PD_3-GROUPS AND HNN EXTENSIONS

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Abstract. We show that if a PD_3-group G splits as an HNN extension $A*_C\varphi$ where C is a PD_2-group then the Poincaré dual in $H^1(G;\mathbb{Z}) = Hom(G,\mathbb{Z})$ of the homology class $[C]$ is the epimorphism $f : G \to \mathbb{Z}$ with kernel the normal closure of A. We also make several other observations about PD_3-groups which split over PD_2-groups.

In this note we shall give algebraic analogues of some properties of Haken 3-manifolds. We are interested in the question "when does a PD_3-group split over a PD_2-group?". In §2 we show that such splittings are minimal in a natural partial order on splittings over more general subgroups. In the next two sections we consider PD_3-groups G which split as an HNN extension $A *_C \varphi$ with A and C finitely generated. In §3 we show that A and C have the same number of indecomposable factors. Our main result is in §4, where we show that if C is a PD_2-group then the Poincaré dual in $H^1(G;\mathbb{Z}) = Hom(G,\mathbb{Z})$ of the homology class $[C]$ is the epimorphism $f : G \to \mathbb{Z}$ with kernel the normal closure of A. In §5 we extend an argument from [7] to show that no FP_2 subgroup of a PD_3-group is a properly ascending HNN extension, and in §6 we show that if G is residually finite and splits over a PD_2-group then G has a subgroup of finite index with infinite abelianization. Our arguments extend readily to PD_n-groups with PD_{n-1}-subgroups, but as our primary interest is in the case $n = 3$, we shall formulate our results in such terms.

1. terminology

We mention here three properties of 3-manifold groups that are not yet known for all PD_3-groups: coherence, residual finiteness and having subgroups of finite index with infinite abelianization. Coherence may often be sidestepped by requiring the subgroups in play to be FP_2 rather than finitely generated. If every finitely generated subgroup of a group G is FP_2 we say that G is *almost coherent*.

We shall say that a group G is *split* over a subgroup C if it is either a generalized free product with amalgamation (GFPA) $G = A *_C B$, where $C < A$ and $C < B$, or an HNN extension $G = HNN(A;\alpha,\gamma : C \to A)$, where α and γ are monomorphisms. (We may also write $G = A *_C \varphi$, where $\varphi = \gamma \circ \alpha^{-1}$.) An HNN extension is *ascending* if one of the associated subgroups is the base. In that case we may assume that $\alpha = id_A$, and $\varphi = \gamma$ is an injective endomorphism of A.

The *virtual first Betti number* $v\beta(G)$ of a finitely generated group is the least upper bound of the first Betti numbers $\beta_1(N)$ of normal subgroups N of finite index in G. Thus $v\beta(G) > 0$ if some subgroup of finite index maps onto $\mathbb{Z}$.

A group G is *large* if it has a subgroup of finite index which maps onto a non-abelian free group. It is clear that if G is large then $v\beta(G) = \infty$.

1991 *Mathematics Subject Classification.* 57N13.

Key words and phrases. fundamental class, HNN extension, PD_3-group, surface group.

2. Comparison of splittings

Let G be a group which is a GFPA $A *_C B$ or an HNN extension $A *_C \varphi$. If we identify the groups A, B and C with subgroups of G then inclusion defines a partial order on such splittings: $A *_C B \leqslant A' *_{C'} B'$ if $A \leqslant A', B \leqslant B'$ and $C \leqslant C'$, and $A *_C \varphi \leqslant A' *_{C'} \varphi'$ if $A \leqslant A', C \leqslant C'$ and $\varphi'|_C = \varphi$, and the stable letters coincide. (In the HNN case we are really comparing splittings compatible with a given epimorphism $G \to \mathbb{Z} \cong G/\langle\langle A \rangle\rangle$.)

