOTENTS

2.1. **The generic Temperley Lieb category.** Let $m, n \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$ be such that $n - m$ is even. An (m, n)-*diagram* consists of the following data:

(1) A closed rectangle R in the plane with two opposite edges designated as top and bottom.
(2) m marked points (vertices) on the top edge and n marked points on the bottom edge.
(3) $(n + m)/2$ smooth curves (or "strands") in R such that for each curve γ, $\partial\gamma = \gamma \cap \partial R$ consists of two of the $n + m$ marked points, and such that the curves are pairwise non-intersecting.

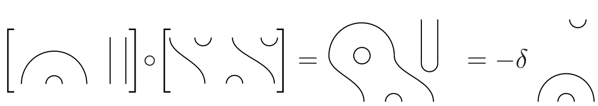
Two such diagrams are *equivalent* if they induce the same pairing of the $n + m$ marked points. We call a (m, n)-*crossingless matching* one such equivalence class.

Let δ be an indeterminate over $\mathbf{Q}$. The *generic Temperley Lieb category* $\mathcal{TL}(\delta)$ (as defined in [GW03]) is a strict monoidal category defined as follows. The objects are the elements of $\mathbf{N}_0$. If $m - n$ is odd, $\mathrm{Hom}(m, n)$ is the zero vector space. If $m - n$ is even, $\mathrm{Hom}(m, n)$ is the $\mathbf{Q}(\delta)$ vector space with basis (m, n)-*crossingless matchings*. The composition of morphisms is first defined on the level of diagrams. The composition $g \circ f$ of an (n, m)-diagram g and an (m, k)-diagram f is defined by the following steps:

(1) Put the rectangle of g on top of that of f, identifying the top edge of f (with its m marked points) with the bottom edge of g (with its m marked points).
(2) Remove from the resulting rectangle any closed loops in its interior. The result is a (n, k)-diagram h.
(3) The composition $g \circ f$ is $(-\delta)^r h$, where r is the number of closed loops removed.

This composition clearly respects equivalence of diagrams. The *tensor product of objects* in $\mathcal{TL}$ is given by $n \otimes n' = n + n'$. The *tensor product of morphisms* is defined by horizontal juxtaposition. With this we end the definition.

Example 2.1. Vertical composition in $\mathcal{TL}(\delta)$:



Consider the *flip involution*, a contravariant functor $\overline{}: \mathcal{TL}(\delta) \to \mathcal{TL}(\delta)$ defined as the identity on objects and by flipping the diagrams upside down on morphisms.

For any natural number n, the *Temperley-Lieb algebra on n strands* is defined to be the $\mathbf{Q}(\delta)$-algebra $TL_n(\delta) := \mathrm{End}_{\mathcal{TL}(\delta)}(n)$.

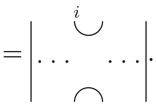
Example 2.2. A generator of $TL_{12}(\delta)$ as a $\mathbf{Q}(\delta)$-module: 

2.2. **Jones-Wenzl projectors.** Let n be a natural number. Let $TL_n(2)$ be the Temperley-Lieb algebra specialised at $\delta \rightsquigarrow 2$.

Proposition 2.3. *There is a unique non-zero idempotent* $\mathrm{JW}_n \in TL_n(2)$*, called the* Jones-Wenzl projector on n strands*, such that*

$$e_i \circ \mathrm{JW}_n = \mathrm{JW}_n \circ e_i = 0,$$

for all $1 \le i \le n-1$*, where* $e_i =$

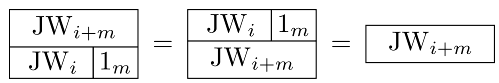


It is easy to see that when the JW_n is expressed in the $\mathbf{Q}$-basis of (n, n)-diagrams, the coefficient of the identity is 1.

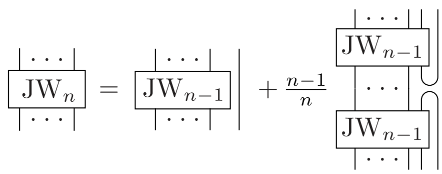
The following proposition adds-up the most important properties of the Jones-Wenzl projectors. We will prove a p-analogue of these properties later in the paper.

Proposition 2.4. *The Jones-Wenzl projectors satisfy:*

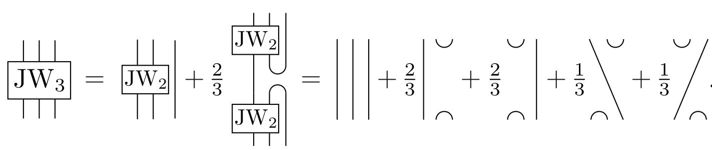
(1) ***Absorption.***



(2) ***Recursion.***



As an example of the recursion,



The following equality follows easily from the definitions:

$$\mathrm{JW}_m \circ \mathrm{Hom}_{\mathcal{TL}}(n, m) \circ \mathrm{JW}_n = \begin{cases} \{0\}, & \text{if } n \neq m, \\ \mathrm{span}_{\mathbf{Q}}\{\mathrm{JW}_n\}, & \text{if } n = m. \end{cases} \tag{2.1}$$

2.3. **Definition of the p-Jones Wenzl projectors.** Let us fix a prime number p for the rest of this section. We will introduce the p-analogue of the Jones-Wenzl idempotents defined above.

