SUPERSYMMETRIC W-ALGEBRAS

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Abstract. We develop a general theory of W-algebras in the context of supersymmetric vertex algebras. We describe the structure of W-algebras associated with odd nilpotent elements of Lie superalgebras in terms of their free generating sets. As an application, we produce explicit free generators of the W-algebra associated with the odd principal nilpotent element of the Lie superalgebra $\mathfrak{gl}(n+1|n)$.

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1. Introduction

The W-algebras first appeared in relation with the conformal field theory in the work of Zamolodchikov [23] and Fateev and Lukyanov [10]. These algebras were studied intensively by physicists, both at the classical level through Hamiltonian reduction of Wess–Zumino–Novikov–Witten models and their connection with affine Lie algebras, see e.g. [4, 11, 13], but also using BRST formalism [6, 7]. For an extensive review on physicists works, see [5] and references therein. A definition of the W-algebras in the context of the vertex algebra theory and quantized Drinfeld–Sokolov reduction was given by Feigin and Frenkel [12]; see also the book by Frenkel and D. Ben-Zvi [14, Ch. 15]. A more general family of W-algebras $W^k(\mathfrak{g}, f)$ was introduced by Kac, Roan and Wakimoto [20], which depends on a simple Lie (super)algebra $\mathfrak{g}$, an (even) nilpotent element $f \in \mathfrak{g}$ and the *level* $k \in \mathbb{C}$. In the particular case of the principal nilpotent element $f = f_{\text{prin}}$ this reduces to the definition of [12]; see also a recent expository article by Arakawa [1] where basic structure theorems and representation theory of W-algebras are reviewed.

In the present paper we will be concerned with supersymmetric counterparts of the W-algebras which can be defined by analogy with [14, Ch. 15]. Such W-algebras have already been studied, mostly in the physics literature; see [9, 16, 17]. Moreover, a supersymmetric quantum hamiltonian reduction approach was developed in the work of Madsen and the second author [22]. We will rely on this work and the supersymmetric vertex algebra theory developed by Heluani and Kac [15, 18] to describe the structure of the W-algebras associated with odd nilpotent elements of Lie superalgebras. Our main structural result is Theorem 4.11 which describes free generating sets of the W-algebras.

We will then apply the main result to the case of the general linear Lie superalgebras. It is well-known that the Lie superalgebra $\mathfrak{gl}(m|n)$ contains an odd principal nilpotent element if and only if $m = n \pm 1$. We take $m = n + 1$ (this can be done without a real loss of generality) and produce explicit free generators of the W-algebra as coefficients of a certain noncommutative characteristic polynomial (Theorems 5.1 and 5.3). These formulas can be regarded as supersymmetric analogues of the generators of the principal W-algebra associated with the Lie algebra $\mathfrak{gl}(n)$ produced by Arakawa and the first author [2]. Furthermore, we show that the Miura transformation used in [2] can also be applied in the supersymmetric context to recover the generators of the W-algebra appeared in [9, 16, 17].

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2. Supersymmetric Vertex Algebras

In this section, we introduce supersymmetric vertex algebras following [15] and [18]. Proofs and additional details can be found in these references. Note that in the terminology of the paper [15] these objects are called $N_K = 1$ *supersymmetric vertex algebras.*

2.1. **Notation and basic definitions.** We will be considering two couples of coordinates

$$Z = (z, \theta), \quad W = (w, \zeta),$$

where z and w are even and θ and ζ are odd. Introduce the notation

$$\mathbb{C}[\![Z]\!] := \mathbb{C}[\![z]\!] \otimes \mathbb{C}[\theta], \quad \mathbb{C}(\!(Z)\!) := \mathbb{C}(\!(z)\!) \otimes \mathbb{C}[\theta].$$

Since $\theta^2 = 0$ we have $\mathbb{C}[\theta] = \mathbb{C} \oplus \mathbb{C}\theta$. Similarly,

$$\mathbb{C}[Z, Z^{-1}] := \mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\theta], \quad \mathbb{C}[\![Z, Z^{-1}]\!] := \mathbb{C}[\![z, z^{-1}]\!] \otimes \mathbb{C}[\theta].$$

Furthermore, set

$$\begin{aligned} Z - W &:= (z - w - \theta\zeta, \theta - \zeta), \\ Z^{j_0|j_1} &:= z^{j_0}\theta^{j_1} \quad \text{for } \ j_0 \in \mathbb{Z}, \ j_1 = 0, 1, \\ (Z - W)^{j_0|j_1} &:= (z - w - \theta\zeta)^{j_0}(\theta - \zeta)^{j_1}. \end{aligned}$$

Let $\mathcal{U} = \mathcal{U}_{\bar{0}} \oplus \mathcal{U}_{\bar{1}}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space which we will also call a *vector superspace*. Accordingly, elements $a \in \mathcal{U}_{\bar{0}}$ (resp. $a \in \mathcal{U}_{\bar{1}}$) are called *even* (resp. *odd*) with the parity $p(a) = \bar{0}$ (resp. $p(a) = \bar{1}$). The corresponding endomorphism algebra $\operatorname{End}\mathcal{U} = (\operatorname{End}\mathcal{U})_{\bar{0}} \oplus (\operatorname{End}\mathcal{U})_{\bar{1}}$ is a superalgebra, where

$$f \in (\operatorname{End}\mathcal{U})_{\bar{\imath}} \quad \Longleftrightarrow \quad f\big((\operatorname{End}\mathcal{U})_{\bar{\jmath}}\big) \subset (\operatorname{End}\mathcal{U})_{\bar{\imath}+\bar{\jmath}}$$

for any $\bar{\imath}, \bar{\jmath} \in \mathbb{Z}/2\mathbb{Z}$.

Any element of the vector superspace $\mathcal{U}[\![Z, Z^{-1}]\!] := \mathcal{U} \otimes \mathbb{C}[\![Z, Z^{-1}]\!]$ is called a $\mathcal{U}$*-valued formal distribution*. It has the form

$$a(Z) = \sum_{j_0 \in \mathbb{Z},\, j_1 = 0,1} Z^{j_0|j_1} a_{j_0|j_1} \in \mathcal{U}[\![Z, Z^{-1}]\!], \qquad a_{j_0|j_1} \in \mathcal{U}. \tag{2.1}$$

The *super residue* of a formal distribution $a(Z)$ is defined by

$$\operatorname{res}_Z a(Z) := a_{-1|1} \in \mathcal{U}.$$

Since $\operatorname{res}_Z Z^{j_0|j_1} a(Z) = a_{-1-j_0|1-j_1}$, it is convenient to use the notation

$$a_{(j_0|j_1)} := \operatorname{res}_Z Z^{j_0|j_1} a(Z)$$

so that $a_{j_0|j_1} = a_{(-1-j_0|1-j_1)}$ and the distribution $a(Z)$ in (2.1) takes the form

$$a(Z) = \sum_{j_0 \in \mathbb{Z},\, j_1 = 0,1} Z^{-1-j_0|1-j_1} a_{(j_0|j_1)}.$$

An $\operatorname{End}\mathcal{U}$-valued formal distribution $a(Z)$ is called a *super field* if for any given $v \in \mathcal{U}$ there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that

$$a_{(j_0|j_1)} v = 0 \quad \text{for all} \quad j_0 \geqslant N\,, \;\; j_1 = 0, 1.$$

Similarly, a $\mathcal{U}$*-valued formal distribution in two variables* is an element of the vector superspace $\mathcal{U}[\![Z, Z^{-1}, W, W^{-1}]\!]$:

$$a(Z, W) = \sum_{\substack{j_0, k_0 \in \mathbb{Z}, \\ j_1, k_1 = 0,1}} Z^{j_0|j_1} W^{k_0|k_1} a_{j_0|j_1, k_0|k_1} \in \mathcal{U}[\![Z, Z^{-1}, W, W^{-1}]\!]$$

with $a_{j_0|j_1, k_0|k_1} \in \mathcal{U}$. A formal distribution $a(Z, W)$ is called *local* if

$$(z - w)^n a(Z, W) = 0$$

for some $n \in \mathbb{Z}_{\geqslant 0}$. We let the *formal δ-distribution* be defined by

$$\delta(Z, W) = (\theta - \zeta) \sum_{n \in \mathbb{Z}} z^n w^{-n-1}.$$

Note that for any $f \in \mathcal{U}[\![Z, Z^{-1}]\!]$ we have

$$\mathrm{res}_Z \delta(Z,W) f(Z) = f(W).$$

Since $(z-w)\delta(Z,W) = 0$, the formal δ-distribution is local.

The differential operators ∂_z, ∂_θ, ∂_w and ∂_ζ act naturally on $\mathbb{C}[\![Z, Z^{-1}, W, W^{-1}]\!]$. Consider two more odd differential operators

$$D_Z = \partial_\theta + \theta\partial_z, \quad D_W = \partial_\zeta + \zeta\partial_w.$$

Then $[D_Z, D_Z] = 2\partial_z$. Set

$$D_Z^{j_0|j_1} = \partial_z^{j_0} D_Z^{j_1}, \quad D_Z^{(j_0|j_1)} = (-1)^{j_1} \frac{1}{j_0!} D_Z^{j_0|j_1}.$$

Lemma 2.1. *Let $a(Z,W)$ be a local formal distribution. Then*

$$a(Z,W) = \sum_{\substack{j_0 \in \mathbb{Z}_{\geqslant 0},\\ j_1 = 0,1}} D_W^{(j_0|j_1)} \delta(Z,W)\, c_{j_0|j_1}(W),$$

where the sum is finite, and

$$c_{j_0|j_1}(W) = \mathit{res}_Z (Z-W)^{j_0|j_1} a(Z,W).$$

Definition 2.2. A *supersymmetric vertex algebra* is a tuple $(V, |0\rangle, S, Y)$ where V is a vector superspace, $|0\rangle \in V$ is a *vacuum vector*, S is an odd endomorphism of V, and the *state-field correspondence* Y is a parity preserving linear map from V to the space of $\mathrm{End}\, V$-valued super fields

$$Y : V \to \mathrm{End}\, V[\![Z, Z^{-1}]\!], \quad a \mapsto a(Z)$$

satisfying the following axioms:

- (*vacuum*) $a(Z)\,|0\rangle\,|_{z=0,\,\theta=0} = a$, $S\,|0\rangle = 0$,
- (*translation covariance*) $[S, a(Z)] = (\partial_\theta - \theta\partial_z) a(Z)$,
- (*locality*) for any $a, b \in V$ there exists $N \in \mathbb{Z}_+$ such that $(z-w)^N [a(Z), b(W)] = 0$.

