Problem Set 6

Q42 Let X be a topological space.

(a) Show that $Y \subseteq X$ is dense if and only if for each non-empty, open subset $U \subseteq X$: $Y \cap U \neq \emptyset$.

(b) Show that the intersection of finitely many open and dense sets in X is open and dense in X.

(c) Show that $Y \subseteq X$ is open and dense if and only if its complement is closed and has empty interior.

Q43 Show that Version 2 of the Baire Category Theorem implies Version 1.

Q44 Let X be a topological space and $Y \subseteq X$. Show that the following are equivalent:

(a) Y is nowhere dense in X.

(b) $X \setminus \overline{Y}$ is open and dense in X.

(c) $X \setminus Y$ contains an open and dense set.

(d) $X \setminus \overline{(X \setminus \overline{Y})} = \emptyset$

(e) $Y \subset \overline{(X \setminus \overline{Y})}$

Q45 Show that the union of finitely many nowhere dense sets is nowhere dense.

Q46 (a) Let X be a normed space and $S \subseteq X$. Show that if $\{f(x) \mid x \in S\}$ is bounded for each $f \in X^*$, then S is bounded.

(b) Use the previous part to show that if two norms on a vector space V are not equivalent, then there is a linear functional on V which is continuous with respect to one of the norms and discontinuous with respect to the other.

Q47 [*How does l_p sit in l_q?*]
Let $1 \leq p < q \leq \infty$. Define $T_{p,q}\colon l_p \to l_q$ by $T_{p,q}(x) = x$, and let $X_p = \mathrm{Im}(T_{p,q}) \subseteq l_q$ and $B_p = T_{p,q}(B(l_p)) \subseteq l_q$.

(a) Show that $T_{p,q} \in \mathfrak{B}(l_p, l_q)$ with $||T_{p,q}|| = 1$.

(b) Show that B_p is closed in l_q and has empty interior.

(c) Show that X_p is meagre in l_q.

(d) Conclude that

$$\{x \in l_q \mid \sum_{i=1}^{\infty} |x_i|^p = \infty\}$$

is dense in l_q.

(e) Show that X_p as a normed space with the induced norm from l_q is not complete.