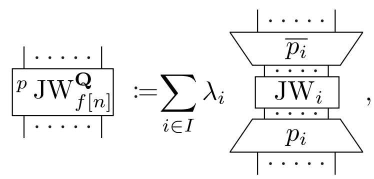
If $n \in \mathbf{N}$ is an integer and $a_i p^i + a_{i-1}p^{i-1} + \cdots + a_1 p + a_0$ is the p-adic expansion of $n+1$, we define the *support of* n to be the following set of natural numbers

$$\mathrm{supp}(n) = \{a_i p^i \pm a_{i-1}p^{i-1} \pm \cdots \pm a_1 p \pm a_0\}.$$

Definition 2.5. Let n be a natural number. If $n+1$ has at least two non-zero coefficients in its p-adic expansion, we define *the father of* n to be the natural number $f[n]$ obtained by replacing the right-most non-zero coefficient in the p-adic expansion of $n+1$ by zero and then substracting 1. In formulas, if $n+1 = \sum_{i=m}^{r} a_i p^i$ with $a_m \neq 0$, then $f[n] := (\sum_{i=m+1}^{r} a_i p^i) - 1$. If $n+1$ has only one non-zero coefficient in its p-adic expansion, then $n+1 = jp^i$ for some $0 < j < p$ and some $i \in \mathbb{N}$. In that case, we say that n is a *p-Adam* (because it has no father).

Let us start by defining the *rational p-Jones-Wenzl projector on n strands*, denoted by ${}^p\mathrm{JW}_n^{\mathbf{Q}}$. It will be defined by induction on the number of non-zero coefficients in the p-adic expansion of $n+1$. If n is a p-Adam, we define ${}^p\mathrm{JW}_n^{\mathbf{Q}} := \mathrm{JW}_n$.

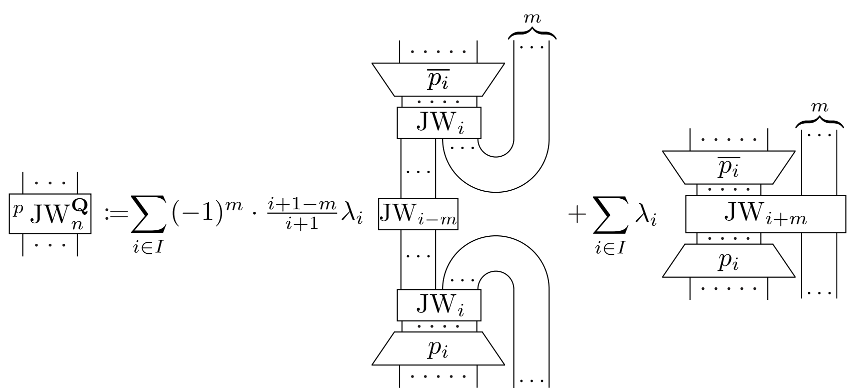
If n is not a p-Adam, suppose that



with $I = \mathrm{supp}(f[n]) - 1$, $\lambda_i \in \mathbf{Q}$, $p_i \in \mathrm{Hom}_{\mathcal{TL}}(f[n], i)$ and $\overline{p_i}$ the image of p_i under the flip involution (in the case where $f[n]$ is a p-Adam, we have that $\lambda_{f[n]} = 1$ and $p_{f[n]} = \mathrm{id}$).

Let $m := n - f[n]$. We define

(2.2)



The "new" set of p_i and λ_i should be clear from the picture. The new index set is $(I-m) \sqcup (I+m)$ which is exactly $\mathrm{supp}(n) - 1$. With this we finish the definition of ${}^p\mathrm{JW}_n^{\mathbf{Q}} := \sum \lambda_i \, (\overline{p_i} \circ \mathrm{JW}_i \circ p_i)$. For notational convencience,we will sometimes use the notation $U_n^i := \overline{p_i} \circ \mathrm{JW}_i \circ p_i$, so we have

$$ {}^p\mathrm{JW}_n^{\mathbf{Q}} := \sum \lambda_i U_n^i. \tag{2.3} $$

Theorem 2.6. *For all $n \in \mathbf{N}$, the morphism ${}^p\mathrm{JW}_n^{\mathbf{Q}} \in TL_n(2)$ is an idempotent. Furthermore, if we express ${}^p\mathrm{JW}_n^{\mathbf{Q}}$ in the $\mathbf{Q}$-basis of crossingless matchings, and write each of its coefficients as an irreducible fraction a/b, then p does not divide b.*

We remark that in the definition of the Temperley-Lieb algebra one could have used any other commutative ring $\mathbf{R}$ instead of $\mathbf{Q}$. We denote by $TL_n(2)_{\mathbf{R}}$ the corresponding algebra. Now we can state the main definition of this paper.

