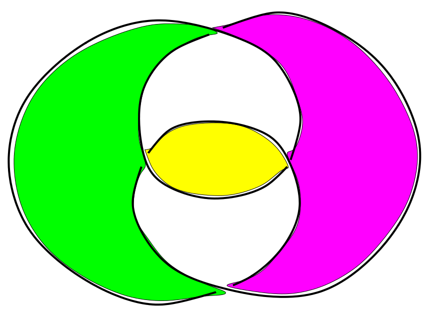
Varieties associated to 3–manifolds:

Finding Hyperbolic Structures of and Surfaces in 3–Manifolds



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CONTENTS

These notes were compiled for a short course given at the University of Melbourne, 5-8 February 2001, and much of the material has been taken from or is based on the following references:

(1) W.D. Neumann: *Notes on Geometry and 3–Manifolds* [23].
(2) G. Baumslag: *Topics in Combinatorial Group Theory* [2].
(3) P.B. Shalen: *Representations of 3–manifold groups* [27].
(4) D. Cooper, M. Culler, H. Gillett, D.D. Long, P.B. Shalen: *Plane curves associated to character varieties of 3–manifolds* [7].

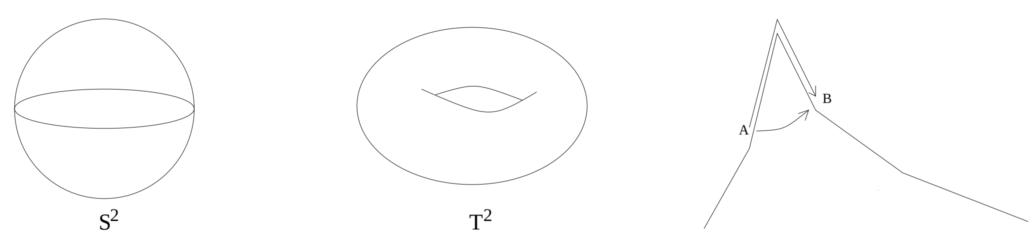
This compilation is intended as a hands–on guide to the computation of character varieties and A–polynomials, as well as an exposition to the construction by Culler and Shalen which associates essential surfaces to ideal points of curves in the aforementioned algebraic varieties. It is also described how one can decide whether a connected essential surface is associated to an ideal point, thus providing a tool useful in the study of examples.

1. Geometry and Surfaces

Seen locally, a manifold of dimension n looks like $\mathbb{R}^n$. The surface of the moon may be thought of as a sphere, which is a 2–dimensional manifold denoted by S^2. Similarly, the surface of a doughnut is a 2–manifold called torus T^2. One 3–manifold is our universe. But which one?

In these notes, a 2–manifold is often called a surface and a 3–manifold simply a manifold. Sometimes our manifolds have boundary, i.e. there are points whose neighbourhood looks like a (closed) half space $\mathbb{R}^n_+$.

Reference. [23], *Chapter 1.*



Sphere and torus — Non–uniform Geometry

Figure 1

1.1. Geometry, Curvature and Orientability.

Manifolds can have a geometry, which is often very uniform. To picture a non–uniform geometry, think of the peak of a mountain, where going "straight" over the top might be a longer way between two points than going "around" the peak. In our mathematical models, we consider completely uniform geometries. In dimension two, there are three possibilities:

- $\mathbb{S}^2$: spherical geometry (such as the geometry on the smooth round sphere of unit radius) of curvature $K = +1$,
- $\mathbb{E}^2$: flat or Euclidean geometry of curvature $K = 0$,
- $\mathbb{H}^2$: hyperbolic geometry of curvature $K = -1$.

The torus T^2 has a flat geometry, even though we cannot realise it in $\mathbb{R}^3$. It can be realised abstractly by thinking of quadrilaterals with face identifications:

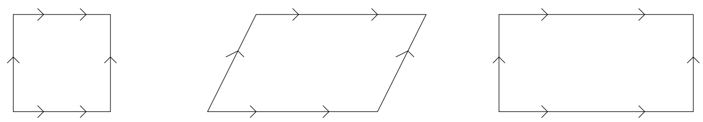


Figure 2

These three pictures give different flat geometries on the torus, for example we find closed geodesics of different lengths. Geometry and curvature are intrinsic properties of a manifold.

The average curvature in a triangle is given by the formula $\frac{\alpha+\beta+\gamma-\pi}{Area\Delta}$, where α, β and γ are the angles of the triangle Δ. The torus as realised in $\mathbb{R}^3$ therefore has areas of positive curvature and areas of negative curvature. We like geometries on manifolds to be locally homogeneous, so they have the same curvature everywhere.

Closed orientable surfaces have natural uniform geometries, where the sphere and the torus appear as special cases:

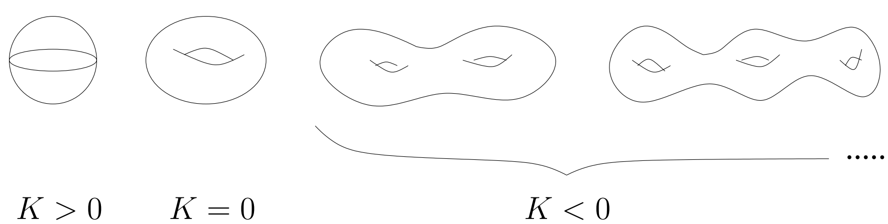


Figure 3

Another important intrinsic property of manifolds is orientability. An example of a non–orientable 2–manifold is the Möbius strip. A 2–dimensional creature pointing to the boundary component returns "other–handed" after travelling once around the strip.

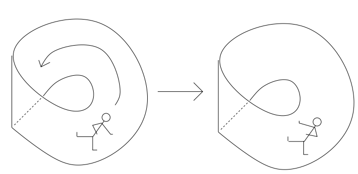


Figure 4

We will confine ourselves to orientable manifolds. The orientable surfaces without boundary are precisely the ones in Figure 3.

1.2. **Classifying manifolds.** How can we tell surfaces apart? How can we tell manifolds apart? These questions lead to the invention of invariants, quantities which can help to classify manifolds. For surfaces there is a convenient invariant: subdivide a surface S into polygons, and let V be the number of vertices, E be the number of edges, and F be the number of faces in the subdivision, then

$$V - E + F = \chi(S)$$

is the so–called *Euler characteristic* of S. The Euler characteristic of a sphere is 2, that of a torus is 0, that of a surface with "two holes" is -2, and in general that of a surface with "g holes" is $2-2g$. The number g is called the genus of the surface. Given the above classification, the Euler characteristic is a perfect invariant for classifying closed orientable surfaces.

In a manifold of constant curvature K, we have (Area of S) $\times K = 2\pi\chi(S)$. In a non–uniform geometry, the Gauß–Bonnet formula

$$\int_S K dA = 2\pi\chi(S)$$

has to be applied, which explains why the torus embedded in $\mathbb{R}^3$ has places of positive and negative curvature.

In the case of 3–manifolds, we don't know how to classify them (yet). One approach is to look at essential surfaces in a 3–manifold, another is to use geometry. The varieties referred to in the title provide a connection between these two methods. Most 3–manifold topologists believe the following fact:

Thurston's Geometrisation Conjecture. *If you can't cut a 3–manifold along essential two–spheres or tori, then it has a uniform geometry.*

The essential spheres and tori seem to cause trouble. We will shortly explain what to do with them after cutting.

There are eight geometries in dimension 3: $\mathbb{S}^3$, $\mathbb{E}^3$ and $\mathbb{H}^3$ look the same in every direction; one direction is singled out in the geometries $\mathbb{S}^2 \times \mathbb{E}$ and $\mathbb{H}^2 \times \mathbb{E}$ (an "up–and–down" direction if you like); and there are the twisted geometries *Nil*, *Sol* and *PSL*. Seven of these geometries can be studied in terms of surfaces and are hence classified in terms of 2–manifolds, which was mostly done in the 1930s. The eighth is hyperbolic geometry.

1.3. **Hyperbolic Geometry.** How can we understand hyperbolic geometry? We change from the geometry to a mathematical model. An example is the upper half space model:

$$\mathbb{C} \times \mathbb{R}_+ = \{(z, r) \mid z = x + iy, r > 0\},$$

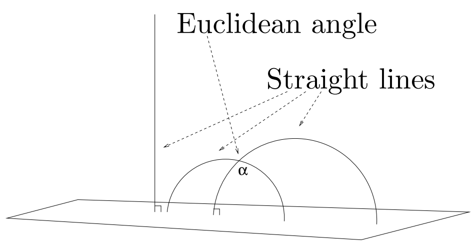
where a measure for arc length is given by taking the Euclidean measure

$$ds = \sqrt{dx^2 + dy^2 + dr^2}$$

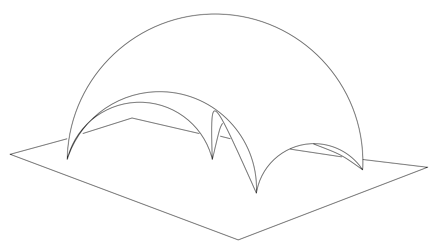
and dividing by the height:

$$ds_{hyp} = \frac{1}{r} ds = \frac{1}{r}\sqrt{dx^2 + dy^2 + dr^2}.$$

Hyperbolic planes are hemispheres, and we have a concept of points at infinity, which lie on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. A nice feature of hyperbolic geometry is that infinite objects can have finite volume, such as ideal tetrahedra.



