Lecture 29: Integration on surfaces

Answers

Let S denote the torus obtained by revolving the circe of radius b centered at distance a from the origin around the z axis. From assignments 3 and 4, you know that S has a parameterisation given by

$$\Phi(u,v) = \Big((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u\Big).$$

You have already computed the derivatives of the parameterisation:

$$\Phi_u = \Big(-b\sin u\cos v, -b\sin u\sin v, b\cos u\Big), \quad \text{and} \quad \Phi_v = \Big(-(a+b\cos u)\sin v, (a+b\cos u)\cos v, 0\Big),$$

as well as the (matrix of the) first fundamental form:

$$Q_I = \begin{pmatrix} \Phi_u\cdot\Phi_u & \Phi_u\cdot\Phi_v \\ \Phi_u\cdot\Phi_v & \Phi_v\cdot\Phi_u \end{pmatrix} = \begin{pmatrix} b^2 & 0 \\ 0 & (a+b\cos u)^2 \end{pmatrix}.$$

One computes:

$$\Phi_{uu} = \Big(-b\cos u\cos v, -b\cos u\sin v, -b\sin u\Big)$$
$$\Phi_{uv} = \Big(b\sin u\sin v, -b\sin u\cos v, 0\Big)$$
$$\Phi_{vv} = \Big(-(a+b\cos u\cos v, -(a+b\cos u)\sin v, 0\Big)$$

$$||\Phi_u\wedge\Phi_v|| = b(a+b\cos u)$$

$$N = \frac{\Phi_u\wedge\Phi_v}{||\Phi_u\wedge\Phi_v||} = \Big(\cos u\cos v, \cos u\sin v, \sin u\Big)$$

The (matrix of the) second fundamental form (with respect to Φ) is:

$$Q_{II} = \begin{pmatrix} \Phi_{uu}\cdot N & \Phi_{uv}\cdot N \\ \Phi_{uv}\cdot N & \Phi_{vv}\cdot N \end{pmatrix} = \begin{pmatrix} -b & 0 \\ 0 & -\cos u(a+b\cos u) \end{pmatrix}.$$

(1) The Gaussian curvature of S is given by the formula

$$K(u,v) = \frac{\cos u}{b(a+b\cos u)}.$$

(2) The "top" and "bottom" circles are precisely the areas with Gaussian curvature zero, and the "half facing the z–axis" has negative Gaussian curvature, and the "half facing outside" has positive Gaussian curvature.

(3) The Gaussian curvature of S is invariant under all isometries of $\mathbb{R}^3$ that preserve S. Hence it is rotationally symmetric, as well as symmetric with respect to reflection in the xy–plane and reflection in any plane containing the z-axis.

(4) The torus can be written as an admissible union of the following four regions:

$$R_1 = \{(u,v) \mid 0 \le u \le \pi,\ 0 \le v \le \pi\}$$
$$R_2 = \{(u,v) \mid 0 \le u \le \pi,\ \pi \le v \le 2\pi\}$$
$$R_3 = \{(u,v) \mid \pi \le u \le 2\pi,\ 0 \le v \le \pi\}$$
$$R_4 = \{(u,v) \mid \pi \le u \le 2\pi,\ \pi \le v \le 2\pi\}$$

(5) Using the above four regions and additivity of the integral, the area of S is

$$\int_S dS = \int_{R_1} dS + \int_{R_2} dS + \int_{R_3} dS + \int_{R_4} dS$$

$$= \int_0^{2\pi} \int_0^{2\pi} b(a + b\cos u)dudv$$

$$= (2\pi a)(2\pi b)$$

$$= (\text{length of core circle}) \times (\text{length of cross section}).$$

(6) The integral of Gaussian curvature is:

$$\int_S KdS = \int_{R_1} K\ dS + \int_{R_2} K\ dS + \int_{R_3} K\ dS + \int_{R_4} K\ dS$$

$$= \int_0^{2\pi} \int_0^{2\pi} \cos u\ dudv$$

$$= 0.$$