By Lemma 2.1, the locality axiom implies a finite sum decomposition

$$[a(Z), b(W)] = \sum_{\substack{j_0 \in \mathbb{Z}_{\geqslant 0},\\ j_1 = 0,1}} \left(D_W^{(j_0|j_1)} \delta(Z,W)\right) a(W)_{(j_0|j_1)} b(W)$$

for $a(W)_{(j_0|j_1)} b(W) := \mathrm{res}_Z (Z-W)^{j_0|j_1} [a(Z), b(W)]$. The expression $a(W)_{(j_0|j_1)} b(W)$ is called the $(j_0|j_1)$*-th product of the super fields* $a(W)$ *and* $b(W)$.

Definition 2.3. (1) The *normally ordered product* of two $\operatorname{End} V$-valued formal distributions $a(Z)$ and $b(Z)$ is defined by

$$:a(Z)b(Z):= a_+(Z)b(Z) + (-1)^{p(a)p(b)}b(Z)a_-(Z),$$

where

$$a_+(Z) = \sum_{j_0\in\mathbb{Z}_{\geqslant 0},\, j_1=0,1} Z^{j_0|j_1}a_{j_0|j_1} \quad \text{and} \quad a_-(Z) = \sum_{j_0\in\mathbb{Z}_{<0},\, j_1=0,1} Z^{j_0|j_1}a_{j_0|j_1}.$$

(2) If $j_0 \leqslant -2$ and $j_1 = 0, 1$, or $j_0 = -1$ and $j_1 = 0$, then $a(Z)_{(j_0|j_1)}b(Z)$ is given by

$$a(Z)_{(j_0|j_1)}b(Z) = (-1)^{1-j_1} :\big(D_Z^{(-1-j_0|1-j_1)}a(Z)\big)b(Z): .$$

Remark 2.4. One can check that

$$:a(Z)b(Z):|0\rangle\,|_{z=0,\theta=0} = a_{(-1|1)}b$$

and

$$a(Z)_{(j_0|j_1)}b(Z)\,|0\rangle\,|_{z=0,\theta=0} = a_{(j_0|j_1)}b$$

for (j_0, j_1) as in part (2) of Definition 2.3.

Lemma 2.5 (Dong's lemma)**.** *Let $a(Z), b(Z), c(Z)$ be pairwise local formal distributions. Then $\big(a(Z), (b_{(j_0|j_1)}c)(Z)\big)$ is local for any $j_0 \in \mathbb{Z}$ and $j_1 = 0, 1$.*

Lemma 2.6 (Uniqueness lemma)**.** *Let V be a supersymmetric vertex algebra. If $a(Z)$ is a super field such that $(a(Z), b(Z))$ is local for every $b \in V$ and $a(Z)\,|0\rangle = 0$ then $a(Z) = 0$.*

By the uniqueness lemma and Remark 2.4,

$$a(Z)_{(j_0|j_1)}b(Z) = (a_{(j_0|j_1)}b)(Z),$$

and we set

$$:ab: = a_{(-1|1)}b = :a(Z)b(Z):|0\rangle\,|_{z=0,\,\theta=0}.$$

Note that for a given supersymmetric vertex algebra V, the state-field correspondence map

$$Y : V \to (\operatorname{End} V)[\![Z, Z^{-1}]\!], \quad a \mapsto a(Z),$$

is injective. Hence a supersymmetric vertex algebra V can be considered as a set of super fields $Y(V)$. In the following theorem, we construct a vertex algebra as a set of super fields.

Theorem 2.7 (Existence theorem)**.** *Let V be a vector superspace and $\widehat{V}$ be a set of pairwise local* $\operatorname{End} V$*-valued super fields. Suppose $Id \in \widehat{V}$ is the constant field and $\widehat{V}$ is invariant under the operator $D = \partial_\theta + \theta\partial_z$ and all $(j_0|j_1)$-products. Then the superspace V with the vacuum vector Id, the operator S given by $Sa(Z) = D(a(Z))$ and the $(j_0|j_1)$-products is a supersymmetric vertex algebra.*

2.2. **Supersymmetric Lie conformal algebras.** Recall that a Lie conformal algebra (LCA) R gives rise to a vertex algebra called a universal enveloping vertex algebra $V(R)$ [3, 18]. Now we introduce its supersymmetric analogue: that is, a supersymmetric LCA and the corresponding universal enveloping supersymmetric vertex algebra. Consider two superalgebras:

- Let $\mathcal{L}$ be the associative superalgebra generated by a pair of elements $\Lambda = (\lambda, \chi)$, where λ is even and χ is odd, such that
$$[\lambda, \chi] = 0, \quad [\chi, \chi] = 2\chi^2 = -2\lambda.$$
- Let $\mathcal{K}$ be another associative superalgebra generated by a pair of elements $\nabla = (T, S)$, where T is even and S is odd, such that
$$[T, S] = 0, \quad [S, S] = 2S^2 = 2T.$$

Note that $\mathcal{L}$ and $\mathcal{K}$ are isomorphic via the map $\lambda \mapsto -T$ and $\chi \mapsto -S$.

Set
$$(Z - W)\Lambda = (z - w - \theta\zeta)\lambda + (\theta - \zeta)\chi.$$
Given a formal distribution $a(Z, W)$ of two variables Z and W, consider the *formal Fourier transformation*
$$\mathcal{F}^{\Lambda}_{Z,W}\, a(Z, W) = \operatorname{res}_Z \exp\big((Z - W)\Lambda\big) a(Z, W)$$
which can be expanded as
$$\mathcal{F}^{\Lambda}_{Z,W}\, a(Z, W) = \sum_{j_0 \in \mathbb{Z}_{\geqslant 0},\, j_1 = 0,1} (-1)^{j_1} \Lambda^{(j_0|j_1)} c_{j_0|j_1}(W),$$
where
$$\Lambda^{(j_0|j_1)} = (-1)^{j_1} \frac{\lambda^{j_0}\chi^{j_1}}{j_0!}$$
and $c_{j_0|j_1}(W)$ is defined in Lemma 2.1.

Define the Λ-*bracket* $(a, b) \to [a_\Lambda b]$ of a local pair $\big(a(Z), b(Z)\big)$ by
$$[a_\Lambda b](W) := \mathcal{F}^{\Lambda}_{Z,W}[a(Z), b(W)].$$

Proposition 2.8. *The Λ-bracket satisfies the following properties for all pairwise local distributions $(a(Z), b(Z), c(Z))$:*

(1) *(sesquilinearity)*

$$[Sa_\Lambda b] = \chi[a_\Lambda b], \quad [a_\Lambda Sb] = -(-1)^{p(a)}(S+\chi)[a_\Lambda b];$$

(2) *(skew-symmetry)*

$$[b_\Lambda a] = (-1)^{p(a)p(b)}[a_{-\Lambda-\nabla} b],$$

where

$$[a_{-\Lambda-\nabla} b] = \sum_{j_0\in\mathbb{Z}_{\geqslant 0},\, j_1=0,1} (-1)^{j_1}(-\Lambda-\nabla)^{(j_0|j_1)} a_{(j_0|j_1)} b$$

for $-\Lambda-\nabla = (-\lambda - T, -\chi - S)$ *with*

$$[\chi, S] = 2\lambda \qquad \textit{and} \qquad [\chi, T] = [\lambda, T] = [\lambda, S] = 0;$$

(3) *(Jacobi identity)*

$$[a_\Lambda [b_\Gamma c]] = -(-1)^{p(a)}[[a_\Lambda b]_{\Lambda+\Gamma} c] + (-1)^{(p(a)+1)(p(b)+1)}[b_\Gamma [a_\Lambda c]],$$

where
 (i) $\Gamma = (\gamma, \eta)$ *with* $[\gamma, \eta] = [\gamma, \gamma] = 0$ *and* $[\eta, \eta] = -2\gamma$,
 (ii) $\Lambda + \Gamma = (\lambda+\gamma, \zeta+\eta)$ *with* $[\lambda, \eta] = [\lambda, \gamma] = [\zeta, \gamma] = [\zeta, \eta] = 0$.

This motivates the following definition.