Upper half space model



Ideal tetrahedron

FIGURE 5

Consider again the flat torus. A 2–dimensional creature living on the torus (with a particular flat geometry) must feel like living in the plane where things are copied and move in parallel lines as shown in Figure 6. The group of motions Γ takes a *fundamental domain* into another copy, and is a subgroup of $Isom(\mathbb{E}^2)$. The given torus is obtained as $T^2 = \mathbb{E}^2/\Gamma$. This picture works for any geometry and dimension:

Definition 1. [23] *A geometric manifold (or manifold with a geometric structure) is a manifold of finite volume of the form $\mathbb{X}/\Gamma$, where $\mathbb{X}$ is a geometry and Γ is a discrete subgroup of the isometry group $Isom(\mathbb{X})$.*

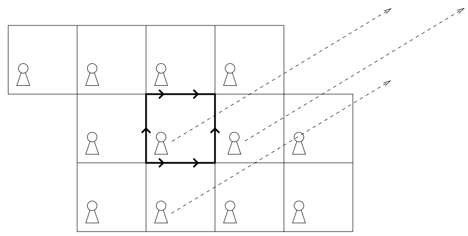


FIGURE 6

When we are interested in orientable hyperbolic 3–manifolds, we are interested in $M = \mathbb{H}^3/\Gamma$, where Γ is a discrete subgroup of

$$Isom^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1\}/\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

An element in the orientation preserving isometry group is determined by how points at infinity get moved:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

A hyperbolic structure on a 3–manifold M is a complete Riemannian metric of constant sectional curvature -1. If M admits such a structure, its universal cover is isometrically identified with hyperbolic 3–space $\mathbb{H}^3$. If M is orientable, the action of $\pi_1(M)$ by deck transformations on $\mathbb{H}^3$ defines a representation of $\pi_1(M)$ into $Isom^+\mathbb{H}^3 \cong PSL_2(\mathbb{C})$.

1.4. **Decomposing manifolds.** Following Thurston's Geometrisation Conjecture, we cut a given manifold along essential spheres and tori. We then take a connected component which results from this process, fill in the spheres with balls and add "tori at infinity" to obtain the so–called ends of the manifold.

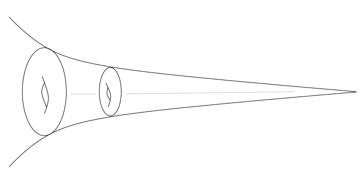


FIGURE 7. End of a hyperbolic manifold

Adding "tori at infinity" is the same as taking the interior of a manifold with boundary. Hyperbolic manifolds have no boundary; however, we may reverse the above process to obtain compact cores (which have isomorphic fundamental groups) and may talk about the boundary thereof.

This gives nice examples, e.g. knot complements, where we think of the knot as embedded in the 3–sphere S^3 rather than $\mathbb{R}^3$. The figure eight knot $\mathfrak{k}$ is shown in Figure 8. Its complement $S^3 - \mathfrak{k}$ admits a hyperbolic structure, and so in fact do "most" knot complements. Associated to the complete hyperbolic structure, we have a discrete subgroup Γ of $PSL_2(\mathbb{C})$. It is a convenient fact that we may think of Γ as a subgroup of $SL_2(\mathbb{C})$.

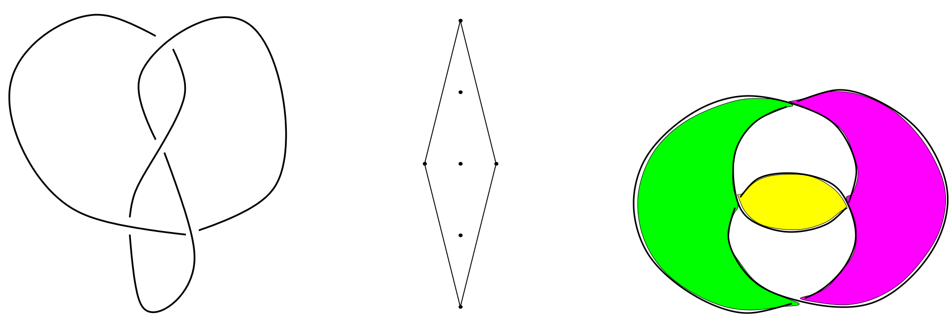


Figure eight knot　　Newton polygon　　Surface with slope ± 4

FIGURE 8. The figure eight knot complement $\mathcal{F}$

1.5. **Deforming the geometry detects surfaces.** When we cut off the cusp from the manifold, we may think of this as cutting along an Euclidean plane in the upper half space model, and see the two motions associated to the boundary torus. This gives $\lambda, \mu \in SL_2(\mathbb{C})$, and we can take the eigenvalues l and m corresponding to a common invariant subspace, since the group of motions on the torus is the abelian group $\mathbb{Z} \oplus \mathbb{Z}$.

Complete geometries are rigid, but we can deform them into non–complete geometries. The difference is that in a complete geometry, geodesics go to infinity, whilst in a non–complete geometry, they may fall over the edge. We can think of this as (in)completeness of a metric space with respect to the metric given by the geometry.

Deforming the geometry gives a complex curve of values for $(l, m) \in (\mathbb{C} - \{0\})^2$. We call the "simplest" polynomial defining this curve the A–polynomial of the manifold. This is a very powerful invariant, even though it is not yet fully understood. The A–polynomial is related to the geometry of the manifold and to essential surfaces with boundary contained in the manifold, thus providing one of the links between geometry and topology we wish to talk about.

The A–polynomial of the figure eight knot complement $\mathcal{F}$ for example is given by the equation

$$A(l, m) = -lm^8 + lm^6 + l^2m^4 + 2lm^4 + m^4 + lm^2 - l,$$

where the variables l and m correspond to eigenvalues of the standard longitude $\mathcal{L}$ and meridian $\mathcal{M}$ of the knot. As m and l are units, we can also write the above polynomial as

$$A(l, m) = -m^4 + m^2 + l + 2 + l^{-1} + m^{-2} - m^{-4}.$$

The Newton polygon of A is the convex hull of the set of monomial exponents of A in the plane. In our example, edges of this polygon have slopes ± 4. Incidentally, there are essential surfaces in the manifold $S^3 - \mathfrak{k}$ with boundary curves isotopic to $\mathcal{M}^{\pm 4}\mathcal{L}$ (see Figure 8). With respect to the fixed basis $\{\mathcal{M}, \mathcal{L}\}$, we say that these surfaces have boundary slopes ± 4. The connection is stated in the following:

Theorem. [7] *The slopes of edges of the Newton polygon of the A–polynomial are the boundary slopes of essential surfaces in the knot complement.*

An obvious surface in the figure eight knot complement, the Seifert surface with boundary curve isotopic to the longitude, is not detected by this Newton polygon. This is due to the fact that we have only considered representations into $SL_2(\mathbb{C})$ with certain geometric meaning. If we consider all representations, we see the slope 0 arising from a factor $l - 1$ in the resulting A–polynomial.

1.6. **Outline.** The relationship between "representation curves" and essential surfaces is explained in Sections 2-5. Both essential surfaces in M and hyperbolic structures on M are related to actions of $\pi_1(M)$ on trees. In Section 2, a surface S in M gives rise to a (canonical) action of $\pi_1(M)$ on a (canonical) dual tree $\mathcal{T}_S$. This process is somewhat reversed in Section 3, where we start with an action of

$\pi_1(M)$ on a tree $\mathcal{T}$ and then construct an associated surface S in M. However, an associated surface S is not canonical, and we will compare the action on $\mathcal{T}$ which we started from to the action on the dual tree $\mathcal{T}_S$. This leads to some first applications, and we will also characterise surfaces that can be associated to a given action.

The representation and character varieties are defined and explored a little in Section 4, and the construction by Culler and Shalen is completed in Section 5 by associating an action on a tree to an ideal point of a representation curve.

In the remaining five sections, various applications are given. We first prove the Weak Neuwirth Conjecture in Section 6, whose proof leads to information about boundary slopes of associated surfaces, and hence to the above mentioned "boundary slopes" theorem in Section 7, where we define the A–polynomial. The material on roots of the Alexander polynomial and eigenvalues of metabelian representations in Section 8 does not use Culler–Shalen theory, and can readily be understood with little knowledge about representations. However, it fits into the general theme of trying to give representations geometric meaning. In the last two sections, more information about associated surfaces is gained, which leads to the roots of unity phenomenon and necessary and sufficient conditions on connected associated surfaces.

Acknowledgements. The introduction is inspired by a lecture Walter Neumann gave at Columbia University on 5 June 2000. The material on Culler–Shalen theory is mostly taken from Shalen [27]. Some of the original material contained in these notes now forms part of [32, 33].

Much of the contents had been covered in a working seminar at Columbia University, in which Abhijit Champanerkar, Brian Mangum, Walter Neumann and Stephan Tillmann lectured. I would like to thank these co–lecturers, as well as Radu Popescu and the other participants of this seminar. Furthermore, I would like to thank Craig Hodgson for helpful conversations.

2. From Surfaces to Actions on Trees

This section reviews some material on groups acting on trees, defines essential surfaces in 3–manifolds, and gives some flavour of the interplay between topology and algebra.

Reference. [27]*, Section 1.*

2.1. **Group actions.** An *action* of a group Γ on a set X is a function $\cdot : \Gamma \times X \to X$ that satisfies $1 \cdot x = x$ and $(\gamma\delta) \cdot x = \gamma \cdot (\delta \cdot x)$ for all $x \in X$ and all elements $\gamma, \delta \in \Gamma$. An action gives a partition of the set X into *orbits*. Two elements $x, y \in X$ are contained in the same orbit if and only if there is an element $\gamma \in \Gamma$ such that $\gamma \cdot x = y$. We denote the orbit of x by $\Gamma \cdot x$. The *stabilizer* Γ_x of $x \in X$ is the collection of all elements in Γ such that $\gamma \cdot x = x$.

Exercise 1. *Show that the stabilizer of $x \in X$ is a subgroup of Γ, and that for each $x \in X$ the map $\Gamma \to X$ defined by $\gamma \to \gamma \cdot x$ induces a bijection between the set of cosets $\gamma\Gamma_x$ and the orbit $\Gamma \cdot x$ of x.*

We call an action *free* if the stabiliser of every element of X is the trivial subgroup of X. A subset $S \subseteq X$ is *invariant* under the action if $\gamma \cdot s \in S$ for each $s \in S$ and for each $\gamma \in \Gamma$. That is, S is a union of orbits. An element is *fixed* by Γ if $\{x\}$ is invariant.

