HOMOTOPY TYPES OF 4-MANIFOLDS WITH 3-MANIFOLD FUNDAMENTAL GROUPS

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ABSTRACT. We show that the homotopy type of a 4-manifold M whose fundamental group is a finitely presentable PD_3-group π, and with $w_1(M) = w_1(\pi)$ is determined by π, $\pi_2(M)$, $k_1(M)$ and the equivariant intersection pairing λ_M.

The basic algebraic invariants of a closed 4-manifold M are the fundamental group $\pi = \pi_1(M)$, the $\mathbb{Z}[\pi]$-module $\Pi = \pi_2(M)$, the equivariant homotopy intersection pairing λ_M on Π, the first k-invariant $\kappa = k_1(M) \in H^3(\pi; \Pi)$, the Euler characteristic $\chi(M)$, and the Stiefel-Whitney classes $w = w_1(M)$ and $w_2(M)$. (Strictly speaking, λ_X determines Π, and it also determines w if $\pi \neq 0$.) These invariants determine the stable homeomorphism type of M (with respect to sums with $S^2 \times S^2$), if π is the group of an aspherical closed orientable 3-manifold [14]. (The k-invariant is determined by the other data in this situation.)

We shall show that the homotopy type of a PD_4-complex X whose fundamental group is a finitely presentable PD_3-group π, and with $w_1(X) = w_1(\pi)$ is determined by the invariants π, Π, κ and λ_X. (However we do not yet know the possible values of κ or λ_X.) This leads to an alternative approach to the results of [14], and also to an unstabilized result when π is solvable.

1. NOTATION AND TERMINOLOGY

We shall assume throughout that X is a PD_4-complex such that $\pi = \pi_1(X)$ is a finitely presentable PD_3-group and $w_1(X) = w_1(\pi)$. We shall write $w = w_1(\pi)$, for simplicity. Let $p : \widetilde{X} \to X$ be the universal covering. The homology of $\widetilde{X}$ is given by $H_i(\widetilde{X}; \mathbb{Z}) = H_i(X; \mathbb{Z}[\pi])$, for all i. We assume that π acts on the left of $\widetilde{X}$, and so these are left $\mathbb{Z}[\pi]$-modules. Since π has one end, $H_i(X; \mathbb{Z}[\pi]) = 0$ if $i \neq 0, 2$. Therefore $\Pi = H_2(X; \mathbb{Z}[\pi])$ is not 0, since π is not a PD_4-group.

The homologies of $\widetilde{X}$ and X are related by the Cartan-Leray spectral sequence for the covering, which has the form

$$E^2_{p,q} = H_p(\pi; H_q(X; \mathbb{Z}[\pi])) \Rightarrow H_{p+q}(X; \mathbb{Z}).$$

(Note that the groups $H_*(\pi; A) = Tor^{\mathbb{Z}[\pi]}_*(\mathbb{Z}, A)$ with coefficients in a left module A are defined via a resolution of $\mathbb{Z}$ by *right* $\mathbb{Z}[\pi]$-modules.) There is also a Universal Coefficient spectral sequence

$$E^{p,q}_2 = Ext^q_{\mathbb{Z}[\pi]}(H_p(X; \mathbb{Z}[\pi]), \mathbb{Z}[\pi]) \Rightarrow H^{p+q}(X; \mathbb{Z}[\pi]),$$

which relates the homology and equivariant cohomology of the universal cover.

Let $\varepsilon : \mathbb{Z}[\pi] \to \mathbb{Z}$ be the augmentation homomorphism and $I_\pi = \text{Ker}(\varepsilon)$ be the augmentation ideal. Clearly $H^0(\pi; I_\pi) = 0$. Applying the fixed point functor $Hom_{\mathbb{Z}[\pi]}(\mathbb{Z}, -)$ to the augmentation exact sequence

$$0 \to I_\pi \to \mathbb{Z}[\pi] \xrightarrow{\ \varepsilon\ } \mathbb{Z} \to 0$$