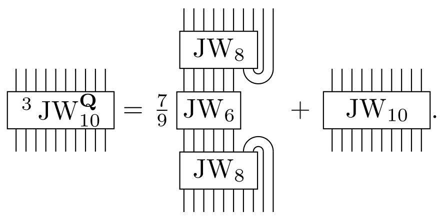
Definition 2.7. We define the *p-Jones-Wenzl projector on n-strands* ${}^p\mathrm{JW}_n \in TL_n(2)_{\mathbf{F}_p}$ as the expansion of ${}^p\mathrm{JW}_n^{\mathbf{Q}} \in TL_n(2)$ in the $\mathbf{Q}$-basis of crossingless matchings but replacing each of the coefficients a/b (expressed as irreducible fractions) by $\overline{a} \cdot \left(\overline{b}\right)^{-1} \in \mathbf{F}_p$, where the bar means reduction modulo p.

Remark 2.8. A more elegant way to define the p-Jones Wenzl projector is to lift ${}^p\mathrm{JW}_n^{\mathbf{Q}} \in TL_n(2)_{\mathbf{Q}} \subset TL_n(2)_{\mathbf{Q}_p}$ to an idempotent in $TL_n(2)_{\mathbf{Z}_p}$ and then project to $TL_n(2)_{\mathbf{F}_p}$.

Remark 2.9. One can define an analogue of ${}^p\mathrm{JW}_n^{\mathbf{Q}} \in TL_n(2)$ in the generic Temperley-Lieb algebra $TL_n(\delta)$. This is done by replacing natural numbers by quantum numbers in the coefficients of the formula. For instance, $(i+1-m)/(i+1)$ must be changed by $[i+1-m]_q/[i+1]_q \in \mathbf{Q}(\delta)$. The projectors thus defined satisfy all properties in Section 3.1 essentially with the same proof.

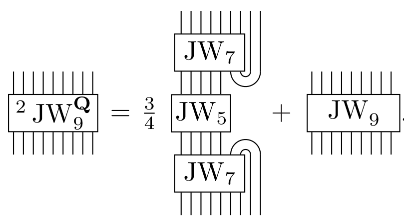
Example 2.10 (Example of a rational 3-Jones-Wenzl projector)**.** Let us compute ${}^3\mathrm{JW}_{10}^{\mathbf{Q}}$. We notice that $f_3[10] = 8$ and that 8 is a 3-Adam. Using (2.2) we have,

(2.4)



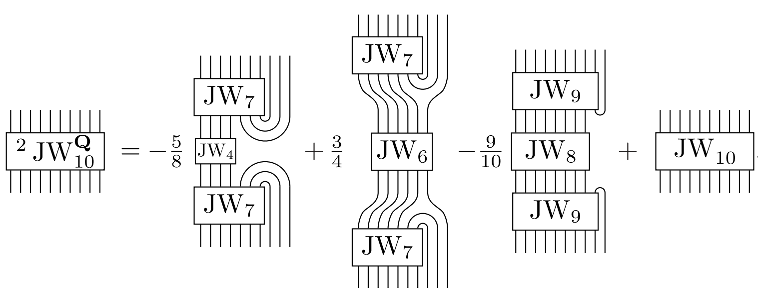
Example 2.11 (Example of rational 2-Jones-Wenzl)**.** To calculate ${}^2\mathrm{JW}_{10}^{\mathbf{Q}}$, first we note that $f_2[10] = 9$, $f_2[9] = 7$ and 7 is a 2-Adam. Using (2.2) we have,

(2.5)



Using (2.2) again we obtain,

(2.6)

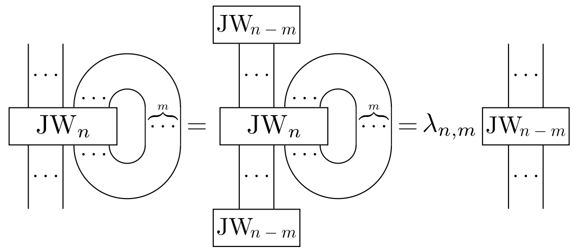


Note that ${}^3\mathrm{JW}_{10}^{\mathbf{Q}}$ and ${}^2\mathrm{JW}_{10}^{\mathbf{Q}}$ are quite different.

3. Some properties of the p-Jones-Wenzl projectors

3.1. The following lemma, although simple, will prove to be useful.

Lemma 3.1. *Let $0 \leqslant m \leqslant n$. In $TL_n(2)$ we have the equality*



where $\lambda_{n,m} := (-1)^m \cdot \frac{n+1}{n+1-m}$.