Lemma 1. *Let $G = A' *_C \varphi$ be an HNN extension, with stable letter t, and let $A \leqslant A'$ be a subgroup such that $C \cup \varphi(C) \leqslant A$. If $G = \langle A, t\rangle$ then $A = A'$.*

Proof. Let $\alpha \in A'$. Then we may write $\alpha = a_0 t^{\varepsilon_1} a_1 \dots t^{\varepsilon_n} a_n$ where $a_i \in A$ and $\varepsilon_i = \pm 1$, for all i, since $G = \langle A, t\rangle$. We may clearly assume that n is minimal. Hence there are no substrings of the form tct^{-1} or $t^{-1}\varphi(c)t$, with $c \in C$, in this expression for α (since any such may be replaced by $\varphi(c)$ or c, respectively). But it then follows from Britton's Lemma for the HNN extension $A' *_C \varphi$ that $n = 0$, and so $\alpha = a_0$ is in A. □

If G is a PD_3-group then we would like to know when C can be chosen to be a PD_2-group.

Lemma 2. *Let G be a PD_3-group which is a generalized free product with amalgamation $A *_C B$ or an HNN extension $A *_C \varphi$, with C a PD_2-group. Then the splitting is minimal in the partial order determined by inclusions.*

Proof. Suppose that $A' *_{C'} B' \leqslant A *_C B$ or $A' *_{C'} \varphi' \leqslant A *_C \varphi$ (respectively), is another splitting for G. Then C' is either a free group or has finite index in C. The inclusions induce a commuting diagram relating the Mayer-Vietoris sequences associated to the splittings. In each case, the left hand end of the diagram is

$$\begin{array}{ccc} 0 \to H_3(G;\mathbb{Z}) & \xrightarrow{\delta'} & H_2(C';\mathbb{Z}) \\ \downarrow{=} & & \downarrow \\ 0 \to H_3(G;\mathbb{Z}) & \xrightarrow{\delta} & H_2(C;\mathbb{Z}). \end{array}$$

Since the connecting homomorphisms δ' is injective, $H_2(C';\mathbb{Z}) \neq 0$, and so C' cannot be a free group. Hence it is a PD_2-group, and so δ and δ' are isomorphisms [3]. Since the inclusion of C' into C has degree 1, we see that $C' = C$. If $G = A' *_C \varphi$ it then follows from Lemma 1 that $A' = A$. If $G = A *_C B$ and $G = A' *_C B'$ then a similar argument based on normal forms shows that $A' = A$ and $B' = B$. □

If $f : G \to \mathbb{Z}$ is an epimorphism then $G \cong A *_C \varphi$ with $\mathrm{Ker}(f) = \langle\langle A \rangle\rangle$ and stable letter represented by $t \in G$ with $f(t) = 1$. For instance, we may take $C = A = \mathrm{Ker}(f)$ and φ to be conjugation by t. If $\mathrm{Ker}(f)$ is finitely generated, this is the only possibility (up to the choice of t with $f(t) = \pm 1$), but in general there are other ways to do this. If G is FP_2 then we may choose A and C finitely generated [4], and if G is almost coherent then A and C are also FP_2. The construction of [4] gives a pair (A, C) with A generated by $C \cup \varphi(C)$, which is usually far from minimal in this partial order. (See below for an example.) If G is FP then A is FP_k if and only if C is FP_k, for any $k \geqslant 1$ [2, Proposition 2.13].

If G is FP_2 and $\mathrm{Ker}(f)$ is not finitely generated then any HNN structure for G with finitely generated base and associated subgroups is the initial term of an infinite increasing chain of such structures, obtained by applying the construction of [4]. If $G = A*_A\varphi$ is a properly ascending HNN extension, so that $\varphi(A) < A$, then G has a doubly infinite chain of HNN structures, with bases the subgroups t^nAt^{-n}, for $n \in \mathbb{Z}$. However PD_n-groups are never properly ascending HNN extensions. (See Theorem 5 below.) Does every descending chain of HNN structures for a PD_3-group terminate? Do any PD_3-groups which are HNN extensions have minimal splittings over FP_2-groups which are not PD_2-groups?