gives $H^i(\pi; \mathbb{Z}) \cong H^{i+1}(\pi; I_\pi)$ for $i \leqslant 2$, while applying the functor $Hom_{\mathbb{Z}[\pi]}(-, \mathbb{Z}[\pi])$ gives $Hom_{\mathbb{Z}[\pi]}(I_\pi, \mathbb{Z}[\pi]) \cong \mathbb{Z}[\pi]$ and $Ext^i_{\mathbb{Z}[\pi]}(I_\pi, \mathbb{Z}[\pi]) \cong H^{i+1}(\pi; \mathbb{Z}[\pi])$, for $i > 0$. (In fact every homomorphism from I_π to $\mathbb{Z}[\pi]$ is the restriction of right multiplication by an element of $\mathbb{Z}[\pi]$.)

If M is a left $\mathbb{Z}[\pi]$-module let $M^w = \mathbb{Z}^w \otimes M$ be the left $\mathbb{Z}[\pi]$-module with the same underlying abelian group and diagonal left π-action, given by $g(1 \times x) = w(g)(1 \times gx)$ for all $g \in \pi$ and $x \in \Pi$. Then $\mathbb{Z}[\pi]^w \cong \mathbb{Z}[\pi]$, since we may define an isomorphism $f : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]^w$ by $f(g) = w(g) \otimes g$ for all $g \in \pi$. The linear extension of w defines the *w-twisted augmentation* $\varepsilon_w : \mathbb{Z}[\pi] \to \mathbb{Z}^w$, with kernel I^w_π. Arguments similar to those of the previous paragraph apply to I^w_π.

If R is a right $\mathbb{Z}[\pi]$-module let $\overline{R}$ be the left module with the same underlying group and $\mathbb{Z}[\pi]$-action determined by $g.r = w(g)rg$ for all $r \in R$ and $g \in \pi$. We use a similar strategy and notation to obtain a right module $\overline{L}$ from a left $\mathbb{Z}[\pi]$-module L. Free right modules give rise to free left modules of the same rank, and conversely. We may define the dual of a left module M as the left module $M^\dagger = \overline{Hom_{\mathbb{Z}[\pi]}(M, \mathbb{Z}[\pi])}$.

Two (left) $\mathbb{Z}[\pi]$-modules L and L' are *stably projective equivalent* if $L \oplus P \cong L' \oplus P'$ for some finitely generated projective $\mathbb{Z}[\pi]$modules P, P'. They are *stably equivalent* if we may assume that P and P' are each free modules. We shall let $[L]_{pr}$ and $[L]$ denote the equivalence classes corresponding to these two equivalence relations. As our concern in this paper is mainly with finite PD_4-complexes, which correspond most closely to manifolds, stable equivalence is the more useful notion. However our arguments apply with little change to the more general setting of finitely dominated PD_4-complexes (and even

PD_4-spaces in the sense of [13]), for which the broader notion of stably projective equivalence is needed.

If A is an abelian group let $rk(A) = \dim_{\mathbb{Q}} \mathbb{Q} \otimes A$ be its rank.

2. THE BASIC EXAMPLES

Let $Y = K(\pi, 1)$. We may assume that $Y = Y_o \cup D^3$, where (Y_o, S^2) is a PD_3-pair and Y_o is cohomologically 2-dimensional. Let τ be a self-homeomorphism of $S^2 \times S^1$ which does not extend over $S^2 \times D^2$. (There is an unique isotopy class of such maps τ.) Then $X(\pi) = (Y_o \times S^1) \cup S^2 \times D^2$ and $X(\pi)_\tau = (Y_o \times S^1) \cup_\tau S^2 \times D^2$ are PD_4-complexes with fundamental group π, orientation character $w = w_1(Y)$ and Euler characteristic 2. Since $X(\pi) = \partial(Y_o \times D^2)$, it retracts onto Y_o.