Proof. The first equality is a consequence of Proposition 2.4; moreover, from (2.1) we can deduce the existence of some coefficients $\lambda_{n,m} \in \mathbf{Q}$ satisfying the second equality. We only need to calculate these $\lambda_{n,m}$ to finish the proof. Let us observe that $\lambda_{n,0} = 1$ and $\lambda_{n,m} = \lambda_{n,k} \cdot \lambda_{n-k,m-k}$, for all $0 \leqslant k \leqslant m$. We prove the result by induction on m. For $m = 1$ we have that $\lambda_{n,1} = -(n+1)/n$, by [EL17, Eq. (2.8)]. Let $m > 1$. By our inductive hypothesis we obtain

$$\lambda_{n,m} = \lambda_{n,1} \cdot \lambda_{n-1,m-1} = -\frac{(n+1)}{n} \cdot \frac{(-1)^{m-1} \cdot n}{n-(m-1)} = (-1)^m \cdot \frac{n+1}{n+1-m}.$$

□

Proposition 3.2. *The element ${}^p\mathrm{JW}_n^{\mathbf{Q}} \in TL_n(2)$ is an idempotent. Moreover, $\{\lambda_i U_n^i\}_{i \in I}$ is a set of mutually orthogonal idempotents.*

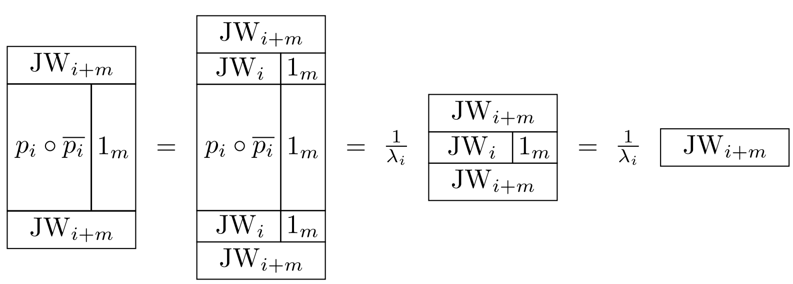
Proof. We will prove it by induction in the number of non-zero terms that $n+1$ has in the p-adic expansion. If n is a p-Adam, then ${}^p\mathrm{JW}_n^{\mathbf{Q}} = \mathrm{JW}_n$, which is an idempotent. Consider now n not to be a p-Adam. Let

$${}^p\mathrm{JW}_{f[n]}^{\mathbf{Q}} = \sum_{i \in \mathrm{supp}(f[n])-1} \lambda_i \ (\overline{p_i} \circ \mathrm{JW}_i \circ p_i).$$

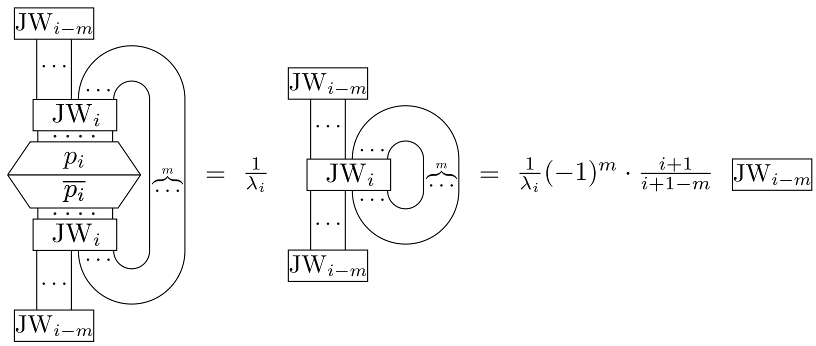
By our inductive hypothesis and Equation (2.1), we have that

$$\mathrm{JW}_i \circ p_i \circ \overline{p_i} \circ \mathrm{JW}_i = \frac{1}{\lambda_i} \mathrm{JW}_i \tag{3.1}$$

and $\mathrm{JW}_i \circ p_i \circ \overline{p_j} \circ \mathrm{JW}_j = 0$, for all $i \neq j \in \mathrm{supp}(f[n]) - 1$. Then, absortion and Equation (3.1) give

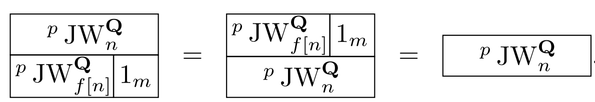


Equation (3.1) and Lemma 3.1 give

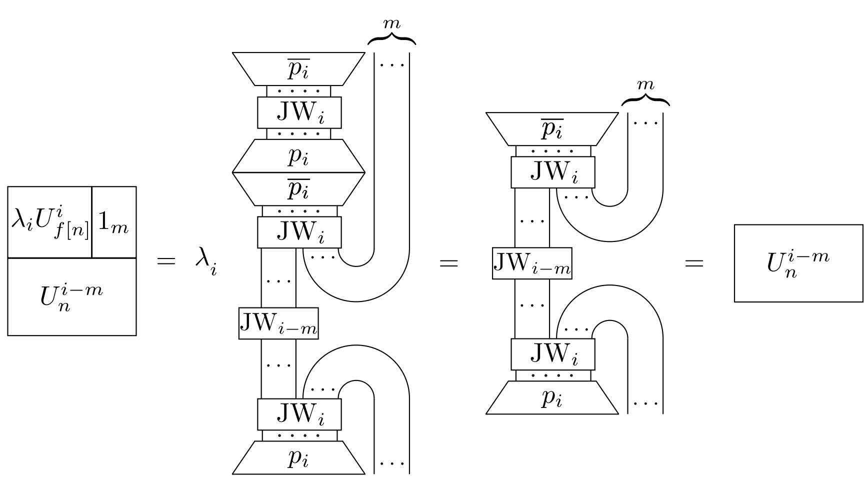


These two formulas prove the idempotence of the summands in ${}^p\mathrm{JW}_n^{\mathbf{Q}}$. Since $i \pm m \neq j \pm m$ for all $i, j \in \mathrm{supp}(f[n]) - 1$, $i \neq j$, by (2.1) we finish the proof. □

