Problem Set 5

Q45 Show that the function $\mathbb{R} \to [-1, 1]$ defined by $x \mapsto \sin(x)$ is open but not closed with respect to the usual topologies on the spaces involved.

Q46 Find more examples of functions that are open but not closed. Similarly, find more examples of functions that are closed but not open.

Q47 Suppose $f : Y \to X_1 \times \ldots \times X_n$ is a function, where Y and each X_k are topological spaces and the range is given the product topology. Show that f is continuous if and only if each coordinate function $p_k \circ f : Y \to X_k$ is continuous.

Q48 Let $f : X \to Y$ be a function of topological spaces and give $X \times Y$ the product topology. Let $G(f) = \{(x, f(x)) \mid x \in X\}$ be the graph of f. Show that f is continuous if and only if the function $p_X|_{G(f)} : G(f) \to X$ is a homeomorphism, where $G(f)$ is given the subspace topology and $p_X : X \times Y \to X$ is the coordinate projection onto X.

Q49 Let X be a set. Given any collection $\mathcal{S} \subseteq \mathcal{P}(X)$, show that

$$\{U_1 \cap \ldots U_n \mid U_k \in \mathcal{S}, n \in \mathbb{N}\} \cup \{X\}$$

is a basis for a topology on X.

Q50 The following is an additional example to Question 3 on Assignment 3. Consider:

$$G = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Q} \setminus \{0\}\}.$$

Show that $\overline{X}$ is not T_0, but it is uncountable, there are precisely six open subsets of $\overline{X}$ and there are points in $\overline{X}$ which are dense.

Discussion: The sets G consist of homeomorphisms of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{E}})$, and each set has the structure of a group. The quotient space is often denoted $\mathbb{R}^2/G$, and one says that G acts on $\mathbb{R}^2$. Such group actions on spaces occur naturally in many different contexts, and the examples show that the topological properties of the quotient spaces depend on the group.

Q51 Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces and $y_0 \in Y$. Show that $(X, \mathcal{O}_X)$ is homeomorphic with $X \times \{y_0\}$, where the latter is given the subspace topology from the product topology on $X \times Y$.

Q52 Let $(X_\alpha, \mathcal{O}_\alpha)$ be a connected topological space for each $\alpha \in I$, where I is an arbitrary index set. Show that the product space

$$(\prod_{\alpha \in I} X_\alpha, \mathcal{O}_{prod})$$

is also connected.

Hint: Given $x \in X = \prod_{\alpha \in I} X_\alpha$, consider the set

$$S_x = \{y \in X \mid p_\alpha(y) = p_\alpha(x) \text{ for all but finitely many } \alpha\}.$$

If Y is the component of X containing x, show that $S_x \subseteq Y$. Also show that $\overline{S_x} = X$.