The arguments of [15, §2] for the case when Y is a 3-manifold are essentially homological and apply equally well in our situation. Let $U = Y_o \times S^1$. The long exact sequence of the pair $(X(\pi), U)$ with coefficients $\mathbb{Z}[\pi]$ gives a five-term exact sequence

$$H_3(X(\pi), U; \mathbb{Z}[\pi]) \to H_2(U; \mathbb{Z}[\pi]) \to H_2(X(\pi); \mathbb{Z}[\pi]) \to$$

$$H_2(X(\pi), U; \mathbb{Z}[\pi]) \to H_1(U; \mathbb{Z}[\pi]) \to 0,$$

since $H_i(X; \mathbb{Z}[\pi]) = 0$ for $i \neq 0, 2$. Now $H_3(X(\pi), U; \mathbb{Z}[\pi]) = 0$ and $H_2(X(\pi), U; \mathbb{Z}[\pi]) \cong \mathbb{Z}[\pi]$, by excision, while $H_2(U; \mathbb{Z}[\pi]) \cong H_2(Y_o; \mathbb{Z}[\pi])(\cong \pi_2(Y_o)) \cong \mathbb{Z}[\pi]$ and $H_1(U; \mathbb{Z}[\pi]) \cong \mathbb{Z}$. Hence this sequence reduces to

$$0 \to \pi_2(Y_o) \cong \mathbb{Z}[\pi] \to \pi_2(X(\pi)) \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0.$$

Hence $\pi_2(X(\pi))$ is an extension of I_π by $\pi_2(Y_o) \cong \mathbb{Z}[\pi]$. The extension splits, since $Ext^1_{\mathbb{Z}[\pi]}(I_\pi, \mathbb{Z}[\pi]) = H^2(\pi; \mathbb{Z}[\pi]) = 0$, and so $\pi_2(X(\pi)) \cong \mathbb{Z}[\pi] \oplus I_\pi$. The retraction of $X(\pi)$ onto Y_o determines a splitting.

A similar argument shows that $\pi_2(X(\pi)_\tau) \cong \mathbb{Z}[\pi] \oplus I_\pi$ also, but we do not know whether $X(\pi)_\tau$ retracts onto Y_o.

If π has a balanced presentation then there is a closed 4-manifold with $\pi_1(M) \cong \pi$ and $\chi(M) = 2$. Let K be a finite 2-complex with $\pi_1(K) \cong \pi$ and $\chi(K) = 1$ and let N be a 4-dimensional handlebody thickening of K. Then the double of N is a closed 4-manifold M with $\pi_1(M) \cong \pi$ and $\chi(M) = 2$, and which retracts onto K.

If Y is a closed 3-manifold then the corresponding closed 4-manifolds are the manifolds obtained by elementary surgery on the second factor of $Y \times S^1$. (There are two possible framings of the normal bundle.) Plotnick uses manifold topology, first to define a splitting of the above exact sequence and then to show that the image of $\pi_2(Y_o)$ in $\pi_2(X)$ is self-annihilating, with respect to the equivariant homotopy intersection pairing λ_X, in either case, and that $\pi_2(X(\pi))$ is the direct sum of two

self-annihilating summands. The equivariant homotopy intersection pairings of these 4-manifolds are not isometric [15, Theorem 3.1].

3. Π, χ AND λ_X

In this section we shall summarize the key properties of Π and χ, which were determined in [11, Theorem 3.13], and define the equivariant intersection pairing, using the cohomological formulation.

We shall first state without proof a result from [11].

Theorem. [11, Theorem 3.13] *Let X be a PD_4-complex such that $\pi = \pi_1(X)$ is a finitely presentable PD_3-group and $w_1(X) = w_1(\pi)$. Then $\chi(X) \geqslant 2$ and $[\Pi]_{pr} = [I_\pi]_{pr}$.* $\square$

The Euler characteristic is in fact determined by π and Π. This follows easily from the invariance of χ between the pages of a spectral sequence, and the special nature of Π.

Corollary 1. $\chi(X) = rk(\Pi) + 1 - \beta_1(\pi)$.