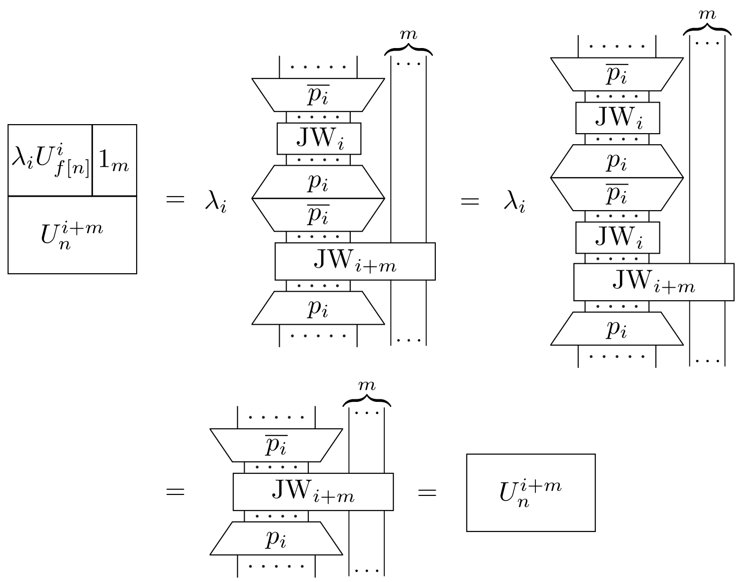
Proposition 3.3. *The idempotent* ${}^p\mathrm{JW}_n^{\mathbf{Q}}$ *satisfies the following absorption property:*



Proof. We prove only the second equality, the first one being analogous. Recall the notation expressed by Equation (2.3) and remark that



By the first part of Proposition 2.4 and Equation (3.1) we have that



The remaining terms appearing in the expansion of the left-hand side are all zero by Equation (2.1). ☐

4. THE HECKE CATEGORY OF $\tilde{A}_1$-SOERGEL BIMODULES

4.1. **Hecke algebra.** The *infinite dihedral group* U_2 (of type $\tilde{A}_1$) is the group with presentation $U_2 = \langle s, t : s^2 = t^2 = e \rangle$. We denote the length function by ℓ and the Bruhat order by $\leq$. An *expression* is an ordered tuple $\underline{w} = (s_1, s_2, \ldots, s_r)$ of elements of S.

We denote by $w \in W$ the corresponding product of simple reflections $w = s_1 s_2 \cdots s_r$.

Consider the ring $\mathcal{L} = \mathbf{Z}[v^{\pm 1}]$ of Laurent polynomials with integer coefficients in one variable v. The *Hecke algebra* $\mathbf{H}$ of the infinite dihedral group is the free $\mathcal{L}$-module with basis $\{H_w \mid w \in U_2\}$ and multiplication given by:

$$H_w H_s = \begin{cases} H_w H_s, & \text{if } w < ws; \\ H_{ws} + (v^{-1} - v) H_w, & \text{if } ws < w, \end{cases}$$

for all $w \in U_2$. The set $\{H_w : w \in U_2\}$ is called the *standard basis* of $\mathbf{H}$. On the other hand, $\mathbf{H}$ has the *Kazhdan-Lusztig basis* (or *KL-basis*) that we call $\{b_w : w \in U_2\}$. In the literature this basis is also denoted by $\underline{H}_w$ (see [JW17]) or C'_w in the original paper by Kazhdan and Lusztig [KL79]. The following formula has an easy proof (all the calculations with the infinite dihedral group are explicit).

Lemma 4.1. *Let $s_1 \cdots s_k$ be a reduced expression of $x \in U_2$ and r a simple reflection. Then*

$$b_w b_r = \begin{cases} (v + v^{-1}) b_w & \text{if } r = s_k; \\ b_{wr} + b_{ws_k} & \text{if } k > 1 \text{ and } r = s_{k-1}; \\ b_{wr} & \text{otherwise.} \end{cases}$$

4.2. **The p-canonical basis.** Consider the Coxeter system $W = U_2$. Let $\mathcal{H}$ be the Hecke category (as defined in [EW16]) which is minimal over $\mathbf{Z}$ with Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