Let T_2 be the orientable surface of genus 2. The PD_2-group $H = \pi_1(T_2)$ has a standard presentation

$$\langle a, b, c, d \mid [a, b][c, d] = 1\rangle.$$

We may rewrite this presentation as

$$\langle a, b, c, t \mid tct^{-1} = aba^{-1}b^{-1}c\rangle,$$

which displays H as an HNN extension $F(a, b, c) *_{\langle c\rangle} \varphi$, split over the PD_1-group $\langle c\rangle \cong \mathbb{Z}$. The associated epimorphism $f : H \to \mathbb{Z}$ is determined by $f(a) = f(b) = f(c) = 0$ and $f(d) = 1$. In this case the algorithm from [4] would suggest taking $C = \langle a, b, c\rangle$ and $A = \langle a, b, c, tat^{-1}, tbt^{-1}\rangle$, giving an HNN extension with base $A \cong F(5)$ and split over $C \cong F(3)$. Taking products, we see then that the PD_3-group $G = \pi_1(T_2 \times S^1) = H \times \mathbb{Z}$ splits over the PD_2-group $\mathbb{Z}^2$, and is also an HNN extension with base $F(5)\times\mathbb{Z}$ and associated subgroups $F(3)\times\mathbb{Z}$. The latter groups have one end, but are not PD_2-groups.

Splittings over PD_2-groups need not be unique. Let W be an aspherical orientable 3-manifold with incompressible boundary and two boundary components U, V. Let $M = DW$ be the double of W along its boundary. Then M splits over copies of U and V, and $[U] = [V]$ in $H_2(M; \mathbb{Z})$. If U and V are not homeomorphic the corresponding (minimal) splittings of $G = \pi_1(M)$ are evidently distinct. For instance, we may start with the hyperbolic 3-manifold of [11, Example 3.3.12], which is the exterior of a knotted θ-curve $\Theta \subset S^3$. Let W be obtained by deleting an open regular neighbourhood of a meridian of one of the arcs of Θ. Then W is aspherical, $\partial W = T \amalg T_2$ and each component of ∂W is incompressible in W.

3. INDECOMPOSABLE FACTORS

If G is a PD_3-group then $c.d.A = c.d.C = 2$, since these subgroups have infinite index in G, and $H_2(C; \mathbb{Z}) \neq 0$, as observed in Lemma 2. A simple Mayer-Vietoris argument shows that $H^1(A; \mathbb{Z}[G]) \cong H^1(C; \mathbb{Z}[G])$ as right $\mathbb{Z}[G]$-modules, since $H^i(G; \mathbb{Z}[G]) = 0$ for $i \leqslant 2$. (Note that the latter condition fails for PD_2-groups.) The isomorphism is given by the difference $\alpha_* - \gamma_*$ of the homomorphisms induced by α and γ.

We shall assume henceforth that A and C are finitely generated. Then these modules may be obtained by extension of coefficients from the "end modules" $H^1(A; \mathbb{Z}[A])$ and $H^1(C; \mathbb{Z}[C])$. If one is 0 so is the other, and so A has one end if and only if C has one end. If A and C are FP_2 and have one end then they are 2-dimensional duality groups, and we may hope to apply the ideas of [9].

Can G have splittings with base and associated subgroups having more than one end? The next lemma implies that the subgroups A and C must have the same numbers of indecomposable factors. (The analogous statement for PD_2-groups is false, as may be seen from the example in §2 above!)

Lemma 3. *Let $K = (*_{i=1}^m K_i) * F(n)$ be the free product of $m \geqslant 1$ finitely generated groups K_i with one end and $n \geqslant 0$ copies of $\mathbb{Z}$. Then $H^1(K;\mathbb{Z}[K]) \cong \mathbb{Z}[K]^{r-1}$, where $r = m+n$ is the number of indecomposable factors of K.*

Proof. If $n = 0$ the result follows from the Mayer-Vietoris sequence for the free product, with coefficients $\mathbb{Z}[K]$.