Proof. The only nonzero entries in the Cartan-Leray homology spectral sequence for the universal cover of X are when $0 \leqslant p \leqslant 3$ and $q = 0$ or 2, and then $E_{p,0}^2 = H_p(\pi; \mathbb{Z})$, while $E_{0,2}^2 = \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \Pi$ and $E_{p,2}^2 = H_{p+1}(\pi; \mathbb{Z})$ for $p > 0$, since $[\Pi]_{pr} = [I_\pi]_{pr}$. Since $\chi(X) = \Sigma_{p,q}(-1)^{p+q} rk(E_{p,q}^2)$, the corollary follows. $\square$

If $\chi(X) = 2$ then Π is stably projective equivalent to $\mathbb{Z}[\pi] \oplus I_\pi$. Hence $H^3(\pi; \Pi) \cong \mathbb{Z} \oplus H_1(\pi; \mathbb{Z}^w)$, since $H^3(\pi; \mathbb{Z}[\pi]) \cong \mathbb{Z}$ and $H^3(\pi; I_\pi) \cong H^2(\pi; \mathbb{Z}) \cong H_1(\pi; \mathbb{Z}^w)$, by Poincaré duality. If $w = 1$ this is just π^{ab}.

If X is a closed 4-manifold, $\pi = \pi_1(X)$ and $w = w_1(X)$ then geometric intersection numbers can be used to define a w-hermitean equivariant intersection pairing on Π, with values in $\mathbb{Z}[\pi]$. In the Poincaré duality complex case we cannot count geometric intersection numbers and so weshall use the cohomological formulation of the intersection pairing instead. In fact this cohomological formulation is well suited to application of [12, Theorem 2] in Theorem 8 below.

Lemma 2. *There is an exact sequence of left $\mathbb{Z}[\pi]$-modules*

$$0 \to \overline{H^2(X; \mathbb{Z}[\pi])} \xrightarrow{ev} \Pi^\dagger \to \mathbb{Z} \to 0.$$

Proof. This follows from the Universal Coefficient spectral sequence, since $H^2(\pi; \mathbb{Z}[\pi]) = 0$, $H^2(\pi; \mathbb{Z}[\pi]) \cong \mathbb{Z}$ and $\Pi \cong H_2(X; \mathbb{Z}[\pi]) \cong \overline{H^2(X; \mathbb{Z}[\pi])}$, by Poincaré duality. (See [11, Lemma 3.3].) $\square$

Let $D : \Pi \to H^2(X; \mathbb{Z}[\pi])$ be the isomorphism given by Poincaré duality. Then the intersection pairing may be defined by

$$\lambda_X(u, v) = ev(v)([X] \cap u), \quad \forall \, u, v \in H^2(X; \mathbb{Z}[\pi]).$$

(See [16, Proposition 4.58].) Then $\lambda_X(gu, gv) = w(g)\lambda_X(u, v)$ for all $g \in \pi$ and $u, v \in H^2(X; \mathbb{Z}[\pi])$. Since $\Pi \neq 0$ and ev is a monomorphism, λ_X is non-zero, and so w is determined by λ_X. If $r : X \to Y_o$ is a retraction then $r_*[X] = 0$, and so $r^*H^2(Y_o; \mathbb{Z}[\pi]) \cong \mathbb{Z}[\pi]$ is a direct summand of Π which is self-annihilating under λ_X.

It is clear from the argument in [11, Theorem 3.13] that if X is finite and π is of type FF then $\Pi \oplus \mathbb{Z}[\pi]^r \cong \mathbb{Z}[\pi]^{\chi(X)-1} \oplus I_\pi \oplus \mathbb{Z}[\pi]^r$ for r large, and so $[\Pi] = [I_\pi]$. The minimal value $\chi(X) = 2$ is realized by the complexes $X(\pi)$ and $X(\pi)_\tau$ defined above.

In the 3-manifold group case $\widetilde{K}_0(\mathbb{Z}[\pi]) = 0$, by work of Farrell and Jones, anticipating the Geometrization Theorem [7]. In this case we may again assume that $[\Pi] = [I_\pi]$, and some of our statements can be simplified. We shall assume that $\widetilde{K}_0(\mathbb{Z}[\pi]) = 0$ wherever convenient. However even with this assumption there may be difficulties. If π is polycyclic but not abelian then there are ideals $J < \mathbb{Z}[\pi]$ which are not free but for which $\mathbb{Z}[\pi] \oplus J \cong \mathbb{Z}[\pi]^2$ [1].