This defines a category $\mathcal{H}^{\Bbbk}$ by base change, for any ring $\Bbbk$. The main result of that paper [EW16, Theorem 6.26] (although in that paper they consider general Coxeter systems and a big range of realizations) is the following. The category $\mathcal{H}$ is a Krull-Remak-Schmidt $\mathbf{Z}$-linear category with a grading shift functor $[1]$. The indecomposable objects B_w are indexed by $w \in W$ (modulo grading shift) and $B_w \overset{\oplus}{\subset} \underline{w}$ is the only summand of $\underline{w}$ (where $\underline{w}$ is any reduced expression of $w \in W$) that does not appear in any reduced expression $\underline{u}$, with $u \leq w$. Furthermore, there is an isomorphism of $\mathbf{Z}[v^{\pm 1}]$-algebras called the *character*

$$\begin{aligned} \mathrm{ch}\colon \langle \mathcal{H} \rangle &\longrightarrow \mathbf{H} \\ \langle B_s \rangle &\longmapsto H_s, \end{aligned}$$

where $s \in S$, and $\langle \mathcal{H} \rangle$ denotes the split Grothendieck group of $\mathcal{H}$. The group $\langle \mathcal{H} \rangle$ has a $\mathbf{Z}[v^{\pm 1}]$-algebra structure as follows: the monoidal structure on $\mathcal{H}$ induces a unital, associative multiplication and v acts via $v[B] := [B(1)]$ for an object B of $\mathcal{H}$.

It is a fundamental theorem by Elias and Williamson [EW14] (again, for any Coxeter group) conjectured by Soergel that in $\mathcal{H}^{\mathbf{R}}$, the image of the indecomposable objects are the Kazhdan-Lusztig basis. In formulas:

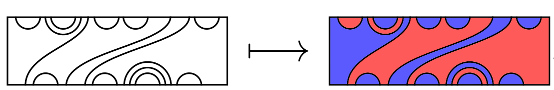
$$\mathrm{ch}(\langle B_w \rangle) = b_w.$$

However, in $\mathcal{H}^{\mathbb{F}_p}$ this is not the case. In this latter category, to emphasize the dependence on p, we will denote by ${}^pB_w$ the indecomposable object. Let us define

$$\mathrm{ch}(\langle {}^pB_w \rangle) := {}^pb_w.$$

The set $\{{}^pb_w\}_{w \in W}$ is another $\mathbf{Z}[v^{\pm 1}]$-basis of $\mathbf{H}$. It is called the *p-canonical basis of* $\mathbf{H}$.

4.3. **The Jones-Wenzl idempotents as Soergel bimodules.** For the infinite dihedral group, let us color the set $S = \{s, t\}$. Let $\mathcal{TL}_c$ be the 2-colored Temperley-Lieb category as defined in [EL17, Section 2.1] (this is just the generic Temperley-Lieb category defined in Section 2.1, specialised at $\delta \rightsquigarrow 2$, with regions colored by elements of S in such a way that adjacent regions always have different colours). Take a diagram $\mathcal{E}$ in $\mathcal{TL}$ and colour its regions accordingly using the set S with s colouring its left-most region. This new diagram ${}^s\mathcal{E}$ is a morphism in $\mathcal{TL}_c$. By abuse of notation we will just call it $\mathcal{E}$. For example,



By the main theorem of Elias and the second author's paper [EL17, Proposition 3.10], there is an additive $\mathbf{Q}$-linear monoidal faithful functor (fully faithful if one only considers degree zero morphisms in $\mathcal{H}^{\mathbf{Q}}$) $\mathcal{F}\colon \mathrm{Kar}(\mathcal{TL}_c) \longrightarrow \mathcal{H}^{\mathbf{Q}}$, where $\mathrm{Kar}(-)$ is the Karoubi envelope functor. The functor $\mathcal{F}$ is essentially deformation retract. It takes JW_n into the indecomposable object $B_{\underline{n+1}}$, where $\underline{n}$ is the unique element w of length n such that $sw < w$ in U_2.

4.4. **Categorification.** By [JW17, Lemma 5.1] we have that

$$ {}^{p}b_{\underline{n+1}} = \sum_{i \in \mathrm{supp}(n)} b_{\underline{i}}. \tag{4.1} $$

We will categorify this formula. Recall that $\sum_{i\in \mathrm{supp}(n)-1} \lambda_i U_n^i$ is the orthogonal decomposition of ${}^{p}\mathrm{JW}_n^{\mathbf{Q}}$, as in (2.3), where $U_n^i := \overline{p_i} \circ \mathrm{JW}_i \circ p_i$.

Proposition 4.2. *In the category* $\mathrm{Kar}(\mathcal{TL}_c)$ *there is an isomorphism*

$$ {}^{p}\mathrm{JW}_n^{\mathbf{Q}} \cong \bigoplus_{i\in \mathrm{supp}(n)-1} \mathrm{JW}_i. $$

Proof. Since $\{\lambda_i U_n^i\}_{i\in\mathrm{supp}(n)}$ is a set of mutually orthogonal projectors, it is enough to prove that $\lambda_i U_n^i \cong \mathrm{JW}_i$. Consider the map

$$ f = \lambda_i(\mathrm{JW}_i \circ p_i) : (n, \lambda_i U_n^i) \to (i, \mathrm{JW}_i). $$

By Equation (3.1) one can see that f is indeed a map in the Karoubi envelope, i.e., $f = \mathrm{JW}_i \circ f \circ (\lambda_i U_n^i)$. It is easy to prove that $g = \overline{p_i} \circ \mathrm{JW}_i$ is the inverse of f, thus proving the proposition. □

Applying the functor $\mathcal{F}$ one obtains

$$ \mathcal{F}({}^{p}\mathrm{JW}_n^{\mathbf{Q}}) \cong \bigoplus_{i\in\mathrm{supp}(n)} B_{\underline{i}}. $$