Definition 2.9. A *supersymmetric Lie conformal algebra (LCA)* $\mathcal{R}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{K}$-module endowed with odd bilinear map $\mathcal{R}\otimes\mathcal{R} \to \mathcal{L}\otimes\mathcal{R}$, called Λ-bracket, given by a finite sum expansion

$$a\otimes b \mapsto [a_\Lambda b] = \sum_{j_0\in\mathbb{Z}_{\geqslant 0},\, j_1=0,1} (-1)^{j_1}\Lambda^{(j_0|j_1)} a_{(j_0|j_1)} b$$

with $a_{(j_0|j_1)}b \in \mathcal{R}$, satisfying the following properties:

(1) (sesquilinearity) In $\mathcal{L}\otimes\mathcal{R}$ we have

$$[Sa_\Lambda b] = \chi[a_\Lambda b], \quad [a_\Lambda Sb] = -(-1)^{p(a)}(S+\chi)[a_\Lambda b]\,,$$

where S and χ obey the relation $[S, \chi] = 2\lambda$;

(2) (skew-symmetry) In $\mathcal{L}\otimes\mathcal{R}$ we have

$$[b_\Lambda a] = (-1)^{p(a)p(b)}[a_{-\Lambda-\nabla} b],$$

where

$$[a_{-\Lambda-\nabla} b] = \sum_{j_0\in\mathbb{Z}_{\geqslant 0},\, j_1=0,1} (-1)^{j_1}(-\Lambda-\nabla)^{(j_0|j_1)} a_{(j_0|j_1)} b$$

for $-\Lambda-\nabla=(-\lambda-T,-\chi-S)$ satisfying

$$[\chi,S]=2\lambda \qquad \text{and} \qquad [\chi,T]=[\lambda,T]=[\lambda,S]=0;$$

(3) (Jacobi-identity) In $\mathcal{L}\otimes\mathcal{L}'\otimes\mathcal{R}$ we have

$$[a_\Lambda[b_\Gamma c]]=-(-1)^{p(a)}[[a_\Lambda b]_{\Lambda+\Gamma}c]+(-1)^{(p(a)+1)(p(b)+1)}[b_\Gamma[a_\Lambda c]],$$

where
(i) $\Gamma=(\gamma,\eta)$ such that $[\gamma,\eta]=[\gamma,\gamma]=0$ and $[\eta,\eta]=-2\gamma$,
(ii) $\Lambda+\Gamma=(\lambda+\gamma,\zeta+\eta)$ such that $[\lambda,\eta]=[\lambda,\gamma]=[\zeta,\gamma]=[\zeta,\eta]=0$.

Note that the tensor product sign is often omitted in the notation.

The next theorem provides an equivalent definition of supersymmetric vertex algebras in terms of Λ-brackets; cf. [19, Thm. 4.1].

Theorem 2.10. *A supersymmetric vertex algebra is a tuple* $(V,S,[\ {}_\Lambda\ ],|0\rangle,:\ :)$ *such that*

(i) $(V,S,[\ {}_\Lambda\ ])$ *is a supersymmetric Lie conformal algebra.*
(ii) $(V,S,|0\rangle,:\ :)$ *is a unital differential superalgebra, where* S *is an odd derivation of the product* $:\ :$*, and the following properties hold:*

$$(2.2)\qquad \begin{aligned} &:ab:-(-1)^{p(a)p(b)}:ba:=(-1)^{p(a)p(b)}\sum_{j\geqslant 1}\frac{(-T)^j}{j!}(b_{(-1+j|1)}a),\\ &::ab:c:-:a:bc::=\sum_{j\geqslant 0}a_{(-2-j|1)}(b_{(j|1)}c)+(-1)^{p(a)p(b)}\sum_{j\geqslant 0}b_{(-2-j|1)}(a_{(j|1)}c).\end{aligned}$$

(iii) The Λ*-bracket and the product* $:\ :$ *are related by the non-commutative Wick formula:*

$$(2.3)\qquad [a_\Lambda :bc:]=\sum_{k\geqslant 0}\frac{\lambda^k}{k!}[a_\Lambda b]_{(k-1|1)}c+(-1)^{(p(a)+1)p(b)}:b[a_\Lambda c]:\ .$$

The properties (2.2) of the product $:\ :$ are referred to as the *quasi-commutativity* and *quasi-associativity*, respectively.

Definition 2.11. (1) A set $\mathcal{B}=\{a_i\mid i\in I\}$ of elements in a supersymmetric vertex algebra V *strongly generates* V if the set of monomials

$$\{:a_{j_1}a_{j_2}\dots a_{j_s}:\ |\ j_1,\dots,j_s\in I,\ s\in\mathbb{Z}_{\geqslant 0}\}$$

spans V. If $s=0$, the monomial is understood as $|0\rangle$. For $s>2$ the product in the monomial is applied consecutively from right to left.

(2) An ordered set $\mathcal{B} = \{a_i \mid i \in I\} \subset V$ *freely generates* a supersymmetric vertex algebra V if the set of monomials

$$\{ : a_{j_1} a_{j_2} \dots a_{j_s} : \mid j_r \leqslant j_{r+1} \text{ and } j_r < j_{r+1} \text{ if } p(a_{j_r}) = \bar{1} \}$$

forms a basis of V over $\mathbb{C}$.

Theorem 2.12. *Let $\mathcal{R}$ be a supersymmetric Lie conformal algebra with an ordered $\mathbb{C}$-basis $\mathcal{B} = \{a_i \mid i \in I\}$. Then there exists a unique supersymmetric vertex algebra $V(\mathcal{R})$ such that*

(i) $V(\mathcal{R})$ is freely generated by $\mathcal{B}$,
(ii) the operator S on $V(\mathcal{R})$ is defined by $S(: ab :) =: (Sa)b : +(-1)^{p(a)} : a(Sb) :$,
(iii) the Λ-bracket on $\mathcal{R}$ extends to the Λ-bracket on $V(\mathcal{R})$ via the Wick formula (2.3).

Definition 2.13. For a given supersymmetric Lie conformal algebra $\mathcal{R}$, the supersymmetric vertex algebra $V(\mathcal{R})$ in Theorem 2.12 is called the *universal enveloping supersymmetric vertex algebra* associated to $\mathcal{R}$.

2.3. **Supersymmetric nonlinear LCAs.** In this section we follow Section 3 of [8] to introduce *nonlinear* supersymmetric LCAs. We omit the arguments which are straightforward supersymmetric analogues of those in [8].

For a positive integer n, consider a $\mathcal{K}$-module $\mathcal{R} = \bigoplus_{\zeta \in \mathbb{N}/n} \mathcal{R}_\zeta$ with $(\mathbb{N}/n)$-grading so that $\mathrm{gr}(a) = \zeta$ for $a \in \mathcal{R}_\zeta$. The grading gr is naturally extended to the grading of the tensor algebra $\mathcal{T}(\mathcal{R})$ by

$$\mathrm{gr}(a \otimes b) = \mathrm{gr}(a) + \mathrm{gr}(b).$$

Set

$$\mathcal{T}(\mathcal{R})_{(\zeta)-} = \bigoplus_{\zeta' < \zeta} \mathcal{T}(\mathcal{R})_{\zeta'}.$$

Definition 2.14. Suppose that $\mathcal{R}$ is endowed with a *nonlinear Λ-bracket*

$$[\mathcal{R}_{\zeta\,\Lambda} \mathcal{R}_{\zeta'}] \subset \mathcal{L} \otimes \mathcal{T}(\mathcal{R})_{(\zeta+\zeta')_-},$$

satisfying skew-symmetry, sesquilinearity and Jacobi identity in Definition 2.9. Then $\mathcal{R}$ is called *supersymmetric nonlinear Lie conformal algebra.*

Proposition 2.15. *Let $\mathcal{R}$ be a supersymmetric nonlinear LCA. Then the normally ordered product and Λ-bracket admit unique extensions to the linear maps*

$$\mathcal{T}(\mathcal{R}) \otimes \mathcal{T}(\mathcal{R}) \to \mathcal{T}(\mathcal{R}), \quad A \otimes B \mapsto : AB :,$$
$$\mathcal{T}(\mathcal{R}) \otimes \mathcal{T}(\mathcal{R}) \to \mathcal{L} \otimes \mathcal{T}(\mathcal{R}), \quad A \otimes B \mapsto [A_\Lambda B],$$

in such a way that for any $a, b \in \mathcal{R}$ and $A, B, C \in \mathcal{T}(\mathcal{R})$ we have

(i) $[a_\Lambda b]$ *is defined by the* Λ*-bracket on* $\mathcal{R}$*,*
(ii) $: aB := a \otimes B$*,*
(iii) $: 1A :=: A1 := A$*,*
(iv) $: (a \otimes B)C : - : a : BC ::$ *is defined by the quasi-associativity,*
(v) $[A_\Lambda (b \otimes C)]$ *and* $[(a \otimes B)_\Lambda C]$ *are defined by the Wick formula.*

For a given supersymmetric nonlinear LCA $\mathcal{R}$, consider the two-sided ideal $\mathcal{J}(\mathcal{R})$ of $\mathcal{T}(\mathcal{R})$ generated by elements of the form

$$(: ab : -(-1)^{p(a)p(b)} : ba :) - (-1)^{p(a)p(b)} \sum_{j \geqslant 1} \frac{(-T)^j}{j!} b_{(-1+j|1)a},$$

where

$$[b_\Lambda a] = \sum_{j_0 \in \mathbb{Z}_{\geqslant 0},\, j_1 = 0,1} (-1)^{j_1} \Lambda^{(j_0|j_1)} b_{(j_0|j_1)} a.$$

Then the Λ-bracket and the product $:\ :$ on $\mathcal{T}(\mathcal{R})$ induce a well-defined Λ-bracket and product on the quotient

$$V(\mathcal{R}) = \mathcal{T}(\mathcal{R}) / \mathcal{J}(\mathcal{R}).$$

Since $V(\mathcal{R})$ satisfies quasi-commutativity, quasi-associativity and Wick formula, it is a supersymmetric vertex algebra which is called the *universal enveloping supersymmetric vertex algebra of* $\mathcal{R}$; cf. Definition 2.13.