Suppose the group Γ acts on two sets X and Y in the ways $\cdot$ and $\bullet$ respectively. A map $f : X \to Y$ is said to be Γ–*equivariant* if

$$f(\gamma \cdot x) = \gamma \bullet f(x) \quad \forall x \in X \quad \forall \gamma \in \Gamma.$$

An action that will appear in the sequel is the action of the fundamental group of a 3–manifold M on its universal covering space $(\tilde{M}, p)$ by *deck transformations*. By the uniqueness of the universal cover, we may suppress base points, since even though $\pi_1(M, x)$ operates on $p^{-1}(x)$, we have $\pi_1(M, x) \cong Aut(\tilde{M}, p)$, which is defined up to equivalence. If M is given a triangulation, then $\tilde{M}$ inherits a triangulation such that the action of $\pi_1(M)$ on $\tilde{M}$ is simplicial. This will be important later, and we will not distinguish between the simplicial complex and its topological realisation.

Exercise 2. *Starting with a triangulation of M, give $\tilde{M}$ an induced triangulation such that the covering projection is a simplicial map.*

2.2. **Graphs and trees.** A *graph* $\mathcal{G}$ consists of a non–empty set of vertices $\mathcal{V} = \mathcal{V}(\mathcal{G})$ and a set of edges $\mathcal{E} = \mathcal{E}(\mathcal{G})$, together with the three maps

$$i : \mathcal{E} \to \mathcal{V}, \quad t : \mathcal{E} \to \mathcal{V}, \quad {}^- : \mathcal{E} \to \mathcal{E},$$

subject to the condition that if $e \in \mathcal{E}$, then

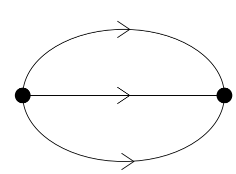
$$e \neq \overline{e} \quad \text{and} \quad \overline{\overline{e}} = e.$$

Thus, the third map is a fixed point free involution on the set of edges, and corresponds to reversal of orientation. The second condition implies that $t(e) = i(\overline{e})$. We call $\overline{e}$ the *inverse* of e, and $t(e)$ the *terminal* and $i(e)$ the *initial* vertex. They are the extremities of an edge, and we call two vertices adjacent if they are the extremities of some edge. If $i(e) = t(e)$, then e is called a *loop*.

When we draw diagrams of graphs, we omit one of e and its inverse, and indicate orientation by an arrow pointing towards the terminus. The simplest example of a graph is a point.

A group Γ acts on a graph if it comes equipped with a homomorphism $\varphi : \Gamma \to Aut(\mathcal{G})$, where $Aut(\mathcal{G})$ consists of invertible morphisms $\mathcal{G} \to \mathcal{G}$ with composition as the binary operation. We denote the image of a vertex v under the action of $\gamma \in \Gamma$ by $\gamma \cdot v$. A group acts on a graph without inversions if $\gamma \cdot e \neq \overline{e}$ for all $e \in \mathcal{E}$ and for all $\gamma \in \Gamma$. We say that the action is trivial if a vertex of $\mathcal{G}$ is fixed by Γ.

Exercise 3. *What is the automorphism group of the following graph?*



Example. *A group acts on its Cayley graph by left multiplication. To construct the Cayley graph, take Γ as the set of vertices, and for a fixed generating set S we get the set of edges from the disjoint union of $\Gamma \times S$ and $S \times \Gamma$. The three functions are defined by $i(\gamma, s) = \gamma$, $t(\gamma, s) = \gamma s$, $\overline{(\gamma, s)} = (s, \gamma)$ and $\overline{(s, \gamma)} = (\gamma, s)$.*

Exercise 4. *What is the Cayley graph of the trivial group, a finite cyclic group, an infinite cyclic group?*

A *path of length* n in $\mathcal{G}$ is a sequence $e_1, \ldots, e_n$ of edges such that $i(e_i) = t(e_{i-1})$ for $i = 2, \ldots, n$. The vertices $i(e_1)$ and $t(e_n)$ are the extremities of the path. A path is closed if $i(e_1) = t(e_n)$. A pair e_i, e_{i+1} is termed a *backtracking* if $e_{i+1} = \overline{e_i}$. A graph $\mathcal{G}$ is connected if any two vertices are the extremities of some path, and it is a tree if it is connected and every closed path of positive length in $\mathcal{G}$ contains a backtracking.

Exercise 5. (1) *A group acts on its Cayley graph without inversions.*
(2) *The Cayley graph is connected.*
(3) *The Cayley graph contains a loop if and only if* $1 \in S$.

If a group Γ acts without inversions on a graph $\mathcal{G}$, the orbit space $\mathcal{G}/\Gamma$ inherits the structure of a graph as follows. Denote the orbits of edges and vertices by $[\cdot]$. We have edges $[e] = \Gamma \cdot e$, and now define the functions by

$$i[e] = [i(e)], \qquad t[e] = [t(e)], \qquad \overline{[e]} = [\overline{e}].$$

Note that the inversion function is fixed point free and of order two since Γ acts without inversions. Then the quotient map $\mathcal{G} \to \mathcal{G}/\Gamma$ defined by $e \to [e]$ and $v \to [v]$ is a morphism of graphs.

Let Γ be a group acting on a tree $\mathcal{T}$. The *translation length* of an element is defined by

$$\ell(\gamma) = \min\{d(v, \gamma \cdot v) \mid v \in \mathcal{T}\}.$$

Group elements which fix a vertex have translation length equal to zero, and are called *elliptic*. The set of fixed points of an elliptic element is a subtree of $\mathcal{T}$. An element with positive translation length is called *loxodromic*. Loxodromic elements act by translations along a unique line, which is called their *axis*, and denoted by $A(\gamma)$. The *invariant set* of an element γ is understood to be its fixed set if it is elliptic and its axis if it is loxodromic.

Lemma 2. [25] *Let* Γ *be a finitely generated group which acts simplicially on a tree. Let the generators of* Γ *be* $\gamma_1, \ldots, \gamma_m$, *and assume that* γ_i *and* $\gamma_i\gamma_j$ *have fixed points for all* i, j. *Then* Γ *fixes a vertex.*

Exercise 6. *Prove the lemma!*

2.3. **Essential surfaces.** A *surface* S in a compact 3–manifold M will always mean a 2–dimensional PL submanifold *properly embedded* in M, that is, a closed subset of M with $\partial S = S \cap \partial M$. If M is not compact, we replace it by a compact core. An embedded sphere S^2 in a 3–manifold M is called *incompressible* if it does not bound an embedded ball in M, and a 3–manifold is *irreducible* if it contains no incompressible 2–spheres.

An orientable surface S without 2–sphere or disc components in an orientable 3–manifold M is called *irreducible* if for each disc $D \subset M$ with $D \cap S = \partial D$ there is a disc $D' \subset S$ with $\partial D' = \partial D$. We will also use the following definition:

Definition 3. [27] *A surface S in a compact, irreducible, orientable 3–manifold is said to be essential if it has the following five properties:*

(1) *S is bicollared;*
(2) *the inclusion homomorphism $\pi_1(S_i) \to \pi_1(M)$ is injective for every component S_i of S;*
(3) *no component of S is a 2–sphere;*
(4) *no component of S is boundary parallel;*
(5) *S is non-empty.*

A *bicollaring* of a surface S in M is a homeomorphism $h : S \times [-1, 1] \to M$ such that $h(x, 0) = x$ for every $x \in S$ and

$$h(\partial S \times [-1, 1]) = \partial M \cap h(S \times [-1, 1]).$$

The existence of a bicollaring implies that the surface is 2–sided, i.e. that S separates any sufficiently small neighbourhood of itself in M. For surfaces in orientable manifolds, 2–sidedness is equivalent to orientability.

The second condition in the above definition is equivalent to saying that there are no compression discs for the surface (cf. [19], Lemma 6.1). A compression disc is an embedded disc in M such that its boundary lies on S and is not contractible on S. If there is such a disc, then the second property clearly fails. Conversely, if the second property fails, then there is a non–trivial simple closed curve on S which is contractible in M. Using the bicollaring, we obtain an embedded annulus which bounds a possibly immersed disc in M. Dehn's lemma yields that there is an embedded disc with the same boundary curve.

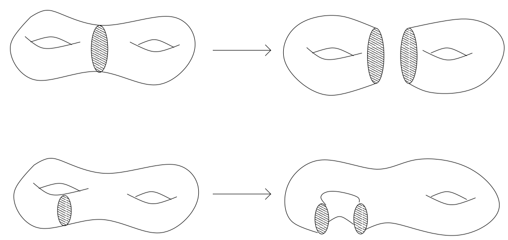


FIGURE 9. Compressions of a surface

If a surface admits a compression disc, we may cut the surface along the boundary of this disc and close the resulting "holes" in our surface with two discs. We call this process a compression of the surface. Two compressions of a genus two surface are shown in Figure 9 – the separating compression results in two tori, whilst the non–separating yields a single torus. The resulting surface tends to be simpler in a sense which we will formalise later.

2.4. **The dual graph of a surface.** If S is an orientable (not necessarily connected) surface in a connected orientable 3–manifold M, we can define the dual graph $\mathcal{G}_S$ of S. The vertices of $\mathcal{G}_S$ are in bijective correspondence to the components M_i of $M - S$, and the edges with the components S_i of S. A vertex v is incident to an edge e if and only if the corresponding component of S is contained in the closure of the component of $M - S$ corresponding to v.

We can make this precise by considering a bicollaring $h : S \times [-1, 1] \to M$, which gives a directed graph structure preserved by the $\pi_1(M)$–action. But the following picture may also suffice, which illustrates that $\mathcal{G}_S$ is a retract of M.