The case $\pi = \mathbb{Z}^3$ is exceptional. All projectives are then free. In this case $\Pi^\dagger$ is stably free, hence free, and it follows from Lemma 2 that $\Pi \cong \mathbb{Z}[\pi]^{\chi(X)-1} \oplus I_\pi$. Moreover, if K is any finite 2-complex with $\pi_1(K) \cong \mathbb{Z}^3$ and $\chi(K) = 1$ then $\pi_2(K)$ is free of rank 1, and so $K \simeq T_o^3$.

4. k_1 AND RETRACTIONS ONTO Y_o

The first k-invariant is an element of $H^3(\pi; \Pi)$, and is well-defined up to the actions of $Aut(\pi)$ and $Aut_\pi(\Pi)$. If Z is a cell complex we may assume that the Postnikov 2-stage $f_2(Z) : Z \to P_2(Z)$ is an inclusion, and that $P_2(Z)$ is obtained from Z by adding cells of dimension $\geqslant 4$.

Lemma 3. *Let G be a PD_n-group and let C_* be a projective resolution of the augmentation module $\mathbb{Z}$ of length n such that $C_n \cong \mathbb{Z}[G]$. Then the class $[C_*]$ of C_* in $H^n(G; C_n) = Ext^n_{\mathbb{Z}[G]}(\mathbb{Z}, C_n) \cong \mathbb{Z}$ is a generator.*

Proof. Let D_* be the chain complex with $D_i = C_i$ for $i < n$ and $D_i = 0$ for $i \geqslant n$, and let E_* be the chain complex with $E_i = D_i$ for $i \neq n - 1$ and $E_{n-1} = C_{n-1}/Z_{n-1} \oplus C_n$. Then D_* and E_* are of type $(\mathbb{Z}, 0, C_n, n - 1)$ as defined in [6, Definition 7.1]. The k-invariant of D_* is represented by the class $[C_*]$, while the k-invariant of E_* is 0.

If $[C_*] = 0$ then there is a c.h.e, $f : D_* \to E_*$ [6, Satz 7.6], and since $H_{n-1}(f)$ is an isomorphism we see that C_* is chain homotopy equivalent

to a sequence in which ∂_n is a split injection. Hence $H^n(C_*; \mathbb{Z}[G]) = 0$, contrary to hypothesis.

Similarly, if $[C_*]$ is a p-fold multiple of some other class for some prime p then $H^n(C_*; \mathbb{F}_p[G]) = 0$, again contrary to hypothesis. Therefore $[C_*]$ is indivisible, and so is a generator of $H^n(G; \mathbb{Z}[G]) \cong \mathbb{Z}$. $\square$

Theorem 4. *Let X, Y and π be as in §1 and §2 above. Then*

(1) *$P_2(X)$ retracts onto $P_2(Y_o)$ if and only if $\Pi \cong \mathbb{Z}[\pi] \oplus L$, for some L, and the image of $k_1(X)$ in $H^3(\pi; \mathbb{Z}[\pi])$ is a generator;*
(2) *if X' is another PD_4-complex with $\pi_1(X') \cong \pi$, $\pi_2(X') \cong \Pi$ and which retracts onto Y_o then $P_2(X') \simeq P_2(X)$.*

Proof. If $P_2(X)$ retracts onto $P_2(Y_o)$ then there is a pair of maps $j : P_2(Y_o) \to P_2(X)$ and $r : P_2(X) \to P_2(Y_o)$ such that $rj \sim id_{Y_o}$. It follows immediately that $\pi_2(Y_o) \cong \mathbb{Z}[\pi]$ is a direct summand of Π, and that $j^* k_1(X) = k_1(Y_o)$, up to the action of automorphisms. The chain complex $C_*(Y_o; \mathbb{Z}[\pi])$ is chain homotopy equivalent to a finite projective complex D_* with $D_i = 0$ for $i > 2$. The complex C_* with $C_i = D_i$ for $i \neq 3$ and $C_3 = H_2(D_*) \cong \pi_2(Y_o)$ is a projective resolution of $\mathbb{Z}$, and $k_1(Y_o)$ is the class of C_* in $H^3(\pi; \mathbb{Z}[\pi])$. Hence $k_1(Y_o)$ is a generator of $H^3(\pi; \mathbb{Z}[\pi])$, by Lemma 3.