Proposition 2.16. *For a given ordered basis* $\mathcal{B}$ *of* $\mathcal{R}$*, the supersymmetric vertex algebra* $V(\mathcal{R})$ *is freely generated by* $\mathcal{B}$*.*

3. Good filtered complexes of supersymmetric nonlinear LCAs

Here we reproduce some useful facts about bigraded complexes. Proofs can be obtained by suitable supersymmetric versions of the arguments in [8, Sec. 4]. Introduce the notation

$$\Gamma = \frac{\mathbb{Z}}{2}, \quad \Gamma_+ = \frac{\mathbb{Z}_{\geqslant 0}}{2}, \quad \Gamma'_+ = \frac{\mathbb{Z}_{>0}}{2}.$$

Let $\mathfrak{g}$ be a graded vector superspace and $\mathcal{R} = \mathcal{K} \otimes \mathfrak{g}$ be a nonlinear Lie conformal algebra such that

$$\mathfrak{g} = \bigoplus_{\substack{p,q \in \Gamma,\ p+q = \mathbb{Z}_+, \\ \Delta \in \Gamma'_+}} \mathfrak{g}^{p,q}[\Delta], \qquad \mathcal{R} = \bigoplus_{\substack{p,q \in \Gamma,\ p+q = \mathbb{Z}_+, \\ \Delta \in \Gamma'_+}} \mathcal{R}^{p,q}[\Delta], \tag{3.1}$$

where

$$\mathcal{R}^{p,q}[\Delta] = \bigoplus_{n \geqslant 0} S^n \otimes \mathfrak{g}^{p,q}\big[\Delta - \frac{n}{2}\big].$$

.

The universal enveloping supersymmetric vertex algebra $V(\mathcal{R})$, which is strongly generated by a basis $\{a_i \,|\, i\in I\}$ of $\mathcal{R}$, has the Γ'_+-grading

$$V(\mathcal{R}) = \bigoplus_{\Delta\in\Gamma'_+} V(\mathcal{R})[\Delta]$$

where

$$V(\mathcal{R})[\Delta] = \mathrm{span}_{\mathbb{C}}\{ :a_{i_1}a_{i_2}\dots a_{i_s}: \,|\, i_k\in I,\, a_{i_k}\in\mathcal{R}[\Delta_k],\, \textstyle\sum_{k=1}^s \Delta_k = \Delta\}.$$

We assume that

$$V(\mathcal{R})[\Delta_1]_{(n_0|n_1)}V(\mathcal{R})[\Delta_1] \subset V(\mathcal{R})\big[\Delta_1+\Delta_2-n_0-\frac{n_1}{2}-\frac{1}{2}\big].$$

Consider a Γ-filtration and a $\mathbb{Z}$-grading of $\mathcal{R}$ induced from (3.1)

$$F^p\mathcal{R} = \bigoplus_{\substack{p'\geqslant p,\\ q,\Delta}} \mathcal{R}^{p',q}[\Delta], \qquad \mathcal{R}^n = \bigoplus_{p+q=n}\mathcal{R}^{p,q},$$

and the corresponding filtration and $\mathbb{Z}$-grading of $V(\mathcal{R})$ defined by

$$V(\mathcal{R})^n = \mathrm{span}_{\mathbb{C}}\{ :a_{i_1}a_{i_2}\dots a_{i_s}: \,|\, i_k\in I,\, a_{i_k}\in\mathcal{R}^{p_k,q_k},\, \textstyle\sum_{k=1}^s p_k+q_k = n\},$$

$$F^pV(\mathcal{R}) = \mathrm{span}_{\mathbb{C}}\{ :a_{i_1}a_{i_2}\dots a_{i_s}: \,|\, i_k\in I,\, a_{i_k}\in\mathcal{R}^{p_k,q_k},\, \textstyle\sum_{k=1}^s p_k \geqslant p\}.$$

Set

$$F^pV(\mathcal{R})^n = F^pV(\mathcal{R})\cap V(\mathcal{R})^n, \quad F^pV(\mathcal{R})^n[\Delta] = F^pV(\mathcal{R})^n\cap V(\mathcal{R})[\Delta]$$

and consider the associated graded algebra

$$\mathrm{gr}\, V(\mathcal{R}) = \bigoplus_{p,q\in\Gamma} \mathrm{gr}^{p,q}V(\mathcal{R}),$$

where

$$\mathrm{gr}^{p,q}V(\mathcal{R})[\Delta] = F^pV(\mathcal{R})^{p+q}[\Delta]/F^{p+\frac{1}{2}}V(\mathcal{R})^{p+q}[\Delta],$$

$$\mathrm{gr}^{p,q}V(\mathcal{R}) = F^pV(\mathcal{R})^{p+q}/F^{p+\frac{1}{2}}V(\mathcal{R})^{p+q} = \bigoplus_{\Delta\in\Gamma'_+}\mathrm{gr}^{p,q}V(\mathcal{R})[\Delta].$$

Suppose a differential map $d: V(\mathcal{R})\to V(\mathcal{R})$ satisfies

$$d(F^pV(\mathcal{R})^n)\subset F^pV(\mathcal{R})^{n+1}, \quad d(V(\mathcal{R}[\Delta])\subset V(\mathcal{R})[\Delta]. \tag{3.2}$$

Then we set for the cohomology spaces

$$F^pH^n(V(\mathcal{R}),d) = \mathrm{Ker}(d|_{F^pV(\mathcal{R})^n})/\mathrm{Im}\, d\cap F^pV(\mathcal{R})^n,$$

$$\mathrm{gr}^{p,q}H(V(\mathcal{R}),d) = F^pH^{p+q}(V(\mathcal{R}),d)/F^{p+\frac{1}{2}}H^{p+q}(V(\mathcal{R}),d).$$

In addition, for the graded differential map $d^{\rm gr} : {\rm gr}\, V(\mathcal{R}) \to {\rm gr}\, V(\mathcal{R})$ induced from d, we define cohomology spaces by

$$H^{p,q}({\rm gr}\, V(\mathcal{R}), d^{\rm gr}) = {\rm Ker}\, d^{\rm gr}|_{{\rm gr}^{p,q} V(\mathcal{R})} / {\rm Im}\, d^{\rm gr} \cap {\rm gr}^{p,q} V(\mathcal{R}).$$

Definition 3.1. Let d be a differential on $V(\mathcal{R})$ satisfying (3.2).

(1) We say d is *almost linear differential of* $\mathcal{R}$ if

$$d^{\rm gr}(\mathfrak{g}^{p,q}[\Delta]) \subset \mathfrak{g}^{p,q+1}[\Delta];$$

or, equivalently, $d(\mathfrak{g}^{p,q}[\Delta]) \subset \mathfrak{g}^{p,q+1}[\Delta] \oplus F^{p+\frac{1}{2}} V(\mathcal{R})^{p+q+1}$.

(2) A differential d is called a *good* almost linear differential of $\mathcal{R}$ if

$$H^{p,q}(\mathfrak{g}, d^{\rm gr}) = 0 \quad \text{ if } \quad p+q \neq 0.$$

In the rest of this section we assume that $V(\mathcal{R})[\Delta]$ has finite dimension for any $\Delta \in \Gamma'_+$ and d is a good almost linear differential of $\mathcal{R}$. Take bases

$$\mathcal{B}^p_{\mathfrak{g}}[\Delta] = \{\, e_i \,|\, i \in \mathcal{I}^p_{\mathfrak{g}}[\Delta] \,\} \qquad \text{for some index sets } \ \mathcal{I}^p_{\mathfrak{g}}[\Delta],$$

$$\mathcal{B}^p_{\mathcal{R}}[\Delta] = \{ e_{(i,n)} \,|\, e_{(i,n)} = S^n e_i,\ e_i \in \mathcal{B}^p_{\mathfrak{g}}[\Delta'],\ \Delta' + \frac{n}{2} = \Delta \},$$

of $\mathfrak{g}^{p,-p}[\Delta] \cap {\rm Ker}\, d^{\rm gr}$ and $\mathcal{R}^{p,-p}[\Delta] \cap {\rm Ker}\, d^{\rm gr} = H^{p,-p}({\rm gr}\, \mathcal{R}, d^{\rm gr})[\Delta]$, respectively. Then

$$\mathcal{B}_{\mathcal{R}} := \bigsqcup_{\Delta \in \Gamma'_+,\, p \in \Gamma} \mathcal{B}^p_{\mathcal{R}}[\Delta] = \{\, e_{(i,n)} \,|\, e_{(i,n)} = S^n e_i,\ i \in \mathcal{I}_{\mathfrak{g}} \,\}$$

is a basis of $H({\rm gr}\, \mathcal{R}, d^{\rm gr})$, where

$$\mathcal{I}_{\mathfrak{g}} := \bigsqcup_{\Delta \in \Gamma'_+,\, p \in \Gamma} \mathcal{I}^p_{\mathfrak{g}}[\Delta].$$

Proposition 3.2.

(1) $H(gr\, V(\mathcal{R}), d^{gr})$ *is freely generated by* $\mathcal{B}_{\mathcal{R}}$.

(2) $H^{p,-p}(gr\, V(\mathcal{R}), d^{gr})[\Delta]$ *has the basis*

$$\mathcal{B}^p_{V(\mathcal{R})}[\Delta] = \Big\{ :e_{(i_1,n_1)} e_{(i_2,n_2)} \dots e_{(i_k,n_k)}: \Big\},$$

where the sets of indices $(i_t, n_t) \in \mathcal{I}^{p_t}_{\mathfrak{g}}[\Delta_t] \times \mathbb{Z}_{\geqslant 0}$ *satisfy the conditions:*

(i) $(i_t, n_t) \leqslant (i_{t+1}, n_{t+1})$,

(ii) if $e_{(i_t,n_t)}$ *and* $e_{(i_{t+1},n_{t+1})}$ *are odd then* $(i_t, n_t) < (i_{t+1}, n_{t+1})$,

(iii) $\sum_{t=1}^k i_t = p$,

(iv) $\sum_{t=1}^k \big(\Delta_t + \frac{n_t}{2}\big) = \Delta$.