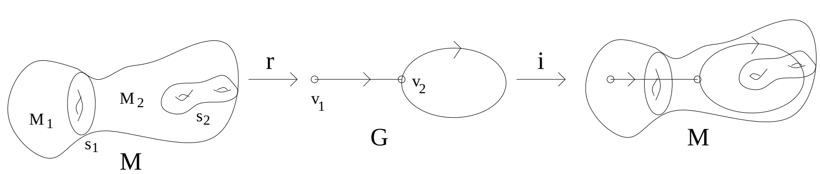


FIGURE 10

The maps $r : M \to \mathcal{G}$ and $i : \mathcal{G} \to M$ are such that $i \circ r$ is homotopy equivalent to the identity of $\mathcal{G}$. So $\pi_1(\mathcal{G})$ is isomorphic to a subgroup and a quotient of $\pi_1(M)$.

Thus, if we consider the universal cover $(\tilde{M}, p)$ of M, and let $\tilde{S} := p^{-1}(S)$, then the dual graph $\mathcal{T}_S$ of $\tilde{S}$ is a tree.

The action of $\pi_1(M)$ on $\tilde{M}$ gives rise to a simplicial action on $\mathcal{T}_S$ as follows. The construction of the dual graph gives us a map $\Phi : \tilde{M} \to \mathcal{T}_S$, the retraction, and we get an action on $\mathcal{T}_S$ by

$$\gamma \cdot \Phi(\tilde{m}) = \Phi(\gamma \cdot \tilde{m}) \quad \forall \gamma \in \pi_1(M) \quad \forall \tilde{m} \in \tilde{M}.$$

Since all manifolds involved are orientable, this action is without inversions, and the quotient of $\mathcal{T}_S$ by the action is the graph $\mathcal{G}_S$.

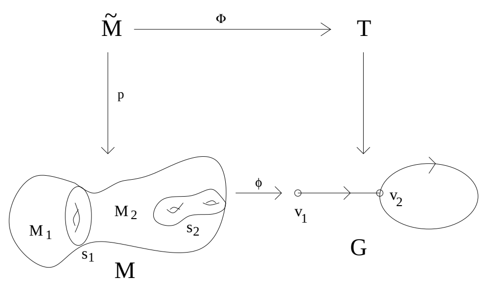


FIGURE 11

Exercise 7. *Do all of this carefully using a bicollaring. To start with, obtain the graph $\mathcal{G}_S$ as a topological space consisting of the sets $M - h(S \times (-1, 1))$ and $S_i \times \{t\}$ for each $t \in (-1, 1)$ with the quotient topology.*

2.5. **Geometric stabilisers.** A vertex v of $\mathcal{T}_S$ corresponds to a connected component $\tilde{K}$ of $\tilde{M} - \tilde{S}$, hence to a connected component in the preimage $p^{-1}(M_i)$ for some i. If we choose a base point in M_i and lift it to a base point in $\tilde{K}$, we see that $\tilde{K}$ is stabilised by the image $\mathrm{im}(\pi_1(M_i) \to \pi_1(M))$ under the inclusion map, and hence the vertex v is stabilised by that group. We conclude that the stabilizer of any vertex of $\mathcal{T}_S$ is conjugate to $\mathrm{im}(\pi_1(M_i) \to \pi_1(M))$ for some component M_i of $M - S$.

Similarly, the stabilizer of an edge of $\mathcal{T}_S$ is conjugate to the image $im(\pi_1(S_i) \to \pi_1(M))$ of a component S_i of S under the inclusion homomorphism.

Having done the above exercise, it is not difficult to see that the action on $\mathcal{T}_S$ is simplicial and without inversions. Furthermore, the dual tree of a given surface is well–defined up to simplicial equivalence since the bicollar is unique up to ambient isotopy.

2.6. **Non–trivial action.** If the surface S is essential, the π_1–injectivity condition for components implies that for any component S_0 of S a connected component of $p^{-1}(S_0)$ is a universal cover of S_0. Applying Van Kampen's theorem to the fundamental group of $\tilde{M}$, it is not too hard to see that for any component M_i of $M - S$ the inclusion $\pi_1(M_i) \to \pi_1(M)$ is injective as well.

Recall that we term an action on a tree non–trivial, if no vertex of $\mathcal{T}_S$ is fixed by all of $\pi_1(M)$. We have the following

Proposition 4. *Let S be an essential surface in a compact, connected, orientable, irreducible 3–manifold M. Then the action of $\pi_1(M)$ on the dual tree $\mathcal{T}_S$ is non–trivial.*

Proof. Assume that the action is trivial, so there is a vertex v such that $Stab(v) = \pi_1(M)$. By our description of the stabilisers, this implies that there is a component M_i of $M - S$ such that the inclusion $\pi_1(M_i) \to \pi_1(M)$ is an isomorphism.

Since S is non–empty, there is a non–trivial component S_0 of S. Since S_0 is not contained in M_i, there is a component M_0 of $M - S_0$ such that the inclusion $\pi_1(M_0) \to \pi_1(M)$ is surjective.

But if S_0 does not separate M, this cannot be true since $\pi_1(M)$ would be a HNN–extension of $\pi_1(M_0)$ across $\pi_1(S_0)$. So assume that S_0 separates M. Then there is another component M_1 of $M - S_0$. Since S_0 is in the closure of both components, there is an edge in the dual tree $\mathcal{T}_S$ such that one vertex is stabilised by $\pi_1(M_0) = \pi_1(M)$. Thus, any element which stabilises the other vertex has to stabilise the whole edge. This implies that $\pi_1(M_1) \cong \pi_1(S_0)$. A theorem of Stallings (see [19], Thm. 10.2) now implies that either M_1 is the interior of a ball, hence $S_0 = S^2$, or $\overline{M_1} \cong (S_0 \times [0,1])$, which implies that S_0 is boundary parallel. Either case violates our choice of S. □

2.7. **Splittings and graphs of groups.** Review amalgamated products, HNN–extensions and graphs of groups. Mention maximal tree and tree of representatives.

3. From Actions on Trees to Surfaces

We would now like to reverse the process described in the previous section. That is, we would like to use a non–trivial simplicial action (without inversions) of the fundamental group of a 3–manifold M on a tree $\mathcal{T}$ to construct an essential surface in the manifold.

A surface *dual to the action* of $\pi_1(M)$ on a tree $\mathcal{T}$ is defined by Culler and Shalen in [14] using a construction due to Stallings (see [27]). If the given manifold M is not compact, replace it by a compact core. Choose a triangulation of M and give the universal cover $\tilde{M}$ the induced triangulation, so that the fundamental group of M acts simplicially on this induced triangulation. One can then construct a simplicial, $\pi_1(M)$–equivariant map f from $\tilde{M}$ to $\mathcal{T}$. The inverse image of midpoints of edges is a surface in $\tilde{M}$ which descends to a surface S in M. It turns out that since the action of $\pi_1(M)$ on $\mathcal{T}$ is non–trivial, if necessary, one can change the map f by homotopy such that the surface S is essential. The dual surface S depends upon the choice of triangulation of M and the choice of the map f, and is therefore not canonical. Furthermore, a dual surface often contains finitely many parallel copies of some of its components. These parallel copies are somewhat redundant, and we implicitly discard them, whilst we still call the resulting surface dual (or associated) to the action.

Reference. [27]*, Section 2.*

3.1. **A $\pi_1(M)$–equivariant map $\tilde{M} \to \mathcal{T}$.** Suppose we have a simplicial action of $\pi_1(M)$ on a tree $\mathcal{T}$. We wish to construct a simplicial, $\pi_1(M)$–equivariant map $\tilde{f}: \tilde{M} \to \mathcal{T}$.

Fix a triangulation of M and give $\tilde{M}$ the induced triangulation. We successively construct maps $\tilde{f}^{(i)}$ from the i–skeleta $\tilde{M}^{(i)}$ of $\tilde{M}$ to $\mathcal{T}$ such that each $\tilde{f}^{(i)}$ extends $\tilde{f}^{(i-1)}$. In the process, we also successively create a subdivision of the triangulation of $\tilde{M}$, which will remain $\pi_1(M)$–invariant, and hence descend to a triangulation of M.

To start, pick a set of orbit representatives $S^{(0)}$ for the action of $\pi_1(M)$ on the set of vertices $\tilde{M}^{(0)}$. Let $h^{(0)}$ be *any* map from $S^{(0)}$ to the vertex set $\mathcal{T}^{(0)}$ of $\mathcal{T}$. We claim that $h^{(0)}$ has exactly one $\pi_1(M)$–equivariant extension $\tilde{f}^{(0)}: \tilde{M}^{(0)} \to \mathcal{T}^{(0)}$.

Indeed, uniqueness and existence follow from the condition

$$\tilde{f}^{(0)}(\gamma \cdot s) = \gamma \cdot h^{(0)}(s) \quad \forall s \in S^{(0)} \quad \forall \gamma \in \pi_1(M).$$

Note that this is well–defined since the action of $\pi_1(M)$ on $\tilde{M}$ is free. So for every vertex v in $\tilde{M}^{(0)}$ there are unique s and γ such that $v = \gamma \cdot s$.

Now then pick a set of orbit representatives $S^{(1)}$ for the action of $\pi_1(M)$ on the set of all 1–simplices of $\tilde{M}$. For each edge $\sigma \in S^{(1)}$, the already constructed map $\tilde{f}^{(0)}$ restricts to a map on the endpoints $\partial\sigma$. Since $\mathcal{T}$ is contractible, this map can be extended to a continuous map $h_\sigma : \sigma \to \mathcal{T}$, since there is a unique path in $\mathcal{T}$ connecting the images of the endpoints. Note that this map may not necessarily be simplicial, since an edge in a triangulation of $\tilde{M}$ could map to a path of length n in the tree, but we may assume that h_σ is linear.

Again, there is a unique, $\pi_1(M)$–equivariant map $\tilde{f}^{(1)} : \tilde{M}^{(1)} \to \mathcal{T}$, which restricts to h_σ for each orbit representative σ, which must be given by

$$\tilde{f}^{(i+1)}(\gamma \cdot x) = \gamma \cdot h_\sigma(x) \quad \forall s \in S^{(i+1)} \quad \forall x \in \sigma \quad \forall \gamma \in \pi_1(M).$$

As above, this map is well–defined and equivariant, and by construction continuous. Since $\pi_1(M)$ acts simplicially on $\mathcal{T}$, each edge contained in an orbit will map to a path of the same length, say n, in $\mathcal{T}$. In order to make $\tilde{f}^{(1)}$ simplicial, it is well–defined and sufficient to subdivide the 1–skeleton accordingly by introducing $n-1$ evenly spaced vertices on the elements of an orbit, which has a representative mapped to a path of length n. Thus, $\tilde{f}^{(1)} : \tilde{M}^{(1)} \to \mathcal{T}$ is $\pi_1(M)$–equivariant and simplicial.