Finally, we decategorify by applying the character ch defined in Section 4.2 and obtain

$$ \mathrm{ch}(\langle \mathcal{F}({}^{p}\mathrm{JW}_n^{\mathbf{Q}})\rangle) = {}^{p}b_{\underline{n+1}}. \tag{4.2} $$

4.5. **The absorption property determines the rational p-Jones Wenzl projector.**

Notation 4.3. Consider $m \in \mathbf{N}$. If n is even, we denote ${}^{p}b_{\underline{n}} b_1^m := {}^{p}b_{\underline{n}} \underbrace{b_s b_t b_s \cdots}_{m \text{ terms}}$. If n is odd, we denote ${}^{p}b_{\underline{n}} b_1^m := {}^{p}b_{\underline{n}} \underbrace{b_t b_s b_t \cdots}_{m \text{ terms}}$.

Lemma 4.4. *If* $n \in \mathbf{N}$ *and* $m := n - f[n]$*, there is a finite set* $K \subset \mathbf{N}$ *such that*

$$ {}^{p}b_{\underline{f[n]+1}} b_1^m = {}^{p}b_{\underline{n+1}} + \sum_{k\in K} c_k\, b_{\underline{k}} $$

where $k \notin \mathrm{supp}(n)$ *and* $c_k \in \mathbf{N}$ *for all* $k \in K$.

Proof. By Equation (4.1) we have that

$$ {}^{p}b_{\underline{f[n]+1}} = \sum_{j\in J} b_{\underline{j}} $$

with $J := \mathrm{supp}(f[n])$.

By using m times Lemma 4.1 we have,

$$\begin{aligned}{}^{p}b_{\underline{f[n]+1}}b_1^m &= \sum_{j\in J} b_{\underline{j}} b_1^m \\ &= \sum_{j\in J}\sum_{r=0}^{m}\binom{m}{r} b_{\underline{j-m+2r}} \\ &= \sum_{j\in \mathrm{supp}(n)} b_{\underline{j}} + \sum_{j\in J}\sum_{r=1}^{m-1}\binom{m}{r} b_{\underline{j-m+2r}} \\ &= {}^{p}b_{\underline{n+1}} + \sum_{j\in J}\sum_{r=1}^{m-1}\binom{m}{r} b_{\underline{j-m+2r}}.\end{aligned}$$

since $0 < r < m$, this concludes the lemma. □

It is well known that in $\mathcal{H}^{\mathbf{Q}}$, the degree zero part of $\mathrm{Hom}(B_x, B_y)$ is either $\mathbf{Q}\cdot \mathrm{id}$ if $x = y$ or zero if $x \neq y$. On the other hand, the absorption property (Proposition 3.3) means that $\mathcal{F}({}^{p}\mathrm{JW}_n^{\mathbf{Q}})$ is a direct summand of $\mathcal{F}({}^{p}\mathrm{JW}_{f[n]}^{\mathbf{Q}}) \otimes \mathrm{id}^m$. So, by Lemma 4.4, the projector $\mathcal{F}({}^{p}\mathrm{JW}_n^{\mathbf{Q}})$ is the unique idempotent in the endomorphism ring of $\mathcal{F}({}^{p}\mathrm{JW}_{f[n]}^{\mathbf{Q}}) \otimes \mathrm{id}^m$ whose image is isomorphic to that of $\mathcal{F}({}^{p}\mathrm{JW}_n^{\mathbf{Q}})$ (or, in other words, whose image categorifies ${}^{p}b_{\underline{n+1}}$). This could be an alternative definition of the rational p-Jones-Wenzl projector.

4.6. **Proof of Theorem 2.6.**

Proof. By abuse of notation, if $b \in \mathcal{H}^{\mathbf{Z}}$ we will denote by b the corresponding object in $\mathcal{H}^{\mathbb{k}}$, for any ring $\mathbb{k}$. By construction of the morphism spaces in $\mathcal{H}^{\mathbb{k}}$ (light leaves are always a $\mathbb{k}$-basis of the Hom spaces between Bott-Samelson objects) we have that

$$\mathrm{Hom}_{\mathcal{H}^{\mathbf{Z}}}(b, b') \otimes_{\mathbf{Z}} \mathbb{k} \cong \mathrm{Hom}_{\mathcal{H}^{\mathbb{k}}}(b, b'). \tag{4.3}$$

Let $\mathbf{F}_p$ be the finite field with p elements, $\mathbf{Z}_p$ the p-adic integers and $\mathbf{Q}_p$ the p-adic numbers. The isomorphism (4.3) gives sense to the following functors

- $\otimes_{\mathbf{Z}_p}\mathbf{Q}_p : \mathcal{H}^{\mathbf{Z}_p} \to \mathcal{H}^{\mathbf{Q}_p}$
- $\otimes_{\mathbf{Z}_p}\mathbf{F}_p : \mathcal{H}^{\mathbf{Z}_p} \to \mathcal{H}^{\mathbf{F}_p}$

Notation 4.5. For the rest of this proof, we will consider objects and morphisms in the Temperley-Lieb category (via the functor $\mathcal{F}$) as if they were objects and morphisms in the Hecke category. For example, ${}^{p}\mathrm{JW}_n^{\mathbf{Q}} \in \mathcal{H}^{\mathbf{Q}}$.