For $e_i \in \mathfrak{g}^{p,-p}[\Delta] \cap \operatorname{Ker} d^{\mathrm{gr}}$ there exists an element $f_i \in F^{p+\frac{1}{2}}V(\mathcal{R})^0[\Delta]$ such that $E_i = e_i + f_i \in F^pV(\mathcal{R})^0[\Delta] \cap \operatorname{Ker} d$. Set

$$H^{p,-p}(\mathfrak{g}, d)[\Delta] = \operatorname{span}\{\, E_i \mid i \in \mathcal{I}^p_{\mathfrak{g}}[\Delta]\,\}, \quad H(\mathfrak{g}, d)[\Delta] = \bigoplus_{p\in\Gamma} H^{p,-p}(\mathfrak{g}, d)[\Delta].$$

Theorem 3.3.

(1) $H(V(\mathcal{R}), d) = H^0(V(\mathcal{R}), d)$.

(2) If the $\mathcal{K}$*-module* $H(\mathcal{R}, d) = \mathcal{K} \otimes H(\mathfrak{g}, d)$ *admits a nonlinear supersymmetric LCA structure, then*

$$H(V(\mathcal{R}), d) \simeq V(H(\mathcal{R}, d)).$$

4. BRST cohomology

We are now in a position to define supersymmetric W-algebras via BRST cohomology following [22]. We will rely on the supersymmetric vertex algebra theory developed by Heluani and Kac [15, 18] to describe the structure of the W-algebras associated with odd nilpotent elements of Lie superalgebras.

4.1. **BRST complex.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra with a $(\frac{1}{2}\mathbb{Z})$-grading $\mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}(i)$ satisfying the following conditions:

(i) There exists $h \in \mathfrak{g}_{\bar{0}}$ such that $\mathfrak{g}(i) = \{a \in \mathfrak{g} \mid \frac{1}{2}[h, a] = ia\}$.

(ii) There are odd elements $f_{\mathrm{odd}} \in \mathfrak{g}(-\frac{1}{2})$ and $e_{\mathrm{odd}} \in \mathfrak{g}(\frac{1}{2})$ such that

$$\operatorname{span}\{e, e_{\mathrm{odd}}, h, f_{\mathrm{odd}}, f\} \simeq \mathfrak{osp}(1|2),$$

where (e, h, f) is an $\mathfrak{sl}_2$-triple.

We will suppose that $\mathfrak{g}$ is equipped with a nondegenerate invariant bilinear form $(\,|\,)$ normalized by the conditions $(e|f) = \frac{1}{2}(h|h) = 1$.

Introduce two supersymmetric vertex algebras.

(1) Let $\bar{\mathfrak{g}} = \{\bar{a} \mid a \in \mathfrak{g}\}$ be the vector superspace defined by $\bar{\mathfrak{g}}_{\bar{1}} = \mathfrak{g}_{\bar{0}}$ and $\bar{\mathfrak{g}}_{\bar{0}} = \mathfrak{g}_{\bar{1}}$. The *supersymmetric current nonlinear LCA* is

$$\mathcal{R}_{\mathrm{cur}} := \mathcal{K} \otimes \bar{\mathfrak{g}}$$

endowed with the Λ-bracket

$$[\bar{a}_\Lambda \bar{b}] = (-1)^{p(a)p(\bar{b})}\overline{[a, b]} + k\,\chi(a|b).$$

(2) Set $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}(i)$ and $\mathfrak{n}_- = \bigoplus_{i<0} \mathfrak{g}(i)$. Then there are bases

$$\{u_\alpha \mid \alpha \in I_+\} \quad \text{and} \quad \{u^\alpha \mid \alpha \in I_+\}$$

of $\mathfrak{n}$ and $\mathfrak{n}_-$, respectively, parameterized by a certain index set I_+, such that $(u^\alpha|u_\beta) = \delta_{\alpha,\beta}$. Introduce two vector superspaces

$$\phi_{\mathfrak{n}} \simeq \mathfrak{n} \subset \mathfrak{g}, \qquad \phi^{\overline{\mathfrak{n}}_-} \simeq \overline{\mathfrak{n}}_- \subset \overline{\mathfrak{g}},$$

spanned by the respective families of elements ϕ_b and $\phi^{\overline{a}}$ with $b \in \mathfrak{n}$ and $\overline{a} \in \overline{\mathfrak{n}}_-$. Consider the supersymmetric nonlinear LCA $\mathcal{R}_{\text{ch}} = \mathcal{K} \otimes (\phi_{\mathfrak{n}} \oplus \phi^{\overline{\mathfrak{n}}_-})$ endowed with the Λ-bracket

$$[\phi^{\overline{a}}\,{}_\Lambda \phi_b] = [\phi_b\,{}_\Lambda \phi^{\overline{a}}] = (a|b).$$

Due to the results of Section 2.3, the two above supersymmetric nonlinear LCAs give rise to respective universal enveloping supersymmetric vertex algebras $V(\mathcal{R}_{\text{cur}})$ and $V(\mathcal{R}_{\text{ch}})$. Their tensor product

$$C(\overline{\mathfrak{g}}, f_{\text{odd}}, k) = V(\mathcal{R}_{\text{cur}}) \otimes V(\mathcal{R}_{\text{ch}})$$

also carries a supersymmetric vertex algebra structure. Introduce the element d by

$$d = \sum_{\alpha \in I_+} : (\overline{u}_\alpha - (f_{\text{odd}}|u_\alpha))\phi^\alpha : + \frac{1}{2} \sum_{\alpha,\beta \in I_+} (-1)^{p(\alpha)p(\overline{\beta})} : \phi_{[u_\alpha, u_\beta]} \phi^\beta \phi^\alpha : , \tag{4.1}$$

where $\phi^\alpha = \phi^{\overline{u}^\alpha}$, $\phi_\alpha = \phi_{u_\alpha}$, $p(\alpha) = p(u_\alpha)$ and $p(\overline{\alpha}) = p(\overline{u}_\alpha)$.

Proposition 4.1. *The Λ-brackets between d and elements in $C(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$ have the form:*

$$[d_\Lambda \overline{a}] = \sum_{\alpha \in I_+} (-1)^{p(\overline{a})p(\alpha)} : \phi^\alpha \overline{[u_\alpha, a]} : + \sum_{\alpha \in I_+} (-1)^{p(\overline{\alpha})} k(\chi + S)\phi^\alpha (u_\alpha|a),$$

$$[d_\Lambda \phi^\alpha] = \frac{1}{2} \sum_{\alpha,\beta \in I_+} (-1)^{p(\overline{\alpha})p(\beta)} : \phi^\beta \phi^{\overline{[u_\beta, u^\alpha]}} :,$$

$$[d_\Lambda \phi_\alpha] = (-1)^{p(\overline{\alpha})} u_\alpha - (f_{\text{odd}}|u_\alpha) + \sum_{\beta \in I_+} (-1)^{p(\overline{\alpha})p(\beta)} : \phi^\beta \phi_{[u_\beta, u_\alpha]} : .$$

Proof. The formulas are verified by a direct calculation in the same way as for the supersymmetric classical W-algebras; see [21]. ☐

Set $Q := d_{(0|0)}$. Then, by the Wick formula (2.3), we have

$$Q(: A\, B :) =: Q(A)\, B : + (-1)^{p(A)} : A\, Q(B) : . \tag{4.2}$$

Proposition 4.2. *The linear map Q on $C(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$ satisfies $Q^2 = 0$.*

Proof. This follows by a direct computation with the use of Proposition 4.1 and property (4.2). ☐

By taking the cohomology of the BRST complex $C(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$ with the differential Q, we can now define the corresponding supersymmetric W-algebra as in [22]; cf. [1] and [14, Ch. 15].

Definition 4.3. The *supersymmetric W-algebra associated to $\overline{\mathfrak{g}}$, f_{odd} and $k \in \mathbb{C}$* is

$$W(\overline{\mathfrak{g}}, f_{\text{odd}}, k) = H(C(\overline{\mathfrak{g}}, f_{\text{odd}}, k), Q).$$

Proposition 4.4. *Let $A, B \in C(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$ satisfy $Q(A) = Q(B) = 0$ and C be any element in $C(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$. Then the following holds:*

(1) $Q(SA) = Q(:AB:) = 0$ and $Q([A_\Lambda B]) = 0$;
(2) $S(QC)$, $:Q(C)\,B:$ and $[Q(C)_\Lambda B]$ belong to the image of Q.