We continue the process in the above manner: if $\tilde{f}^{(i)}$ has been constructed, then pick a set of orbit representatives of the action of $\pi_1(M)$ on the initial set of $(i+1)$–simplices, and for each representative σ, construct a continuous map h_σ to $\mathcal{T}$, which agrees with $\tilde{f}^{(i)}$ on $\partial\sigma$. (Continuity again follows from the fact that $\mathcal{T}$ is simply connected.) Now remember that we did introduce a subtriangulation of $\partial\sigma$ and use this to obtain a generalised barycentric subdivision of σ. Now make use of the general simplicial approximation theorem:

General Simplicial Approximation Theorem (Munkres Thm 2.16.5)**.** *Let K and L be complexes; let $f : |K| \to |L|$ be a continuous map. There exists a subdivision K' of K such that f has a simplicial approximation $h : K' \to L$.*

Thus, we can take h_σ to be this simplicial approximation, and give the $(i+1)$–skeleton of $\tilde{M}$ the induced triangulation which, by construction, is $\pi_1(M)$–invariant. In order to obtain a well–defined map $\tilde{f}^{(i+1)} : \tilde{M}^{(i+1)} \to \mathcal{T}$, we have to know that h_σ has not changed on the boundary of σ whilst we made the map simplicial. This is guaranteed by the following

Lemma 5 (Spanier 3.4.1)**.** *Let $f : |K| \to |L|$ be a map and suppose that for some subcomplex $K_1 \subset K$, $f|_{|K_1|}$ is induced by a simplicial map $K_1 \to L$. If $\varphi : |K| \to |L|$ is a simplicial approximation to f, then $f|_{|K_1|} = \varphi|_{|K_1|}$.*

So there is a unique, $\pi_1(M)$–equivariant map $\tilde{f}^{(i+1)} : \tilde{M}^{(i+1)} \to \mathcal{T}$, which restrics to h_σ for each orbit representative σ. Again, this map must be given by

$$\tilde{f}^{(i+1)}(\gamma \cdot x) = \gamma \cdot h_\sigma(x) \quad \forall s \in S^{(i+1)} \quad \forall x \in \sigma \quad \forall \gamma \in \pi_1(M).$$

As above, this map is well–defined, and by construction simplicial and $\pi_1(M)$–equivariant.

We finally obtain a simplicial, $\pi_1(M)$–equivariant map $\tilde{f}^{(3)} : \tilde{M} \to \mathcal{T}$, which we denote by $\tilde{f}$, along with a new triangulation of $\tilde{M}$ and M. We will freeze all these objects until further notice.

Note that we have started with a given action and a given tree, but obtained $\tilde{f}$ by starting with *any* triangulation of M and *any* map $h^{(0)}$.

3.2. **Constructing a dual surface.** Consider a point $x \in \mathcal{T}$ which is not a vertex. The inverse image $\tilde{f}^{-1}(x) = P$ is a subset of $\tilde{M}$, and we can look at its intersection with a simplex σ of $\tilde{M}$.

If σ is a vertex, then clearly $P \cap \sigma = \emptyset$. If $\tilde{f}$ doesn't map σ onto the edge e of $\mathcal{T}$ containing x, then again $P \cap \sigma = \emptyset$.

If $\tilde{f}$ does map σ onto e, then $P \cap \sigma$ is an $(i-1)$–cell properly embedded in σ which misses the vertices. Embedded in affine space we may think of this as an intersection of the simplex with a hyperplane. It follows that P is locally flat in the interior of each tetrahedron of the triangulation. Is it also locally flat at the intersections with 1–simplices?

Assume σ is such a 1–simplex and $P \cap \sigma = \{z\}$. The 2–simplices incident to σ look like the pages of a cyclic book with σ as its binding. The set P meets each page in a 1–cell which has one endpoint at z and is otherwise disjoint from σ. Since

P meets tetrahedrons in discs, we can obtain the intersection of P with the open star of σ by connecting successive 1–cells on the pages with a 2–cell. This gives an open disc, and P is locally flat at z.

We conclude that $\tilde{f}^{-1}(x)$ is a properly embedded surface in $\tilde{M}$. If we now consider an interval neighbourhood of x and look at its preimage, the above argument shows that the surface is bicollared.

How do we get a surface in M? Let E be the set of midpoints of edges in $\mathcal{T}$. Then $\tilde{f}^{-1}(E) =: \tilde{S}$ is a surface in $\tilde{M}$ since this is true for each $x \in E$.

Since $\pi_1(M)$ acts simplicially on $\mathcal{T}$, this set is invariant under the action. Now then $\tilde{f}$ is $\pi_1(M)$–equivariant, and it follows that $\tilde{S}$ is invariant under the action of the fundamental group by deck transformations. But $\tilde{S}$ is a (not necessarily connected) properly embedded, bicollared surface in $\tilde{M}$, and hence the inverse image under the covering transformation of a properly embedded, bicollared surface S in M. We say that S is *dual (or associated)* to the action of $\pi_1(M)$ on $\mathcal{T}$. Note that S is not canonical since it depends upon choices as emphasised at the end of the previous section.

Now assume that $\pi_1(M)$ acts on $\mathcal{T}$ without inversions. This is equivalent to saying that there is an orientation for edges of $\mathcal{T}$ such that the action of $\pi_1(M)$ preserves this orientation. The fixed–point–free orientation reversing involution on the set of (oriented) edges of $\mathcal{T}$ then acts as an involution on orbits of (oriented) edges of $\mathcal{T}$, and the orbit space $\mathcal{G} = \mathcal{T}/\pi_1(M)$ inherits the structure of a graph. The quotient map $\mathcal{T} \to \mathcal{G}$ is a morphism of graphs and there is a unique map $f : M \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{tikzcd} \tilde{M} \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & \mathcal{T} \arrow[d] \\ M \arrow[r, "f"] & \mathcal{G} \end{tikzcd}$$

If $\overline{E}$ is the set of midpoints of edges in $\mathcal{G}$, then $f^{-1}(\overline{E}) = S$. Since f respects the given triangulations of M and $\mathcal{G}$, and since each point in $\overline{E}$ is 2–sided in $\mathcal{G}$, we may conclude that each component of S is 2–sided in M.

Exercise 8. *Some details have to be supplied in the previous argument. Show that if $\tilde{f}$ is simplicial and the action is without inversions, then the dual surface S is 2–sided.*

Note that if M is orientable, 2–sidedness is equivalent to orientability of S. So now we know that we get a locally flat, 2–sided, properly embedded, bicollared surface. If we didn't like things to be simplicial, we would have needed a property which we got free in the simplicial setting: that $\tilde{f}$ is transversal to the set of midpoints.

3.3. **Algebraic stabilisers.** The vertex and edge stabilisers of the action have the following properties:

Lemma 6. *If S is a dual surface to an action of $\pi_1(M)$ on a tree $\mathcal{T}$, then*

- *for each component M_i of $M - S$, the subgroup $\mathrm{im}(\pi_1(M_i) \to \pi_1(M))$ of $\pi_1(M)$ is contained in the stabilizer of some vertex of $\mathcal{T}$; and*
- *for each component S_i of S, the subgroup $\mathrm{im}(\pi_1(S_i) \to \pi_1(M))$ of $\pi_1(M)$ is contained in the stabilizer of some edge of $\mathcal{T}$.*

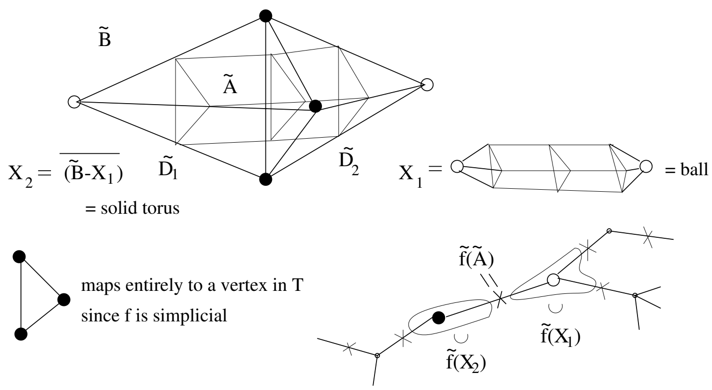
Exercise 9. *Prove the lemma!*

What if the surface associated to our action is empty? Then the only component of $M - S$ is M itself, and by the first part of the lemma, we know that $\pi_1(M)$ stabilises a vertex of $\mathcal{T}$. But this is what we called a trivial action. So if $\pi_1(M)$ acts non–trivially on a tree, then every dual surface is non–empty.

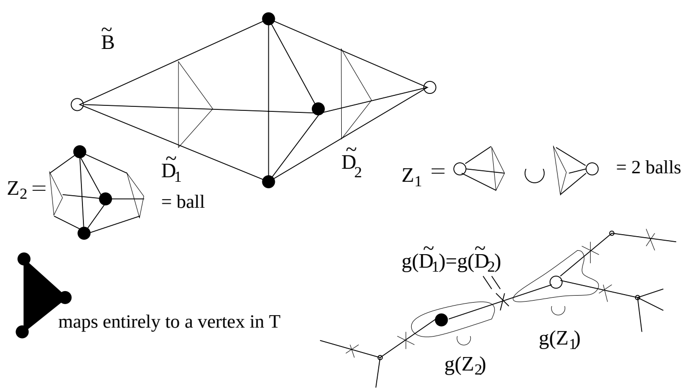
3.4. **Making the dual surface essential.** Given a surface S dual to a non–trivial, simplicial action without inversions on a tree, we know from the previous section that it satisfies two of the five properties of essential surfaces. Can we always choose a dual surface which is essential?