Conversely, if the conditions hold then there are morphisms between the algebraic 2-types $[\pi, \Pi, k_1(X)]$ and $[\pi, \pi_2(Y_o), k_1(Y_o)]$ which can be realized by maps defining a retraction.

Suppose now that $\Pi \cong \mathbb{Z}[\pi] \oplus L$, for some $\mathbb{Z}[\pi]$-module L, and let e generate the first summand. If k and k' in $H^3(\pi; \Pi)$ each represent generators of $H^3(\pi; \mathbb{Z}[\pi]) \cong \mathbb{Z}$ then there is an automorphism ϕ of Π such that $\phi(e) = \pm e + \ell$, for some $\ell \in L$, and $\phi|_L = id_L$, and such that the induced automorphism of $H^3(\pi; \Pi)$ carries k to k'. This proves the second assertion. $\square$

Assertion (2) extends partially an observation in [14], namely that the k-invariant plays no role in their stable classification.

Corollary 5. $P_2(X) \simeq P_2(\partial(Y_o \times D^2)) \Leftrightarrow \Pi \cong \mathbb{Z}[\pi] \oplus I_\pi$. $\square$

If $P_2(X)$ retracts onto $P_2(Y_o)$ then the composite of $f_2(Y_o)$ with the inclusion $P_2(Y_o) \to P_2(X)$ factors through $f_2(X)$, since Y_o has dimension $\leqslant 3$, while $P_2(X) = X \cup e^{\geqslant 4}$.

5. THE MAIN THEOREM

Let Γ be the quadratic functor of Whitehead. Let L be a finitely generated left $\mathbb{Z}[\pi]$-module, and let $Her_w(L^\dagger)$ be the abelian group of w-hermitean pairings on $L^\dagger$. Then there is a natural homomorphism

$B_L : \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma(L) \to Her_w(L^\dagger)$, which is an isomorphism if π is 2-torsion-free and L is projective [12, Theorem 1].

If π is a PD_3-group then it is torsion-free. However the modules of interest to us are not projective, since I_π has projective dimension 2. We shall show that when Π is stably equivalent to I_π then B_Π remains injective. We first recall some details about Γ from [3, Chapter 1.§4]. Let $\gamma_A : A \to \Gamma(A)$ be the canonical quadratic map. We may define a homomorphism $[-] : A \odot A \to \Gamma(A)$ by

$$[a \odot b] = \gamma(a + b) - \gamma(a) - \gamma(b).$$

Then $[a \odot a] = 2\gamma(a)$ for all $a \in A$.

As abelian groups, $\Gamma(A \oplus B) \cong \Gamma(A) \oplus \Gamma(B) \oplus (A \otimes B)$. If A and B are $\mathbb{Z}[G]$-modules the summands are invariant under the action of G, and so this direct sum splitting is a $\mathbb{Z}[G]$-module splitting.

The following lemma is close to the first part of [9, Lemma 2.3] (which considered only finite groups G).

Lemma 6. *Let G be a group. Then $\Gamma(\mathbb{Z}[G]) \cong \mathbb{Z}[G] \oplus \Gamma(I_G)$ as a left $\mathbb{Z}[G]$-module.*

Proof. Let $i_g = g - 1$, for $g \in G$, and let $j : \mathbb{Z} \to \mathbb{Z}[G]$ be the canonical ring homomorphism. Then I_G is free with basis $\{i_g \mid g \in G\}$, and $\mathbb{Z}[G] \cong \mathrm{Im}(j) \oplus I_G$ as abelian groups. Hence $\Gamma(\mathbb{Z}[G])$ splits as a direct sum of abelian groups $\Gamma(\mathbb{Z}) \oplus \Gamma(I_G) \oplus (I_G \otimes \mathbb{Z})$. The middle summand is a $\mathbb{Z}[G]$-submodule, but the others are not.