Proof. By sesquilinearity of supersymmetric LCAs, for any $X \in C(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$ we have $S(QX) = -Q(SX)$. Hence the first properties in (1) and (2) hold. The second properties follow from (4.2). By the Jacobi identity of supersymmetric LCAs, for $X, Y \in C(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$ we have

$$Q([X_\Lambda Y]) = -[Q(X)_\Lambda Y] + (-1)^{p(X)+1}[X_\Lambda Q(Y)]$$

which gives the third properties in (1) and (2). □

Corollary 4.5. *The supersymmetric W-algebra $W(\overline{\mathfrak{g}}, f_{\text{odd}}, k)$ is a supersymmetric vertex algebra.*

4.2. **Building blocks of supersymmetric W-algebras.** For any $\bar{a} \in \bar{\mathfrak{g}}$ set

$$J_{\bar{a}} = \bar{a} + \sum_{\beta \in I_+} (-1)^{p(\bar{a})p(\bar{\beta})} :\phi^\beta \phi_{[u_\beta, a]}: \in C(\overline{\mathfrak{g}}, f_{\text{odd}}, k).$$

Proposition 4.6. *For the element d defined in (4.1) we have*

$$[d_\Lambda J_{\bar{a}}] = \sum_{\beta \in I_+} (-1)^{p(\bar{a})p(\beta)} :\phi^\beta (J_{\overline{\pi_{\leqslant 0}[u_\beta, a]}} + (f_{\text{odd}}|[u_\beta, a])): + \sum_{\beta \in I_+} (-1)^{\bar{\beta}} k(S+\chi)\phi^\beta (u_\beta|a),$$

where $\pi_{\leqslant 0} : \mathfrak{g} \to \oplus_{i \leqslant 0}\mathfrak{g}(i)$ is the projection map with the kernel $\oplus_{i>0}\mathfrak{g}(i)$.

Proof. By the Wick formula,

$$
\begin{aligned}
[d_\Lambda J_{\bar a}] &= [d_\Lambda \bar a] + \sum_{\beta\in I_+} (-1)^{p(\bar a)p(\bar\beta)} [d\,_\Lambda :\phi^\beta \phi_{[u_\beta,a]}:] \\
&= [d_\Lambda \bar a] + \sum_{\beta\in I_+} (-1)^{p(\bar a)p(\bar\beta)} :[d\,_\Lambda \phi^\beta]\phi_{[u_\beta,a]}: \qquad (4.3)\\
&+ \sum_{\beta,\gamma\in I_+,\,k\geqslant 1} \frac{\lambda^k}{2k!}(-1)^{p(\bar\beta)(p(\gamma)+p(a)+1)} \Big(:\phi^\gamma \phi^{\overline{[u_\gamma,u^\beta]}}:\Big)_{(k-1|1)} \phi_{[u_\beta,a]} \qquad (4.4)\\
&+ \sum_{\beta\in I_+} (-1)^{p(\bar a)p(\bar\beta)} :\phi^\beta [d_\Lambda \phi_{[u_\beta,a]}]:.
\end{aligned}
$$

Since the coefficients of $\Lambda^{j_0}\chi$ in $[\phi_{[u_\beta,a]}\,{}_\Lambda :\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:]$ are all zero, the coefficients of $\Lambda^{j_0}\chi$ in

$$[:\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:\,{}_\Lambda \phi_{[u_\beta,a]}] = (-1)^{p(\beta)p(\bar a)}[\phi_{[u_\beta,a]}\,{}_{-\Lambda-\nabla} :\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:]$$

are also 0 so that the expression in (4.4) vanishes. The second term in (4.3) equals

$$\sum_{\beta,\gamma\in I_+} \frac{1}{2}(-1)^{p(\bar\beta)(p(\gamma)+p(a)+1)} ::\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:\phi_{[u_\beta,a]}:.$$

By the quasi-associativity in (2.2) and the fact that $\phi^{\bar n}\,{}_{(j|1)}\phi_m = 0$ for any $n\in\mathfrak{n}$ and $m\in\mathfrak{n}_-$ with $j\geqslant 0$, we have

$$::\phi^\gamma\phi^{\overline{[u_\gamma,u^\beta]}}:\phi_{[u_\beta,a]}: = :\phi^\gamma :\phi^{\overline{[u_\gamma,u^\beta]}}\phi_{[u_\beta,a]}::.$$

The remaining computations are straightforward, they are analogous to the classical case in [21]. □

Proposition 4.7. *If $a,b\in\bigoplus_{i\leqslant 0}\mathfrak{g}(i)$ or $a,b\in\bigoplus_{i>0}\mathfrak{g}$ then*

$$[J_{\bar a}\,{}_\Lambda J_{\bar b}] = (-1)^{p(a)p(\bar b)} J_{\overline{[a,b]}} + k\,(S+\chi)(a|b).$$

Proof. This is verified by a direct computation. □

Introduce the vector superspaces

$$r_+ = \phi_{\mathfrak{n}} \oplus J_{\bar{\mathfrak{n}}} \qquad \text{and} \qquad r_- = J_{\bar{\mathfrak{g}}_{\leqslant 0}} \oplus \phi^{\bar{\mathfrak{n}}_-},$$

where

$$J_{\bar{\mathfrak{n}}} = \operatorname{span}\{J_b \,|\, b\in\bar{\mathfrak{n}}\} \qquad \text{and} \qquad J_{\bar{\mathfrak{g}}_{\leqslant 0}} = \operatorname{span}\{J_{\bar a}\,|\, a\in \bigoplus_{i\in\mathbb{Z}_{\leqslant 0}}\mathfrak{g}(i)\}.$$

It is not difficult to see that both $\mathcal{R}_+ = \mathcal{K} \otimes r_+$ and $\mathcal{R}_- = \mathcal{K} \otimes r_-$ are supersymmetric nonlinear LCAs and that $C(\bar{\mathfrak{g}}, f_{\text{odd}}, k)$ decomposes into the tensor product of supersymmetric vertex subalgebras:

$$C(\bar{\mathfrak{g}}, f_{\text{odd}}, k) = V(\mathcal{R}_+) \otimes V(\mathcal{R}_-).$$

Lemma 4.8 (Künneth lemma)**.** *Let V_1 and V_2 be vector superspaces and $d_i : V_i \to V_i$, $i = 1, 2$, be differentials. If $d : V_1 \otimes V_2 \to V_1 \otimes V_2$ is defined by*

$$d(a \otimes b) = d_1(a) \otimes b + (-1)^{p(a)} a \otimes d_2(b)$$

then

$$H(V, d) \simeq H(V_1, d_1) \otimes H(V_2, d_2).$$

Proposition 4.9. *The differential Q has the properties*

$$Q(V(\mathcal{R}_+)) \subset V(\mathcal{R}_+) \quad \textit{and} \quad Q(V(\mathcal{R}_-)) \subset V(\mathcal{R}_-), \tag{4.5}$$

so that

$$W(\bar{\mathfrak{g}}, f_{\text{odd}}, k) = H(V(\mathcal{R}_+), Q) \otimes H(V(\mathcal{R}_-), Q). \tag{4.6}$$

Proof. The inclusions (4.5) follow from Propositions 4.1 and 4.6. The decomposition (4.6) is then implied by the Künneth lemma. □

4.3. **Generators of supersymmetric W-algebras.** We now aim to describe the cohomologies $H(V(\mathcal{R}_+), Q)$ and $H(V(\mathcal{R}_-), Q)$.

Proposition 4.10. *We have $H(V(\mathcal{R}_+), Q) = \mathbb{C}$ so that $W(\bar{\mathfrak{g}}, f_{\text{odd}}, k) = H(V(\mathcal{R}_-), Q)$.*

Proof. Set $K_{\bar{n}} = (-1)^{p(\bar{n})} J_{\bar{n}} - (f_{\text{odd}}|n)$ for $n \in \mathfrak{n}$ and introduce the superspace

$$r'_+ = \phi_{\mathfrak{n}} \oplus K_{\bar{\mathfrak{n}}}, \qquad K_{\bar{\mathfrak{n}}} = \text{span}\, \{K_{\bar{n}} \,|\, \bar{n} \in \bar{\mathfrak{n}}\}.$$

Then $\mathcal{R}_+ = \mathcal{K} \otimes r'_+$. Define the conformal weight Δ and the bigrading on r'_+ by

$$\Delta(\phi_n) = \Delta(K_{\bar{n}}) = j_n, \quad \text{gr}(\phi_n) = (j_n - 1, -j_n), \quad \text{gr}(K_{\bar{n}}) = (j_n - 1, -j_n + 1),$$

assuming that $n \in \mathfrak{g}(j_n)$. The graded differential Q^{gr} associated with Q is good almost linear (see Section 3) and

$$H({r'}_+, Q^{\text{gr}}) = 0.$$

By Theorem 3.3, we have $H(V(\mathcal{R}_+), Q) = \mathbb{C}$. □

To describe $H(V(\mathcal{R}_-),Q)$, recall that

$$Q(J_{\bar a}) = \sum_{\beta\in I_+}(-1)^{p(\bar a)p(\beta)} :\phi^\beta(J_{\overline{\pi_{\leqslant 0}[u_\beta,a]}} + (f_{\rm odd}|[u_\beta,a])): + \sum_{\beta\in I_+}(-1)^{p(\bar\beta)}kS\phi^\beta(u_\beta|a) \tag{4.7}$$

and

$$Q(\phi^{\bar m}) = \frac{1}{2}\sum_{\beta\in I_+}(-1)^{p(\bar m)p(\beta)} :\phi^\beta\phi^{\overline{[u_\beta,m]}}: . \tag{4.8}$$