In general, let $J = *_{i=1}^m K_i$ and let $C_*(J)$ be a resolution of the augmentation module $\mathbb{Z}$ by free $\mathbb{Z}[J]$-modules, with $C_0(J) = \mathbb{Z}[J]$. Then there is a corresponding resolution $C_*(K)$ of $\mathbb{Z}$ with $C_q(K) \cong \mathbb{Z}[K] \otimes_{\mathbb{Z}[J]} C_q(J)$ if $q \neq 1$ and $C_1(K) \cong \mathbb{Z}[K] \otimes_{\mathbb{Z}[J]} C_q(J) \oplus \mathbb{Z}[K]^n$. Hence there is a short exact sequence of chain complexes (of left $\mathbb{Z}[K]$-modules)

$$0 \to \mathbb{Z}[K] \otimes_{\mathbb{Z}[J]} C_*(J) \to C_*(K) \to \mathbb{Z}[K]^n \to 0,$$

where the third term is concentrated in degree 1. The exact sequence of cohomology with coefficients $\mathbb{Z}[K]$ gives a short exact sequence of *right* $\mathbb{Z}[K]$-modules

$$0 \to \mathbb{Z}[K]^n \to H^1(K;\mathbb{Z}[K]) \to H^1(Hom_{\mathbb{Z}[K]}(\mathbb{Z}[K] \otimes_{\mathbb{Z}[J]} C_*(J), \mathbb{Z}[K]) \to 0.$$

We may identify the right-hand term with $H^1(J;\mathbb{Z}[J]) \otimes_{\mathbb{Z}[J]} \mathbb{Z}[K] \cong \mathbb{Z}[K]^{m-1}$, since J is finitely generated. The lemma follows easily. $\square$

The lemma applies to A and C, since they are finitely generated and torsion-free. The indecomposable factors of C are either conjugate to subgroups of indecomposable factors of A or are infinite cyclic, by the Kurosh subgroup theorem. If A and C have no free factors and the factors of C are conjugate into distinct factors of A then, after modifying φ appropriately, we may assume that $\alpha(C_i) \leqslant A_i$, for all i. However, we cannot expect to also normalize γ in a similar fashion.

4. THE DUAL CLASS

If M is a closed 3-manifold with $\beta_1(M) > 0$ then there is an essential map $f : M \to S^1$. Transversality and the Loop Theorem together imply that there is a closed incompressible surface $S \subset M$ such that $M \setminus S$ is connected. Hence $\pi_1(M)$ is an HNN extension with base $\pi_1(M \setminus S)$ and associated subgroups copies of $\pi_1(S)$. Moreover, the stable letter of the extension is represented by a simple closed curve in M which intersects S transversely in one point. Let $w = w_1(M)$. Then $w_1(S) = w|_S$ and the image of the fundamental class $[S]$ in $H_2(M;\mathbb{Z}^w)$ is Poincaré dual to the image of f in $H^1(M;\mathbb{Z}) = [M, S^1]$.

There is no obvious analogue of transversality in group theory. Nevertheless a similar result holds for PD_3-groups. (We consider only the orientable case, for simplicity.)

Theorem 4. *Let $G = HNN(A;\alpha,\gamma : C \to A)$ be an orientable PD_3-group which is an HNN extension split over a PD_2-group C. Let $f \in H^1(G;\mathbb{Z})$ be the epimorphism with kernel $\langle\langle A\rangle\rangle_G$. Then $f \frown [G]$ is the image of $[C]$ in $H_2(G;\mathbb{Z})$, up to sign.*

Proof. The subgroup C is orientable and the pair $(A;\alpha,\gamma)$ is a PD_3^+-pair [3, Theorem 8.1], and so there is an exact sequence

$$H_3(A,\partial;\mathbb{Z}) \xrightarrow{(1,1)} H_2(C;\mathbb{Z}) \oplus H_2(C;\mathbb{Z}) \xrightarrow{(\alpha_*,-\gamma_*)} H_2(A;\mathbb{Z}) \to H_2(A;\partial;\mathbb{Z}).$$

Hence $\alpha_*[C] = \gamma_*[C]$, and the subgroup they generate is an infinite cyclic direct summand of $H_2(A;\mathbb{Z})$, since $H_2(A;\partial;\mathbb{Z}) \cong H^1(A;\mathbb{Z})$ is free abelian.