Let us assume that the dual surface S admits a compression disc D. We wish to replace S by the surface S' resulting from compression along D such that S' is still dual to the action. So let us try to replace the map $\tilde{f}$ by a map $\tilde{f}'$ with ${\tilde{f}'}^{-1}(E) = p^{-1}(S')$. We illustrate the process with a (simplified) picture in Figure 3.4. Continuity and $\pi_1(M)$–equivariance are the easy properties to obtain, and our picture is generic for their purpose, but simplicial is the crucial point.

Consider a ball neighbourhood B of the compression disc and pick a homeomorphic copy $\tilde{B}$ of B in $p^{-1}(B)$. Our map $\tilde{f}$ restricts to a map $\tilde{B} \to \mathcal{T}$, which we will change on the interior of $\tilde{B}$. Inside our ball $\tilde{B}$, we have an annulus $\tilde{A}$ and a



Before: the map $\tilde{f}$ restricted to $\tilde{B}$



After: the map $\tilde{g}$ restricted to $\tilde{B}$

FIGURE 12

compression disc $\tilde{D}$. Furthermore, there are two copies $\tilde{D}_1$ and $\tilde{D}_2$ of $\tilde{D}$ such that $(\tilde{D}_1 \cup \tilde{D}_2) \cap \partial\tilde{B} = \tilde{A} \cap \partial\tilde{B}$.

The annulus $\tilde{A}$ divides $\tilde{B}$ into a ball X_1 and a solid torus X_2. Since $\tilde{A}$ is 2–sided, X_1 is mapped into the closure of a different component of $\mathcal{T} - E$ than X_2.

Note that the two discs $\tilde{D}_1$ and $\tilde{D}_2$ divide the ball $\tilde{B}$ into three balls, and the boundary of $\tilde{B}$ into an annulus and two discs. The map $\tilde{f}$ maps this annulus into the image of X_2, and the two discs into the image of X_1. Thus, if we let Z_2 be the ball bounded by that annulus and the two discs $\tilde{D}_1$ and $\tilde{D}_2$, we have a continuous map from its boundary to $\tilde{f}(X_2)$, which we can extend continuously to its interior since all our objects are contractible.

Similarly, we have continuous maps from the boundaries of the other two balls into the image of $\tilde{f}(X_1)$, which we can extend continuously to their interior. Call the resulting map $g : \tilde{B} \to T$. By the above construction, g agrees with $\tilde{f}$ on the boundary of $\tilde{B}$, and

$$g^{-1}(\tilde{f}(\tilde{B}) \cap E) = \tilde{D}_1 \cup \tilde{D}_2 \tilde{S}' \cap \tilde{B}.$$

In our illustration, g has also been obtained as a simplicial map. This is not easy in general, but we skip the tedious details here.

So let us extend g uniquely and $\pi_1(M)$–equivariantly to $p^{-1}(B)$ using the action of $\pi_1(M)$ on $\tilde{M}$. Then define the new map $\tilde{f}'$ to agree on $\tilde{M} - p^{-1}(B)$ with $\tilde{f}$ and with g on $p^{-1}(B)$. By the above, this map is well–defined, continuous, $\pi_1(M)$–equivariant, and apparently even simplicial. This shows that compressions on a dual surface result in another dual surface.

What can we do if there are components of S which are boundary parallel or two–spheres? We claim that we can simply omit them. So assume that there is a boundary parallel component S_0 of S, and consider the surface $S' = S - S_0$ and a deformation retract $M' \subset M$ such that $M' \cap S = S'$. Denote the deformation by $d : M \to M'$ and its lift to $\tilde{M}$ by $\tilde{d}$. We have

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\tilde{d}} & \tilde{M} & \xrightarrow{\tilde{f}} & \mathcal{T} \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \\ M & \xrightarrow{d} & M & & \end{array}$$

and the composite $\tilde{f} \circ \tilde{d}$ is the map we are looking for!

Exercise 10. *How do we discard spheres?*

3.5. **Complexity of surfaces.** In some ways, the surface S' is simpler than S. If the surfaces were closed and connected, we could say that simpler means of lower genus. However, we have to define complexity in the following way:

$$c(S) = \sum_{S_i \subset S} (2 - \chi(S_i))^2.$$

Since the Euler characteristic of a compact connected surface is less or equal to 2, we are summing over squares of positive numbers. Note that discarding 2–spheres does not alter the above complexity.

What happens when we compress? Let S_0 be the component which admits a compression disc. Then

$$S'_0 = (S_0 - intA) \cup D_1 \cup D_2,$$

and hence

$$\chi(S'_0) = \chi(S_0 - intA) + \chi(D_1) + \chi(D_2) = \chi(S_0) + 2.$$

If S'_0 is connected, then $c(S') < c(S)$ since $\chi(S'_0) = \chi(S_0) + 2 \leq 2$. If S'_0 is not connected, then there are two components S'_α and S'_β which are not spheres. Put $a = 2 - \chi(S'_\alpha)$ and $b = 2 - \chi(S'_\beta)$. Then a, $b > 0$ and $2 - \chi(S_0) = 4 - \chi(S'_0) = 4 - \chi(S'_\alpha) - \chi(S'_\beta) = a + b$. So in our formula for complexity, we replace $(a+b)^2$ by $a^2 + b^2$, and again $c(S') < c(S)$.

As mentioned before, if we discard components which are spheres, the complexity doesn't change, but if we discard boundary parallel components, it will decrease. We therefore obtain an essential surface by choosing a dual surface of minimal complexity, and amongst those one with minimal number of components.

3.6. **Geometric vs. algebraic.** The preceding sections show that a compact, orientable, irreducible 3–manifold M contains an essential surface S if its fundamental group admits a non–trivial simplicial action without inversions on a tree $\mathcal{T}$.

Conversely, we have seen that an essential surface S gives rise to a non–trivial simplicial action without inversions on a dual tree $\mathcal{T}_S$. What is the relationship between these trees? Since $\mathcal{T}_S$ is a retract of $\tilde{M}$, we may compose the inclusion map $i : \mathcal{T}_S \to \tilde{M}$ with $\tilde{f} : \tilde{M} \to \mathcal{T}$ to obtain a $\pi_1(M)$–equivariant map $\mathcal{T}_S \to \mathcal{T}$. We can now compare the actions of $\pi_1(M)$ on the trees.

Exercise 11. *Show that the vertex and edge stabilisers of the action on $\mathcal{T}_S$ are contained in vertex and edge stabilisers of the action on $\mathcal{T}$ respectively.*

As the following example illustrates, these inclusions are not necessarily equalities, and hence the trees not necessarily $\pi_1(M)$–equivariantly isomorphic. To this end, note that the edge and vertex stabilisers of the action on $\mathcal{T}_S$ are finitely generated. It is quite easy to construct actions on trees which have stabilisers which are not finitely generated.

Consider a homomorphism ρ of $\pi_1(M)$ into $\mathbb{Z}$. Since $\mathbb{Z}$ acts by translations on $\mathbb{R}$, we can pull back to an action of $\pi_1(M)$ on $\mathbb{R}$. Thus the action is non–trivial if ρ is non–trivial. The stabilisers of this action are conjugates of the kernel of ρ. If M was a knot complement, then the kernel is just the commutator group, and there are enough examples where this group is not finitely generated. (In fact: it is finitely generated if and only if the knot is fibered.) Note that non–trivial homomorphisms of $\pi_1(M)$ to $\mathbb{Z}$ correspond to non–trivial elements ϵ of $H^1(M;\mathbb{Z})$. So to each ϵ we can associate an essential surface.

We can say more if $\rho : \pi_1(M) \to \mathbb{Z}$ is an epimorphism. By our construction, we get a map $f : M \to S^1$ which induces an epimorphism $f_* : \pi_1(M) \to \pi_1(S^1)$. The inverse image of some point on S^1 is an essential surface S in M. The homomorphism $\pi_1(M) \to \pi_1(S^1)$ simply gives the algebraic intersection number of loops in M which cross the surface S transversely, i.e. it is the sum of signed intersection numbers. But ρ is onto, so there is a simple closed loop in M which crosses S an odd number of times. Thus $M - S$ must be connected and we have proven the following:

Proposition 7. *If M is a compact, orientable, irreducible 3–manifold with positive first Betti number, then M contains a nonseparating essential surface.*

A *Haken manifold* is a compact, orientable, irreducible 3–manifold which is either a ball or contains an essential surface. If M is a compact, orientable, irreducible 3–manifold with non–empty boundary, we claim that either M is a ball or M has positive first Betti number.

Recall that the q–th Betti number β_q is the rank of the homology group H_q. The third Betti number is equal to 0 since the manifold has boundary, and hence has the homotopy type of a finite 2–dimensional CW–complex. Let DM be the double of M, that is, two copies of M glued along their boundary. Since DM is closed, we have $0 = \chi(DM) = 2\chi(M) - \chi(\partial M)$. Hence $\chi(M) \leq 0$. Now $0 \geq \chi(M) = \beta_0 - \beta_1 + \beta_2 = 1 - \beta_1 + \beta_2$ since M is connected. It follows that $\beta_1 \geq 1 + \beta_2 \geq 1$. So we have the:

Corollary 8. *Every compact, orientable, irreducible 3–manifold with non–empty boundary is a Haken manifold.*

3.7. **Surface detected by an action.** In this subsection, we describe associated surfaces satisfying certain "non–triviality" conditions as in [32]. Any essential surface gives rise to a graph of groups decomposition of $\pi_1(M)$, which shall be denoted by $\langle M_i, S_j, t_k \rangle$, where M_i are the components of $M - S$, S_j are the components of S, and t_k are generators of the fundamental group of the graph of groups arising from HNN–extensions.