Consider the conformal weight Δ and the bigrading on r_- satisfying

$$\Delta(J_{\bar a}) = \frac{1}{2} - j_a, \quad \Delta(\phi^{\bar m}) = -j_m,$$
$$\mathrm{gr}(J_{\bar a}) = (j_a, -j_a), \quad \mathrm{gr}(\phi^{\bar m}) = \big(j_m + \frac{1}{2}, -j_m + \frac{1}{2}\big),$$

where $a\in\mathfrak{g}(j_a)$ and $m\in\mathfrak{g}(j_m)$ for $j_a\leqslant 0$ and $j_m<0$. Note that

$$\Delta(\phi^\beta) = j_\beta, \quad \mathrm{gr}(\phi^\beta) = \big(-j_\beta + \frac{1}{2}, j_\beta + \frac{1}{2}\big),$$

where $u^\beta\in\mathfrak{g}(-j_\beta)$. Since $\Delta(S)=\frac{1}{2}$ and $\mathrm{gr}(S)=(0,0)$. Every term in (4.7) has conformal weight $\frac{1}{2}-j_a$ and every term in (4.8) has conformal weight $-j_m$. The bigradings of terms in (4.7) are given by

$$\begin{aligned} &\mathrm{gr}(:\phi^\beta J_{\overline{\pi_{\leqslant 0}[u_\beta,a]}}:) = \big(j_a + \frac{1}{2}, -j_a + \frac{1}{2}\big),\\ &\mathrm{gr}(\phi^\beta(f_{\rm odd}|[u_\beta,a])) = (j_a, -j_a+1),\\ &\mathrm{gr}(S\phi^\beta(u_\beta|a)) = \big(j_a + \frac{1}{2}, -j_a + \frac{1}{2}\big). \end{aligned} \tag{4.9}$$

The bigradings of terms in (4.8) are

$$\mathrm{gr}(\phi^{\bar m}) = \big(j_m + \frac{1}{2}, -j_m + \frac{1}{2}\big), \qquad \mathrm{gr}(:\phi^\beta\phi^{\overline{[u_\beta,m]}}:) = (j_m+1, -j_m+1). \tag{4.10}$$

Theorem 4.11. *Let* $\mathrm{Ker}\,(\mathrm{ad}\, f_{\rm odd}) = \{\, u_\alpha \,|\, \alpha\in\mathcal{J}\,\}$ *with an index set* $\mathcal{J}$. *Then*

(1) $W(\bar{\mathfrak{g}}, f_{\rm odd}, k)$ *is freely generated by* $|\mathcal{J}|$ *elements as a differential algebra,*

(2) there exists a free generating set of the form

$$\{\, u_\alpha + A_\alpha \,|\, \alpha\in\mathcal{J}\,\},$$

where $A_\alpha\in F^{j_\alpha+\frac{1}{2}}V(\mathcal{R}_-)^0[\frac{1}{2}-j_\alpha]$ *for* $u_\alpha\in\mathfrak{g}(j_\alpha)$.

Proof. Since we know that $W(\bar{\mathfrak{g}}, f_{\rm odd}, k) = H(V(\mathcal{R}_-), Q)$, it is enough to show (1) and (2) for $H(V(\mathcal{R}_-), Q)$. The conformal weight and bigrading on r_- induce those on $V(\mathcal{R}_-)$. With respect to the conformal weight and bigrading, Q induces the graded differential $Q^{\rm gr}$. The bigradings listed in (4.9) and (4.10) show that

$$Q^{\rm gr}(J_{\bar{a}}) = \sum_{\beta\in I_+} (-1)^{p(\bar{a})p(\beta)} \phi^{\beta}(f_{\rm odd}|[u_\beta, a]), \quad Q^{\rm gr}(\phi^{\bar{m}}) = 0.$$

Note that $V(\mathcal{R}_-)^0 \cap r_- = J_{\mathfrak{g}_{\leqslant 0}}$ and $V(\mathcal{R}_-)^1 \cap r_- = \phi^{\bar{\mathfrak{n}}_-}$. Since $Q^{\rm gr}(r_-) = \phi^{\bar{\mathfrak{n}}_-}$, we have $H^{p,q}(r_-, Q^{\rm gr}) = 0$ when $p+q \neq 0$ and so Q is a good almost linear differential map. Furthermore, $\mathrm{Ker}(Q^{\rm gr}|_{r_-}) = \{J_a | a \in \mathrm{Ker}(\mathrm{ad}\, f_{\rm odd})\} \oplus \phi^{\bar{\mathfrak{n}}_-}$, hence

$$H(r_-, Q^{\rm gr}) = \{J_a \,|\, a \in \mathrm{Ker}(\mathrm{ad}\, f_{\rm odd})\}.$$

Thus, using Theorem 3.3, we arrive at (1) and (2). □

5. Generators of $W(\bar{\mathfrak{g}}, f_{\rm prin}, k)$ for $\mathfrak{g} = \mathfrak{gl}(n+1|n)$

Consider the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(n+1|n)$ with the basis $\{E_{i,j} | i,j = 1, \dots, 2n+1\}$ and the $\mathbb{Z}/2\mathbb{Z}$-grading defined by $p(E_{i,j}) = i+j \mod 2$ with the commutation relations

$$[E_{i,j}, E_{i',j'}] = \delta_{j,i'} E_{i,j'} - (-1)^{(i+j)(i'+j')}\, \delta_{i,j'}\, E_{i',j}.$$

Take the odd principal nilpotent element in the form

$$f_{\rm prin} = \sum_{p=1}^{2n} E_{p+1,p}.$$

By Proposition 4.6, for $C(\bar{\mathfrak{g}}, f_{\rm prin}, k)$ and any $m \geqslant l$, we have

$$\begin{aligned} Q(J_{m,l}) = (-1)^m k\, S\, \phi^{l,m} + \sum_{j=l+1}^{m} (-1)^{l+j+1} :\phi^{l,j} J_{m,j}: \\ + \sum_{i=l}^{m-1} (-1)^{(i+m)(m+l+1)} :\phi^{i,m} J_{i,l}: + (-1)^l \phi^{l,m+1} + (-1)^m \phi^{l-1,m}, \end{aligned}$$

where we set $\phi^{j,i} = (-1)^{i+1}\phi^{\overline{E_{ij}}}$ for $i > j$ and $J_{i,j} = J_{\overline{E_{i,j}}}$ for $i \geqslant j$.

We will be working with operators on $C(\bar{\mathfrak{g}}, f_{\rm prin}, k)$ of the form $\sum_{t=0}^{N} A_t S^t$ with $A_t \in C(\bar{\mathfrak{g}}, f_{\rm prin}, k)$, which act on an arbitrary element $X \in C(\bar{\mathfrak{g}}, f_{\rm prin}, k)$ by the rule

$$\sum_{t=0}^{N} A_t S^t(X) = \sum_{t=0}^{N} : A_t(S^t(X)) : .$$

In particular, for the operator $A_{i,j} = \delta_{ij}\, k\, S + (-1)^{i+1} J_{i,j}$ on $C(\bar{\mathfrak{g}}, f_{\text{prin}}, k)$ we have

$$A_{i,j}(X) = \delta_{ij}\, k\, S(X) + (-1)^{i+1} : J_{i,j} X : .$$

Consider the $(2n+1)\times(2n+1)$ matrix

$$\mathcal{A} := \begin{bmatrix} A_{1,1} & -1 & 0 & \cdots & \cdots & 0 \\ A_{2,1} & A_{2,2} & -1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{2n,1} & A_{2n,2} & A_{2n,3} & \cdots & A_{2n,2n} & -1 \\ A_{2n+1,1} & A_{2n+1,2} & A_{2n+1,3} & \cdots & A_{2n+1,2n} & A_{2n+1,2n+1} \end{bmatrix}$$

whose entries are operators on $C(\bar{\mathfrak{g}}, f_{\text{prin}}, k)$. Then the *column* (or *row*) *determinant* of $\mathcal{A}$ is given by the formula

$$\operatorname{cdet} \mathcal{A} = \sum_{N=0}^{2n} \sum_{0=i_0<i_1<\cdots<i_{N+1}=2n+1} A_{i_1,i_0+1} A_{i_2,i_1+1} \dots A_{i_{N+1},i_N+1}. \tag{5.1}$$

Write

$$\operatorname{cdet} \mathcal{A} = W_0 + W_1 S + \dots + W_{2n+1} S^{2n+1}$$

for certain elements $W_p \in C(\bar{\mathfrak{g}}, f_{\text{prin}}, k)$. Clearly, $W_{2n+1} = k^{2n+1}$.

Theorem 5.1. *All elements $W_1, \dots, W_{2n}$ belong to the W-algebra $W(\bar{\mathfrak{g}}, f_{\text{prin}}, k)$.*

Proof. One readily verifies that

$$Q \sum_{p=0}^{2n+1} W_p S^p = \sum_{p=0}^{2n+1} Q(W_p) S^p - W_p S^p Q$$

so that $Q A_{m,l} = (-1)^{m+l+1} A_{m,l} Q + (-1)^{m+1} Q(J_{m,l})$. Therefore,

$$\begin{aligned} &Q\, A_{i_1,i_0+1} \dots A_{i_{p+1},i_p+1} \dots A_{i_{N+1},i_N+1} \\ &\qquad = \sum_{p=0}^{N} (-1)^{i_p} \big(A_{i_1,i_0+1} \dots \big((-1)^{i_{p+1}+1} Q(J_{i_{p+1},i_p+1}) \big) \dots A_{i_{N+1},i_N+1} \big) \\ &\qquad\qquad - A_{i_1,i_0+1} \dots A_{i_{p+1},i_p+1} \dots A_{i_{N+1},i_N+1} Q. \end{aligned}$$

Hence the property $W_p \in W(\mathfrak{g}, f_{\text{prin}}, k)$ will follow if we show that $\sum_{N=0}^{2n} B_N = 0$, where we set

$$B_N = \sum_{p=0}^{N} (-1)^{i_p} \big(A_{i_1,i_0+1} \dots \big((-1)^{i_{p+1}+1} Q(J_{i_{p+1},i_p+1}) \big) \dots A_{i_{N+1} i_N+1} \big).$$