Let $t \in G$ correspond to the stable letter for the HNN extension, and let $A_j = t^j A t^{-j}$, $\alpha_j(c) = t^j \alpha(c) t^{-j}$ and $\gamma_j(c) = t^j \gamma(c) t^{-j}$, for all $c \in C$ and $j \in \mathbb{Z}$. Let K_p be the subgroup generated by $\cup_{|j| \leqslant |p|} A_j$, for $p \geqslant 0$. Then $K_0 = A$ and

$$K_{p+1} = A_{-p-1} *_{\alpha_{-p}=\gamma_{-p-1}} K_p *_{\alpha_{p+1}=\gamma_p} A_{p+1}, \quad \textit{for all } p \geqslant 0,$$

and $K = \langle\langle A \rangle\rangle_G = \mathrm{Ker}(f)$ is the increasing union $K = \cup K_p$ of iterated amalgamations with copies of A over copies of C. Each pair $(K_p; \alpha_{-p}, \gamma_p)$ is again a PD_3^+-pair, and so the images of $[C]$ in $H_2(K;\mathbb{Z})$ under the homomorphisms induced by the α_ns all agree.

Let $\Lambda = \mathbb{Z}[G/K] = \mathbb{Z}[t, t^{-1}]$, and let $\varepsilon : \Lambda \to \mathbb{Z}$ be the augmentation. Then $H_i(K;\mathbb{Z}) = H_2(G;\Lambda)$ is a finitely generated Λ-module, with action deriving from the action of G on K by conjugation. Then $H_2(G;\Lambda) = H_2(K;\mathbb{Z}) = \lim H_2(K_p;\mathbb{Z})$. Since $t.\alpha_{n*}[C] = \alpha_{(n+1)*}[C] = \alpha_{n*}[C]$, for all n, the image of $[C]$ in $H_2(K;\mathbb{Z})$ generates an infinite cyclic direct summand.

Poincaré duality gives an isomorphism $H_2(G;\Lambda) \cong \overline{H^1(G;\Lambda)}$, and this is in turn an extension of $\overline{Hom_\Lambda(K/K', \Lambda)}$ by $\overline{Ext^1_\Lambda}(\mathbb{Z}, \Lambda)$, by the Universal Coefficient spectral sequence. Note that $\overline{Hom_\Lambda(K/K', \Lambda)}$ has no non-trivial Λ-torsion, while $Ext^1_\Lambda(\mathbb{Z}, \Lambda) \cong \Lambda/(t-1)\Lambda = \mathbb{Z}$.

We have a commutative diagram

$$\begin{array}{ccccc}
H^1(\mathbb{Z};\Lambda) & \xrightarrow{H^1(f)} & H^1(G;\Lambda) & \xrightarrow{\frown [G]} & H_2(G;\Lambda) \\
\cong \downarrow \varepsilon_\# & & \downarrow \varepsilon_\# & & \downarrow \varepsilon_\# \\
H^1(\mathbb{Z};\mathbb{Z}) & \xrightarrow{id_{\mathbb{Z}} \mapsto f} & H^1(G;\mathbb{Z}) & \xrightarrow{\frown [G]} & H_2(G;\mathbb{Z})
\end{array}$$

in which the vertical homomorphisms are induced by the change of coefficients ε and the two right hand horizontal homomorphisms are Poincaré duality isomorphisms. Since $H^1(f)$ carries $H^1(\mathbb{Z};\Lambda) = Ext^1_\Lambda(\mathbb{Z}, \Lambda) \cong \mathbb{Z}$ onto the Λ-torsion submodule of $H^1(G;\Lambda)$, a diagram chase shows that $f \frown [G]$ is the image of $[C]$ in $H_2(G;\mathbb{Z})$, up to sign. $\square$