The complement of $\Gamma(I_G)$ in $\Gamma(\mathbb{Z}[G])$ is freely generated (as an abelian group) by the elements $e = \gamma_{\mathbb{Z}[G]}(1)$ and $\{i_g \otimes 1 \mid g \in G\}$, and so the quotient $\Gamma(\mathbb{Z}[G])/\Gamma(I_G)$ is freely generated by the images of these elements. The group G acts on the basis elements by $h.1 = 1 + i_h$ and $h.i_g = i_{hg} - i_h$. Hence

$$2(h.e) = h.2\gamma_{\mathbb{Z}[G]}(1) = h([1 \odot 1]) \equiv [1 \odot 1] + 2i_h \otimes 1 \ mod \ \Gamma(I_G).$$

Since $[1 \odot 1] = 2e$ and $\Gamma(\mathbb{Z}[G])/\Gamma(I_G)$ is torsion-free (as an abelian group),

$$h.e \equiv e + i_h \otimes 1 \ mod \ \Gamma(I_G).$$

We also have

$$h.(i_g \otimes 1) \equiv i_{hg} \otimes 1 - i_g \otimes 1 \ mod \ \Gamma(I_G).$$

Thus the bijection sending i_g to $i_g \otimes 1$ and 1 to e defines an isomorphism $\mathbb{Z}[G] \cong \Gamma(\mathbb{Z}[G])/\Gamma(I_G)$. Hence $\Gamma(\mathbb{Z}[G]) \cong \mathbb{Z}[G] \oplus \Gamma(I_G)$. $\square$

We may strengthen this result as follows.

Lemma 7. *If $\Pi \oplus \mathbb{Z}[\pi]^r \cong \mathbb{Z}[\pi]^s \oplus I_\pi$ then $\Gamma(\Pi)$ is a direct summand of $\Gamma(\mathbb{Z}[\pi]^{s+1})$.*

Proof. Since $\Gamma(\Pi)$ is a direct summand of $\Gamma(\Pi \oplus \mathbb{Z}[\pi]^r)$, it shall suffice to assume that $\Pi \cong \mathbb{Z}[\pi]^s \oplus I_\pi$. We may compare the splittings

$$\Gamma(\mathbb{Z}[\pi]^s \oplus I_\pi) = \Gamma(\mathbb{Z}[\pi]^s) \oplus \Gamma(I_\pi) \oplus (\mathbb{Z}[\pi]^s \otimes I_\pi)$$

and

$$\Gamma(\mathbb{Z}[\pi]^{s+1}) = \Gamma(\mathbb{Z}[\pi]^s) \oplus \Gamma(\mathbb{Z}[\pi]) \oplus (\mathbb{Z}[\pi]^s \otimes \mathbb{Z}[\pi]).$$

If the abelian group underlying a $\mathbb{Z}[\pi]$-module M is free abelian with basis $\{m_i\}$ then the tensor products $M \otimes \mathbb{Z}[\pi]$ and $\mathbb{Z}[\pi] \otimes M$ with the diagonal left $\mathbb{Z}[\pi]$-structures are free $\mathbb{Z}[\pi]$-modules with bases $\{m_i \otimes 1\}$ and $\{1 \otimes m_i\}$, respectively. Hence

$$(\mathbb{Z}[\pi]^s \otimes \mathbb{Z}[\pi]) \oplus \Gamma(\mathbb{Z}[\pi]) \cong (\mathbb{Z}[\pi]^s \otimes I_\pi) \oplus \mathbb{Z}[\pi] \oplus \Gamma(I_\pi) \oplus \mathbb{Z}[\pi],$$

and so $\Gamma(\mathbb{Z}[\pi]^s \oplus I_\pi)$ is a direct summand of $\Gamma(\mathbb{Z}[\pi]^{s+1})$. $\square$

If $[\Pi] = [I]$ then $[\Pi^w] = [I^w]$. Therefore $H_i(\pi; \Pi^w) \cong H_{i+1}(\pi; \mathbb{Z}^w)$ for $i > 0$. Hence $H_2(\pi; \Pi^w) \cong \mathbb{Z}$ and $H_3(\pi; \Pi^w) = 0$.

In the formulation of the next theorem, we have included Π among the relevant invariants, although (as observed earlier) it is determined by the intersection pairing.