Assume that $M - S$ consists of m components. For each component M_i of $M - S$ we fix a representative Γ_i of the conjugacy class of $\mathrm{im}(\pi_1(M_i) \to \pi_1(M))$ as follows. Let $\mathcal{T}' \subset \mathcal{T}_S$ be a tree of representatives, i.e. a lift of a maximal tree in $\mathcal{G}_S$ to $\mathcal{T}_S$, and let $\{s_1, \ldots, s_m\}$ be the vertices of $\mathcal{T}'$, labelled such that s_i maps to M_i under the composite mapping $\mathcal{T}_S \to \mathcal{G}_S \to M$. Then let Γ_i be the stabiliser of s_i.

Any essential surface S which does not contain parallel copies of one of its components is called *detected by an action* of $\pi_1(M)$ on a tree $\mathcal{T}$ if:

- S1. every vertex stabiliser of the action on $\mathcal{T}_S$ is included in a vertex stabiliser of the action on $\mathcal{T}$,
- S2. every edge stabiliser of the action on $\mathcal{T}_S$ is included in an edge stabiliser of the action on $\mathcal{T}$,
- S3. if M_i and M_j, where $i \neq j$, are identified along a component of S, then there are elements $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ such that $\gamma_i \gamma_j$ acts as a loxodromic on $\mathcal{T}$,
- S4. each of the generators t_i can be chosen to act as a loxodromic on $\mathcal{T}$.

Lemma 9. [32] *An essential surface in M detected by an action of $\pi_1(M)$ on a tree $\mathcal{T}$ is dual to the action.*

Proof. Denote the essential surface by S, and choose a sufficiently fine triangulation of M such that the 0–skeleton of the triangulation is disjoint from S, and such that the intersection of any edge in the triangulation with S consists of at most one point. Give $\tilde{M}$ the induced triangulation. There is a retraction $\tilde{M} \to \mathcal{T}_S$, which we may assume to be simplicial, and we now wish to define a map $\mathcal{T}_S \to \mathcal{T}$.

Note that the vertices $\{s_1, \ldots, s_m\}$ of the tree of representatives are a complete set of orbit representatives for the action of $\pi_1(M)$ on the 0–skeleton of $\mathcal{T}_S$. Condition

S3 implies that we may choose vertices $\{v_1, \ldots, v_m\}$ of $\mathcal{T}$ such that v_i is stabilised by Γ_i, and if $M_i \neq M_j$, then $v_i \neq v_j$.

Define a map f^0 between the 0–skeleta of $\mathcal{T}_S$ and $\mathcal{T}$ as follows. Let $f^0(s_i) = v_i$. For each other vertex s of $\mathcal{T}_S$ there exists $\gamma \in \pi_1(M)$ such that $\gamma s_i = s$ for some i. Then let $f^0(s) = \gamma f^0(s^i)$. This construction is well–defined by the condition on the vertex stabilisers, and we therefore obtain a $\pi_1(M)$–equivariant map from $\mathcal{T}_S^0 \to \mathcal{T}^0$. Moreover, this map extends uniquely to a map $f^1 : \mathcal{T}_S \to \mathcal{T}$, since the image of each edge is determined by the images of its endpoints. Since $v_i \neq v_j$ for $i \neq j$, and since each t_k acts as a loxodromic on $\mathcal{T}_v$, the image of each edge of $\mathcal{T}_S$ is a path of length greater or equal to one in $\mathcal{T}$.

If f^1 is not simplicial, then there is a subdivision of $\mathcal{T}_S$ giving a tree $\mathcal{T}_{S'}$ and a $\pi_1(M)$–equivariant, simplicial map $f : \mathcal{T}_{S'} \to \mathcal{T}$. There is a surface S' in M which is obtained from S by adding parallel copies of components such that $\mathcal{T}_{S'}$ is the dual tree of $\tilde{S}'$.

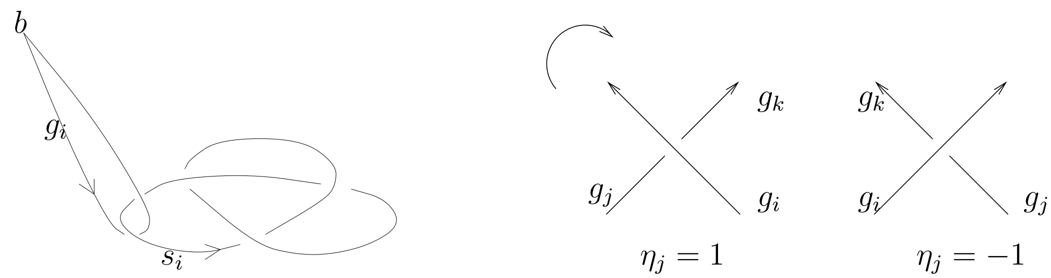
As before, choose a sufficiently fine triangulation of M such that the 0–skeleton of the triangulation is disjoint from S', and such that the intersection of any edge in the triangulation with S' consists of at most one point, and give $\tilde{M}$ the induced triangulation. The composite map $\tilde{M} \to \mathcal{T}_{S'} \to \mathcal{T}$ is $\pi_1(M)$–equivariant and simplicial, and the inverse image of midpoints of edges descends to the surface S' in M. Thus, S' is associated to the action of $\pi_1(M)$ on $\mathcal{T}$. $\square$

Note that if S is dual to the action, then the above lemma shows that the map $\tilde{f} : \tilde{M} \to \mathcal{T}$ factors through a $\pi_1(M)$–equivariant map $\mathcal{T}_S \to \mathcal{T}$, which implies that the vertex and edge stabilisers of the action on $\mathcal{T}_S$ are contained in vertex and edge stabilisers of the action on $\mathcal{T}$ respectively. This gives a different proof of Exercise 11.

4. The Varieties

The construction by Marc Culler and Peter Shalen in [14] associates an action of a finitely generated group Γ on a tree with an ideal point of a curve in the $SL_2(\mathbb{C})$–character variety $X(\Gamma)$. This construction provides the link between geometry and topology in the introduction since $SL_2(\mathbb{C})$–representations of connected manifolds are related to hyperbolic structures, and actions of the fundamental group are related to essential surfaces. Excellent references to the varieties involved are [27], Section 4, as well as Boyer and Zhang [5].

4.1. **Review of knot groups.** The complements of knots in the 3–sphere are nice manifolds to study, and they will be found as examples and in exercises throughout these notes. We recall some basic facts concerning their fundamental groups. If $\mathfrak{k} \subset S^3$ is a knot, we call $\Gamma = \Gamma(\mathfrak{k}) = \pi_1(S^3 - \mathfrak{k})$ a *knot group.*



Generators and arcs

Reading the relations

Figure 13. Wirtinger presentation

Theorem 10. [6] *Let s_i for $i \in \{1, \dots, n\}$ be the overcrossing arcs of a regular projection of a knot $\mathfrak{k}$. Then the knot group admits the following* Wirtinger presentation

$$\Gamma = \pi_1(S^3 - \nu(\mathfrak{k})) =< g_i \mid r_i, i \in \{1, \dots, n\} > .$$

The arc s_i corresponds to the generator g_i as shown in Figure 13, and a crossing with sign $\eta_j = \pm 1$ gives rise to the defining relator

$$r_j = g_j g_i^{-\eta_j} g_k^{-1} g_i^{\eta_j},$$

where we start reading the crossing from the arc s_j and continue in clockwise direction.

It follows from the above theorem that any defining relator is a consequence of all the other defining relators. Given the correspondence between oriented arcs of the projection and generators of the fundamental group, we often label arcs directly with generators.

The abelianisation of a knot group is $\mathbb{Z}$, since adding the commutators $[g_i, g_j]$ to the relations leaves us with one generator and no relations. Thus, $H_1(S^3 - \mathfrak{k}) \cong \mathbb{Z}$. An epimorphism from Γ to $\mathbb{Z}$ arises naturally by considering the *linking number*. For a path γ in the knot complement, consider a regular projection of $\mathfrak{k} \cup \gamma$. After orienting $\mathfrak{k}$, there are two different types of crossings c of γ with $\mathfrak{k}$ to which we associate a sign $\epsilon(c) = \pm 1$ according to Figure 14.

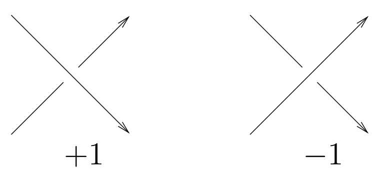


FIGURE 14. Signs of crossings

Let $C(\gamma, \mathfrak{k})$ denote the set of all crossings of γ and $\mathfrak{k}$. We define the linking number

$$\operatorname{lk}(\gamma, \mathfrak{k}) = \frac{1}{2} \sum_{c \in C(\gamma, \mathfrak{k})} \epsilon(c).$$

This number is invariant under ambient isotopy and hence well defined on homotopy classes. This gives us a mapping from Γ to $\mathbb{Z}$, which turns out to be a homomorphism. Generators in the Wirtinger presentation have linking number $+1$ with $\mathfrak{k}$ as can be verified by looking at Figure 13. Hence the linking number defines the categorical epimorphism $\Gamma \to \Gamma/\Gamma'$.

Since the intersection number of meridian and longitude is 1, the linking number of a meridian and the knot $\mathfrak{k}$ is 1, and hence the generators of Γ in the Wirtinger presentation are meridians. Given a knot projection, we obtain the longitude $\mathcal{L}_k$ corresponding to the meridian g_k as follows: Starting at the arc s_k we travel along the knot and write down g_i when undercrossing the arc s_i from left to right and g_i^{-1} when undercrossing from right to left. Finally, we multiply this by g_k with an exponent such that the total exponent sum adds up to zero.

The fact that $\Gamma/\Gamma' \cong \mathbb{Z}$ leads to the following result concerning the structure of the fundamental group of a knot complement:

Proposition 11. *A knot group Γ is a semidirect product $\Gamma = \mathfrak{Z} \ltimes \Gamma'$, where $\mathfrak{Z} \cong \Gamma/\Gamma'$ is infinite cyclic.*

Proof. Note that $<1> \rightarrow \Gamma' \xrightarrow{\iota} \Gamma \xrightarrow{\varphi} \mathbb{Z} \rightarrow <1>$ is an exact sequence. Take any element $g_0 \in \varphi^{-1}(1)$ and define a homomorphism $j : \mathbb{Z} \rightarrow \Gamma$ by $j(1) = g_0$. Then $\varphi j = id$, and the sequence splits. This gives the above result. $\square$

4.2. **Some algebraic geometry.** A subset $I \subseteq \mathbb{C}[X_1, \ldots, X_n]$ is an *ideal* if it satisfies the following properties:

1. $0 \in I$,
2. If $f, g \in I$, then $f + g \in I$,
3. If $f \in I$ and $h \in \mathbb{C}[X_1, \ldots, X_n]$, then $fh \in I$.