In [10] it is shown that if a PD_3-group G has a subgroup S which is a PD_2-group then G splits over a subgroup commensurate with S if and only if an invariant $sing(S) \in \mathbb{Z}/2\mathbb{Z}$ is 0, and then S is maximal among compatibly oriented commensurate subgroups. Theorem 4 suggests a slight refinement of this splitting criterion.

Theorem (Kropholler-Roller [10])**.** *Let G be an orientable PD_3-group and $S < G$ a subgroup which is an orientable PD_2-group. Then*

(1) $G \cong A *_T B$ for some T commensurate with $S \Leftrightarrow sing(S) = 0$ and $[S] = 0$ in $H_2(G;\mathbb{Z})$;
(2) $G \cong A *_T \varphi$ for some T commensurate with $S \Leftrightarrow sing(S) = 0$ and $[S]$ has infinite order in $H_2(G;\mathbb{Z})$;
(3) $G \cong A *_S \varphi \Leftrightarrow sing(S) = 0$ and $[S]$ generates an infinite direct summand of $H_2(G;\mathbb{Z})$.

Proof. The group G splits over a subgroup T commensurate with S if and only if $sing(S) = 0$ [10], and $[S]$ and $[T]$ are then proportional. If $G = A *_T B$ is a generalized free product with amalgamation over a PD_2-group T then the pairs (A, T) and (B, T) are again PD_3^+-pairs [3]. The image of $[T]$ in $H_2(G;\mathbb{Z})$ is trivial, since T bounds each of (A, T) and (B, T), and so $[S] = 0$ also.

If $G \cong A *_T \varphi$ is an HNN extension then the Poincaré dual of $[T]$ is an epimorphism $f : G \to \mathbb{Z}$, by the theorem, and so $[T]$ generates an infinite cyclic direct summand of $H_2(G;\mathbb{Z})$. Hence $[S]$ also has infinite order. $\square$

If $[C] = [S]$ and $sing(S) = 0$ is $sing(C) = 0$ also?

5. no properly ascending HNN extensions

Cohomological arguments imply that no PD_3-group is a properly ascending HNN extension [7, Theorem 3]. A stronger result holds for 3-manifold groups: no finitely generated subgroup can be conjugate to a proper subgroup of itself [6]. We shall adapt the argument of [7] to prove the corresponding result for FP_2 subgroups of PD_3-groups.

Theorem 5. *Let H be an FP_2 subgroup of a PD_3-group G. If $gHg^{-1} \leqslant H$ for some $g \in G$ then $gHg^{-1} = H$.*

Proof. Suppose that $gHg^{-1} < H$. Then $g \notin H$. Let $\theta(h) = ghg^{-1}$, for all $h \in H$, and let $K = H *_H \theta$ be the associated HNN extension, with stable letter t. The normal closure of H in K is the union $\cup_{r\in\mathbb{Z}} t^r H t^{-r}$, and so every element of K has a normal form $k = t^m t^r h t^{-r}$, where m is uniquely determined by k, and h is determined by k, m and r. Let $f : K \to G$ be the homomorphism defined by $f(h) = h$ for all $h \in H$ and $f(t) = g$. If $f(t^m t^r h t^{-r}) = f(t^n t^s h' t^{-s})$ for some m, n, r, s then $g^{n-m} = g^s h' g^{-s} g^t h^{-1} g^{-t}$. After conjugating by a power of g if necessary, we may assume that $s, t \geqslant 0$, and so $g^{n-m} \in H$. But then $H = g^{|n-m|} H g^{-|n-m|}$. Since gHg^{-1} is a proper subgroup of H, we must have $n = m$. It follows easily that f is an isomorphism from K to the subgroup of G generated by g and H.