Theorem 8. *Let X be a PD_4-complex such that $\pi = \pi_1(X)$ is a PD_3-group and $w_1(X) = w_1(\pi)$. Then the homotopy type of X is determined by π, $\Pi = \pi_2(X)$, $\kappa = k_1(X)$ and λ_X.*

Proof. The homotopy type of a PD_4-complex X is determined by $P_2(X)$ and the image of a fundamental class $[X]$ in $H_4(P_2(X); \mathbb{Z}^w)$. where $w = w_1(X)$ [4, Theorem 3.1]. (This was first proven in [9, Theorem 1.1], assuming also that $\beta_2(X; \mathbb{Q}) > 0$.) The invariants π, Π and κ determine $P_2(X)$. We shall show that λ_X determines the image of $[X]$ in $H_4(P_2(X); \mathbb{Z}^w)$.

The Cartan-Leray spectral sequences for the universal covers give epimorphisms $\delta_X : H_4(X; \mathbb{Z}^w) \to H_2(\pi; \Pi^w)$ and $\delta_P : H_4(P_2(X); \mathbb{Z}^w) \to H_2(\pi; \Pi^w)$, since $c.d.\pi = 3$. Since π has one end δ_X is an isomorphism, while there is an exact sequence

$$0 \to \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} H_4(\Pi, 2; \mathbb{Z}) \xrightarrow{\phi} H_4(P_2(X); \mathbb{Z}^w) \xrightarrow{\delta_P} H_2(\pi; \Pi^w) \to 0.$$

The universal cover of $P_2(X)$ is a $K(\Pi, 2)$-space, and the 'boundary" homomorphism $b : H_4(\Pi, 2; \mathbb{Z}) \cong \Gamma(\Pi)$ of Whitehead is an isomorphism, since $\pi_i(K(\Pi, 2)) = 0$ for $i \neq 2$ (see [3, 1.3.7]). Hence $\psi = \mathbb{Z}^w \otimes_\Gamma b$ is also an isomorphism.

Since $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma(\Pi)$ is a direct summand of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma(\mathbb{Z}[\pi]^{s+1})$, by Lemma 7, and since B_M is an isomorphism if M is a finitely generated projective module [12, Theorem 2], B_Π is a monomorphism.

Let $\theta : H_4(P_2(X); \mathbb{Z}^w) \to Her_w(\Pi^\dagger)$ be the function defined by

$$\theta(\xi)(u, v) = v(u \cap \xi) \quad \forall \, u, v \in H^2(X; \mathbb{Z}[\pi]) \text{ and } \xi \in H_4(P_2(X); \mathbb{Z}^w).$$

Then $\theta\phi = B_\Pi\psi$, and so $\theta\phi$ is also a monomorphism.

Suppose that X_1 is a second such PD_4-complex and $h : P_2(X_1) \to P_2(X)$ is a homotopy equivalence which induces an isometry $f : \lambda_{X_1} \cong \lambda_X$. Then the images of $h_*[X_1]$ and $[X]$ in $H_2(\pi; \Pi^w)$ agree, and so $h_*[X_1] - [X]$ is in the image of ϕ. Since h induces an isometry $\lambda_{X_1} \cong \lambda_X$ and since $\theta\phi$ is a monomorphism, it follows that $h_*[X_1] = [X]$. Hence $X_1 \simeq X$ [4, Theorem 3.1]. $\qquad\square$

Corollary 9. *If M is a closed 4-manifold, $\pi = \pi_1(M)$ is a solvable PD_3-group and $w_1(M) = w_1(\pi)$ then the homeomorphism type of M is determined by λ_M.*

Proof. Solvable PD_n-groups are torsion free and polycyclic, and satisfy the Farrell-Jones conjectures [7, 8]. Since such groups π are "good", standard surgery techniques apply in this 4-dimensional setting. $\qquad\square$

With present knowledge, these are the only known good PD_3-groups.

Theorem 8 in conjunction with the Farrell-Jones conjectures for 3-manifold groups [2, Corollary 1.3] suggests an alternative route to some of the results of [14] on the stable homeomorphism classification of closed orientable 4-manifolds with COAT fundamental groups, using stable 4-dimensional surgery [5] rather than the modified surgery of Kreck.

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