We need to prove that ${}^{p}\mathrm{JW}_n^{\mathbf{Q}}$ seen as a morphism in $\mathcal{H}^{\mathbf{Q}_p}$ can be lifted to $\mathcal{H}^{\mathbf{Z}_p}$ using the functor $\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$. We will prove it by induction on the number of non-zero coefficients in the p-adic expansion of $n+1$. If n is a p-Adam, then ${}^{p}\mathrm{JW}_n^{\mathbf{Q}} = \mathrm{JW}_n$. It can easily be deduced from [EL17, Theorem A.2] that JW_n is defined over (can be lifted to) $\mathbf{Z}_p$.

Now we suppose that ${}^{p}\mathrm{JW}_{f[n]}^{\mathbf{Q}}$ can be lifted to $\mathcal{H}^{\mathbf{Z}_p}$. We will prove that ${}^{p}\mathrm{JW}_n^{\mathbf{Q}}$ can also be lifted. Let us say that ${}^{p}\mathrm{JW}_{f[n]}^{\mathbf{Z}_p} \in \mathcal{H}^{\mathbf{Z}_p}$ is this lifting and ${}^{p}\mathrm{JW}_{f[n]}^{\mathbf{F}_p} \in \mathcal{H}^{\mathbf{F}_p}$

is the image of this morphism under the functor $\otimes_{\mathbf{Z}_p}\mathbf{F}_p$. As for any b, b' objects of $\mathcal{H}^{\mathbf{Z}}$ we have

$$\dim(\mathrm{Hom}_{\mathcal{H}^{\mathbf{F}_p}}(b,b')) = \mathrm{rk}(\mathrm{Hom}_{\mathcal{H}^{\mathbf{Z}_p}}(b,b')) = \dim(\mathrm{Hom}_{\mathcal{H}^{\mathbf{Q}_p}}(b,b')).$$

By the formula for ch given in [EW16, Definition 6.23], we obtain at the decategorified level

$$\mathrm{ch}(\langle {}^p\mathrm{JW}^{\mathbf{F}_p}_{f[n]}\rangle) = \mathrm{ch}(\langle {}^p\mathrm{JW}^{\mathbf{Q}}_{f[n]}\rangle) = {}^p b_{\underline{f[n]+1}}.$$

So ${}^p\mathrm{JW}^{\mathbf{F}_p}_{f[n]}$ is isomorphic to the indecomposable object corresponding to the unique word of length $f[n]+1$ starting with s. In formulas

$${}^p\mathrm{JW}^{\mathbf{F}_p}_{f[n]} \cong {}^p B_{\underline{f[n]+1}} \in \mathcal{H}^{\mathbf{F}_p}.$$

Recall that $m = n - f[n]$. The indecomposable object ${}^pB_{\underline{n+1}}$ is a direct summand of ${}^p\mathrm{JW}^{\mathbf{F}_p}_{f[n]}\otimes \mathrm{id}^m$. Let $\pi_{\mathbf{F}_p} \in \mathrm{End}({}^p\mathrm{JW}^{\mathbf{F}_p}_{f[n]}\otimes \mathrm{id}^m)$ be the corresponding projector. Since $\mathrm{End}({}^p\mathrm{JW}^{\mathbf{Z}_p}_{f[n]}\otimes \mathrm{id}^m)$ is a finitely generated $\mathbf{Z}_p$-module, we can use idempotent lifting techniques for complete local rings (see [Lam13, Proposition 21.34 (1)]) and find an idempotent $\pi_{\mathbf{Z}_p} \in \mathrm{End}({}^p\mathrm{JW}^{\mathbf{Z}_p}_{f[n]}\otimes \mathrm{id}^m)$ mapping to $\pi_{\mathbf{F}_p}$.

By applying the corresponding functor one obtains an idempotent

$$(\pi_{\mathbf{Z}_p}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p) \in \mathrm{End}({}^p\mathrm{JW}^{\mathbf{Q}_p}_{f[n]}\otimes \mathrm{id}^m)$$

that (as $\pi_{\mathbf{Z}_p}$ and $\pi_{\mathbf{F}_p}$) decategorifies into ${}^pb_{\underline{n+1}}$. But we have seen ${}^p\mathrm{JW}^{\mathbf{Q}}_n$ is the unique idempotent in the endomorphism ring of ${}^p\mathrm{JW}^{\mathbf{Q}}_{f[n]}\otimes \mathrm{id}^m$ whose image categorifies ${}^pb_{\underline{n+1}}$. Thus, $(\pi_{\mathbf{Z}_p}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p) = {}^p\mathrm{JW}^{\mathbf{Q}}_n$. This implies that ${}^p\mathrm{JW}^{\mathbf{Q}}_n$ can be lifted to $\pi_{\mathbf{Z}_p}\in\mathcal{H}^{\mathbf{Z}_p}$. □

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