Assignment 2

Your solutions should be submitted by the beginning of the lecture on Wednesday, 2 September 2009.

Q1 Let X, Y and Z be normed spaces over the same field, and $S \in \mathfrak{B}(X,Y)$, $T \in \mathfrak{B}(Y,Z)$. Show that $(TS)^* = S^*T^*$.

Q2 Let Y be a subspace of the finite-dimensional normed space X. Show that $(Y^\perp)^\perp$ is isometric to Y.

Q3 Let $\{x_1, \ldots, x_n\}$ be a linearly independent set in the normed space X. Show that there exists a linearly independent set $\{f_1, \ldots, f_n\}$ in X^* such that $f_i(x_j) = \delta_{ij}$ and for each $x \in \operatorname{span}\{x_1, \ldots, x_n\}$,

$$x = \sum_{i=1}^{n} f_i(x) x_i.$$

Q4 Let Y be a subspace of the normed space X and $x \in X$. Show that $\operatorname{dist}(x, Y) \geq 1$ if and only if there exists a bounded linear functional $f \in B(X^*)$ such that $Y \subseteq \ker f$ and $f(x) = 1$.

Q5 Let V be the vector space of all scalar sequences $x = (x_k)_{k=1}^{\infty}$ with at most finitely many non-zero terms. For $1 \leq p \leq \infty$, let $X_p = (V, || \cdot ||_p)$, where

$$||x||_p = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} \qquad (1 \leq p < \infty)$$

and

$$||x||_\infty = \max\{|x_k| \mid 1 \leq k \leq \infty\}.$$

You may assume that each space X_p is a normed space.

For $1 \leq r, s \leq \infty$, let $T_{r,s}\colon X_r \to X_s$ be the formal identity map $T_{r,s}x = x$. Then $T_{r,s}$ is a linear operator.

(a) Show that if $r \leq s$, then $T_{r,s}$ is bounded and $||T_{r,s}|| = 1$.

(b) Show that if $r > s$, then $T_{r,s}$ is unbounded.