Using the relations

$$J_{i,j} = (-1)^{i+1}(A_{i,j} - \delta_{i,j}kS) \quad \text{and} \quad :\phi^{j,i}J_{i',j'} := (-1)^{(i+j+1)(i'+j'+1)} :J_{i',j'}\phi^{j,i}:$$

we find that

$$\begin{aligned}(-1)^{i_{p+1}+1}&Q(J_{i_{p+1},i_p+1})\\ &= -kS(\phi^{i_p+1,i_{p+1}}) + \sum_{j=i_p+2}^{i_{p+1}} (-1)^{i_p+j}\phi^{i_p+1,j}\big(A_{i_{p+1},j} - \delta_{i_{p+1},j}kS\big)\\ &+ \sum_{i=i_p+1}^{i_{p+1}-1}(-1)^{i_p+i}\big(A_{i,i_p+1} - \delta_{i,i_p+1}kS\big)\phi^{i,i_{p+1}} + (-1)^{i_p+i_{p+1}}\phi^{i_p+1,i_{p+1}+1} - \phi^{i_p,i_{p+1}}\end{aligned}$$

and

$$-kS(\phi^{i_p+1,i_{p+1}}) + (-1)^{i_p+i_{p+1}+1}\phi^{i_p+1,i_{p+1}}S + S\phi^{i_p+1,i_{p+1}} = 0.$$

Therefore,

$$\begin{aligned}(-1)^{i_{p+1}+1}Q(J_{i_{p+1},i_p+1}) &= \sum_{j=i_p+2}^{i_{p+1}}(-1)^{i_p+j}\phi^{i_p+1,j}A_{i_{p+1},j}\\ &+ \sum_{i=i_p+1}^{i_{p+1}-1}(-1)^{i_p+i}A_{i,i_p+1}\phi^{i,i_{p+1}} + (-1)^{i_p+i_{p+1}}\phi^{i_p+1,i_{p+1}+1} - \phi^{i_p,i_{p+1}}\end{aligned}$$

so that B_N can be expressed as

$$\begin{aligned}\sum_{p=0}^{N} A_{i_1,i_0+1}\dots A_{i_p,i_{p-1}+1}\Bigg[&\Big(\sum_{j=i_p+2}^{i_{p+1}}(-1)^j\phi^{i_p+1,j}A_{i_{p+1},j} + (-1)^{i_{p+1}}\phi^{i_p+1,i_{p+1}+1}\Big)\\ &+ \Big(\sum_{i=i_p+1}^{i_{p+1}-1}(-1)^iA_{i,i_p+1}\phi^{i,i_{p+1}} - (-1)^{i_p}\phi^{i_p,i_{p+1}}\Big)\Bigg]A_{i_{p+2},i_{p+1}+1}\dots A_{i_{N+1},i_N+1}.\end{aligned}$$

By the quasi-associativity property, we have

$$(\phi^{i_p+1,j}A_{i_{p+1},j})(A_{i_{p+2},i_{p+1}+1}\dots A_{i_{N+1},i_N+1}) = \phi^{i_p+1,j}(A_{i_{p+1},j}(A_{i_{p+2},i_{p+1}+1}\dots A_{i_{N+1},i_N+1})),$$

$$(A_{i,i_p+1}\phi^{i,i_{p+1}})(A_{i_{p+2},i_{p+1}+1}\dots A_{i_{N+1},i_N+1}) = A_{i,i_p+1}(\phi^{i,i_{p+1}}(A_{i_{p+2},i_{p+1}+1}\dots A_{i_{N+1},i_N+1}))$$

for $j = i_p+2,\dots,i_{p+1}$ and $i = i_p+1,\dots,i_{p+1}$, so that vanishing of the telescoping sum implies that $\sum_{N=0}^{2n} B_N = 0$. □

Lemma 5.2. *Suppose that* $\{v_p \,|\, p = 0,\dots,2n\}$ *is a basis of* $\operatorname{Ker}(\operatorname{ad} f_{\text{odd}})$ *such that* $\Delta_{J_{\bar v_p}} = \frac{1}{2}(2n+1-p)$*. Take* $V_p \in W(\bar{\mathfrak{g}}, f_{\text{prin}}, k)$ *of the form* $V_p = J_{\bar v_p} + w_p$ *satisfying the conditions*

(i) V_p *and* w_p *have the conformal weight* $\frac{1}{2}(2n+1-p)$*,*
(ii) w_p *lies in the differential algebra generated by* $J_{\bar{a}}$ *for* $\Delta_{J_{\bar{a}}} < \Delta_{V_p}$*.*

Then the set $\{V_p \,|\, p = 0, \dots, 2n\}$ *freely generates the* W*-algebra* $W(\bar{\mathfrak{g}}, f_{\rm prin}, k)$*.*

Proof. A generating set of the form $\{V'_p = J_{\bar{v}_p} + w'_p \,|\, p = 0, \dots, 2n\}$ satisfying the required conditions (i) and (ii) exists by Theorem 4.11. Set

$$\mathcal{W}_m := \text{subalgebra freely generated by } \{V_m, V_{m+1}, \dots, V_{2n}\},$$

$$\mathcal{W}'_m := \text{subalgebra freely generated by } \{V'_m, V'_{m+1}, \dots, V'_{2n}\}.$$

We will show by a (reverse) induction that $\mathcal{W}_m = \mathcal{W}'_m$ for all $m = 0, \dots, 2n$. Note that $\mathcal{W}_{2n} = \mathcal{W}'_{2n}$, since w_{2n} and w'_{2n} are constants. Now suppose that $\mathcal{W}_p = \mathcal{W}'_p$ for some $p \leqslant 2n$. Then $V_{p-1} - V'_{p-1} \in \mathcal{W}_p = \mathcal{W}'_p$ by condition (ii). Hence we can conclude that $V'_{p-1} = V_{p-1} + (w'_p - w_p) \in \mathcal{W}_{p-1}$ and, similarly, $V_{p-1} \in \mathcal{W}'_{p-1}$. This shows that $\mathcal{W}_{p-1} = \mathcal{W}'_{p-1}$. Thus, $\mathcal{W}'_0 = \mathcal{W}_0$ and since $W(\bar{\mathfrak{g}}, f_{\rm prin}, k) = \mathcal{W}'_0$, the lemma follows. □

Theorem 5.3. *The set of coefficients* $\{W_p \,|\, p = 0, \dots, 2n\}$ *of* cdet $\mathcal{A}$ *freely generates* $W(\bar{\mathfrak{g}}, f_{\rm prin}, k)$ *as a differential algebra.*

Proof. Note that for $i \geqslant j$ we have

$$\Delta_{A_{i,j}(X)} = \frac{1}{2}(i - j + 1) + \Delta_X,$$

and each term in (5.1) satisfies

$$\Delta_{A_{i_1,i_0+1}A_{i_2,i_1+1}\dots A_{i_{N+1},i_N+1}(X)} = \frac{2n+1}{2} + \Delta_X.$$

A direct calculation gives

$$W_{2n-k} = \sum_{l=1}^{2n+1-k} (-1)^{kl} J_{k+l,l} + w_{2n-k} \qquad \text{for} \quad k = 0, 1, \dots, 2n,$$

where $\Delta_{2n-k} = \frac{2n+1}{2} - \frac{2n-k}{2}$ and w_{2n-k} can be expressed as a normally ordered product of the elements $J_{i,j}$ with $0 \leqslant i - j \leqslant k$ and their derivatives. It remains to apply Lemma 5.2. □

Example 5.4. Let $\mathfrak{g} = \mathfrak{gl}(2|1)$. Then $f_{\rm prin} = E_{21} + E_{32}$ and

$$\mathcal{A} = \begin{bmatrix} A_{1,1} & -1 & 0 \\ A_{2,1} & A_{2,2} & -1 \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix}.$$

The column determinant of $\mathcal{A}$ is

$$\begin{aligned}\operatorname{cdet}\mathcal{A} &= A_{1,1}A_{2,2}A_{3,3} + A_{3,1} + A_{2,1}A_{3,3} + A_{1,1}A_{3,2} \\ &= (kS)^3 + W_2 S^2 + W_1 S + W_0.\end{aligned}$$

where

$$\begin{aligned}W_2 &= k^2(J_{1,1} + J_{2,2} + J_{3,3}),\\ W_1 &= k(-J_{1,1}J_{2,2} - J_{1,1}J_{3,3} - J_{2,2}J_{3,3} - J_{2,1} + J_{3,2} - kJ'_{2,2}),\\ W_0 &= -J_{1,1}J_{2,2}J_{3,3} - J_{2,1}J_{3,3} + J_{1,1}J_{3,2} + J_{3,1}\\ &\quad + kJ'_{3,2} + kJ_{1,1}J'_{3,3} - kJ'_{2,2}J_{3,3} + kJ_{2,2}J'_{3,3} + k^2 J''_{3,3},\end{aligned}$$

and $X' := [S, X]$. Hence $W(\overline{\mathfrak{g}}, f_{\mathrm{prin}}, k)$ is freely generated by W_0, W_1 and W_2. ☐

As in [2], by taking the quotient of the W-algebra $W(\overline{\mathfrak{g}}, f_{\mathrm{prin}}, k)$ over the supersymmetric vertex algebra ideal generated by the elements $J_{i,j}$ with $i > j$ we recover the presentation of the W-algebra via the *Miura transformation*; cf. [9, 16, 17]:

$$\operatorname{cdet}\mathcal{A} \mapsto \big(kS + J_{1,1}\big)\big(kS - J_{2,2}\big)\big(kS + J_{3,3}\big)\dots\big(kS - J_{2n,2n}\big)\big(kS + J_{2n+1,2n+1}\big).$$

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