Since K is an ascending HNN extension with FP_2-base, $H^1(K;\mathbb{Z}[K])$ is a quotient of $H^0(H;\mathbb{Z}[K]) = 0$ [5, Theorem 0.1]. Hence it has one end. Since no PD_3-group is an ascending HNN extension [7, Theorem 3], K is a 2-dimensional duality group. Hence it is the ambient group of a PD_3-pair $(K, \mathcal{S})$ [9]. Doubling this pair along its boundary gives a PD_3-group. But this is again a properly ascending HNN extension, and so cannot happen. Therefore the original supposition was false, and so $gHg^{-1} = H$. $\square$

6. residual finiteness, splitting and largeness

The fundamental group of an aspherical closed 3-manifold is either solvable or large [1, Flowcharts 1 and 4]. This is also so for residually finite PD_3-groups containing $\mathbb{Z}^2$ [8, Theorem 11.19]. Here we shall give a weaker result for PD_3-groups which split over other PD_2-groups.

Theorem 6. *Let G be a residually finite orientable PD_3-group which splits over an orientable PD_2-group C. Then either $\beta_1(G) > 0$, or G maps onto D_∞, or G is large. Hence $v\beta(G) > 0$. If G is LERF and $\chi(C) < 0$ then G is large.*

Proof. For the first assertion, we may assume that $\beta_1(G) = 0$, and that $G \cong A *_C B$. Then (A, C) and (B, C) are PD_3-pairs, and so $\beta_1(C) \leqslant 2\beta_1(A)$ and $\beta_1(C) \leqslant 2\beta_1(B)$. Since $\beta_1(C) > 0$, we must have $\beta_1(A) > 0$ and $\beta_1(B) > 0$ also. Moreover $\beta_1(C) = \beta_1(A) + \beta_1(B)$, since $H_1(G)$ is finite and $H_2(G) = 0$. Hence $\beta_1(C) > \beta_1(A)$ and $\beta_1(C) > \beta_1(B)$.

Let $\{\Delta_n | n \geqslant 1\}$ be a descending filtration of G by normal subgroups of finite index. Then $A_n = A/A \cap \Delta_n$, $B_n = B/B \cap \Delta_n$ and $C_n = C/C \cap \Delta_n$ are finite, and

G maps onto $A_n *_{C_n} B_n$, for all n. If $A_n *_{C_n} B_n$ is finite then $C_n = A_n$ or B_n. Thus if all these quotients of G are finite we may assume that $C_n = A_n$ for all n. But then the inclusion of C into A induces an isomorphism on profinite completions, and so $\beta_1(C) = \beta_1(A)$, contrary to what was shown in the paragraph above.

If C_n is a proper subgroup of both A_n and B_n then either $[A_n : C_n] = [B_n : C_n] = 2$, in which case G maps onto D_∞, or one of these indices is greater than 2, in which case $A_n *_{C_n} B_n$ is virtually free of rank > 1, and so G is large. In each case, it is clear that $v\beta(G) \geqslant 1$.

Suppose now that G is LERF. If $[A_n : C_n] \leqslant 2$ then C_n is normal in A_n, and so $C(A \cap \Delta_n)$ is normal in A. Hence if $[A_n : C_n] \leqslant 2$ for all n then $\cap_n C(A \cap \Delta_n)$ is normal in A. Since G is LERF, this intersection is C. Hence if both $[A_n : C_n] \leqslant 2$ and $[B_n : C_n] \leqslant 2$ for all n then C is normal in G, so G is virtually a semidirect product $C \rtimes \mathbb{Z}$, and is a 3-manifold group. If $\chi(C) < 0$ then G is large [1, Flowcharts 1 and 4]. □

Remark. The lower central series of $D_\infty = Z/2Z * Z/2Z$ gives a descending filtration by normal subgroups of finite index which meets each of the free factors trivially.

Is every PD_3-group either solvable or large?

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