A subset $X \subseteq \mathbb{C}^n$ is an *affine algebraic set* if for some ideal $I \subseteq \mathbb{C}[X_1, \ldots, X_n]$

$$X = V(I) = \{z \in \mathbb{C}^n \mid f(z) = 0 \quad \text{for all} \quad f \in I\}.$$

By the Hilbert Basis Theorem, I is finitely generated, so that $V(I)$ is the set of the simultaneous solutions of finitely many polynomial equations. The map V takes ideals in $\mathbb{C}[X_1, \ldots, X_n]$ to subsets of $\mathbb{C}^n$, and in particular $V(0) = \mathbb{C}^n$ and $V(\mathbb{C}[X_1, \ldots, X_n]) = \emptyset$.

The algebraic subsets of $\mathbb{C}^n$ form the closed sets of the so–called *Zariski topology* on $\mathbb{C}^n$. This is well–defined due to the following two facts:

1. $V(I) \cup V(J) = V(I \cap J)$
2. If $\{V(I_j)\}_j$ is any family of algebraic sets, then their intersection is again an algebraic set: $\cap V(I_j) = V(\sum I_j)$.

A nonempty topological space X is called *irreducible* if it is not the union of two proper closed subsets. Then the following holds:

1. If Y is a subspace of a topological space X, then Y is irreducible if and only if $\overline{Y}$ is irreducible.
2. If $\varphi : X \rightarrow Y$ is a continuous map between topological spaces, and X is irreducible, then so is $\varphi(X)$.

Let X be a topological space. By the above, a maximal irreducible subspace is closed. The maximal irreducible subspaces of X are called the *(irreducible) components* of X. It is easy to see that the closure of every point of X is irreducible. Thus X is contained in the union of its components.

An affine algebraic set it called an *algebraic variety* if it is irreducible. We shall use this term in a loose way, and call some sets varieties even though they may not be irreducible.

4.3. **Morphisms.** Let X be an affine algebraic set contained in $\mathbb{C}^n$. A map $\mu : X \to \mathbb{C}$ is called a *polynomial function* if there exists a polynomial $f \in \mathbb{C}[T_1, \ldots, T_n]$ such that $\mu = f|_X$. Thus, the polynomial functions on X are simply the polynomials in $\mathbb{C}[T_1, \ldots, T_n]$ restricted to X.

If $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ are affine algebraic sets, then the map $\varphi : X \to Y$ is called a *morphism* from X to Y if there exist polynomial functions $f_1, \ldots, f_m$ such that

$$\varphi(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n))$$

for all $(a_1, \ldots, a_n) \in X$. A morphism $\varphi : X \to Y$ is a continuous map in the Zariski topology

Two affine algebraic sets X and Y are called *isomorphic*, if there exist morphisms $\varphi : X \to Y$ and $\gamma : Y \to X$ such that $\gamma\varphi = id_Y$ and $\varphi\gamma = id_X$.

4.4. **Representation variety.** Let Γ be a finitely generated group. A representation of Γ into $SL_2(\mathbb{C})$ is a homomorphism $\rho : \Gamma \to SL_2(\mathbb{C})$, and the set of representations is $\mathfrak{R}(\Gamma) = \mathrm{Hom}(\Gamma, SL_2(\mathbb{C}))$. This set is often called the *representation variety* of Γ.

If $\langle \gamma_1, \ldots, \gamma_n \mid r_j \rangle$ is a presentation for Γ, then a representation is uniquely determined by the point $(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \in SL_2(\mathbb{C})^n \subset \mathbb{C}^{4n}$. The latter inclusion introduces affine coordinates. Substituting n general matrices into the relators gives sets of polynomial relations in these affine coordinates, and the Hilbert basis theorem implies that $\mathfrak{R}(\Gamma)$ inherits the structure of an affine algebraic set.

Exercise 12. *Show that $\mathfrak{R}(\Gamma)$ is independent of the chosen presentation for Γ. That is, given two different presentations, show that the associated representation spaces are isomorphic.*

If Γ is a knot group, then any homomorphism into an abelian group H factors through $\Gamma/\Gamma' \cong \mathbb{Z}$, i.e. it can be regarded as the composite $\Gamma \to \mathbb{Z} \to H$. Thus, each abelian representation of a knot group Γ into $SL_2(\mathbb{C})$ corresponds to a single element of $SL_2(\mathbb{C})$, and the set of all abelian representations forms a subvariety of $\mathfrak{R}(\Gamma)$, which is isomorphic to $SL_2(\mathbb{C})$. Since

$$\dim SL_2(\mathbb{C}) = \dim \mathbb{C}[SL_2(\mathbb{C})] = \dim \mathbb{C}[a,b,c,d]/(ac - bd - 1) = 3,$$

we know that in particular $\dim \mathfrak{R}(\Gamma) \geq 3$ for a knot group Γ.

Indeed, for any group Γ with a presentation in g generators and r relations, we have $\dim \mathfrak{R}(\Gamma) \geq 3(g-r)$. For a knot group, we always get a Wirtinger presentation in n generators and n relations where we can omit one of the relations. Thus, it yields the same estimate as above.

Proposition 12. [14] *Let V be an irreducible component of $\mathfrak{R}(\Gamma)$. Then any representation equivalent to a representation in V must itself belong to V.*

Proof. The set $V \times SL_2(\mathbb{C}) \subseteq SL_2(\mathbb{C})^{n+1}$ is a product of two irreducible affine algebraic sets and is therefore an irreducible affine algebraic set. The map $f : V \times SL_2(\mathbb{C}) \to \mathfrak{R}(\Gamma)$ given by $f(X_1, \ldots, X_n, A) = (A^{-1}X_1A, \ldots, A^{-1}X_nA)$ is defined by polynomials in the coordinates and hence a morphism. Thus, f is continuous in the Zariski topology and this implies that $f(V \times SL_2(\mathbb{C}))$ is irreducible. Hence $f(V \times SL_2(\mathbb{C}))$ is contained in a component V' of $\mathfrak{R}(\Gamma)$. But then $V = f(V \times \{E\}) \subseteq V'$. Since V is a component of $\mathfrak{R}(\Gamma)$, this forces $V = V'$. Thus $f(V \times SL_2(\mathbb{C})) \subseteq V$, and this proves the proposition. $\square$

4.5. **Irreducible and reducible.** Two representations are *equivalent* if they differ by an inner automorphism of $SL_2(\mathbb{C})$. For each $\rho \in \mathfrak{R}(\Gamma)$, its character is the function $\chi_\rho : \Gamma \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \operatorname{tr} \rho(\gamma)$. It follows that equivalent representations have the same character since the trace is invariant under conjugation. The converse is not true as the following two representations of $\mathbb{Z}$ illustrate:

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The characters of these representations are identical, but the first representation is faithful, i.e. an injective homomorphism, whilst the latter is trivial. Thus, the representations are not equivalent.

A representation is *irreducible* if the only subspaces of $\mathbb{C}^2$ invariant under its image are trivial. This is equivalent to saying that the representation cannot be conjugated to a representation by upper triangular matrices. Otherwise a representation is *reducible*.

Exercise 13. *Let ρ be a reducible representation of Γ which is not abelian. Show that there is an abelian representation of Γ which has the same character as ρ.*

The following facts imply that we can use characters to study (irreducible) representations modulo equivalence.

Lemma 13. [14]

1. *Let $\rho \in \mathfrak{R}(\Gamma)$. Then ρ is reducible if and only if $\chi_\rho(c) = 2$ for each element c of the commutator subgroup $\Gamma' = [\Gamma, \Gamma]$ of Γ.*
2. *Let $\rho, \sigma \in \mathfrak{R}(\Gamma)$ satisfy $\chi_\rho = \chi_\sigma$ and assume that ρ is irreducible. Then ρ and σ are equivalent.*

The above lemma implies that irreducible representations are determined by characters up to equivalence, and the reducible representations form a closed subset of $\mathfrak{R}(\Gamma)$.

4.6. **Character variety.** The collection of characters $\mathfrak{X}(\Gamma)$ turns out to be an affine algebraic set, which is called the *character variety*. There is a regular map $\mathrm{t} : \mathfrak{R}(\Gamma) \to \mathfrak{X}(\Gamma)$ taking representations to characters. According to [17], affine coordinates of $\mathfrak{X}(\Gamma)$ can be chosen as follows. Number the words

$$\{\gamma_i\gamma_j \mid 1 \leq i < j \leq n\} \cup \{\gamma_i\gamma_j\gamma_k \mid 1 \leq i < j < k \leq n\}$$

starting from $n+1$ onwards and denote them accordingly by $\gamma_{n+1}, \ldots, \gamma_m$. Then a character is uniquely determined by the point $(\operatorname{tr}\rho(\gamma_1), \ldots, \operatorname{tr}\rho(\gamma_m)) \in \mathbb{C}^m$, where $m = n + \binom{n}{2} + \binom{n}{3} = \frac{n(n^2+5)}{6}$.

If $\chi \in \mathfrak{X}(\Gamma)$ is the character of an irreducible representation, then the fibre $\mathrm{t}^{-1}(\chi)$ is at least 3–dimensional. However, the orbit of an abelian representation under conjugation is 2–dimensional. So if κ is a reducible character on an irreducible component of $\mathfrak{X}(\Gamma)$ which contains an irreducible character, then there is a reducible non–abelian representation ρ with $\mathrm{t}(\rho) = \kappa$. This representation is necessarily metabelian, i.e. its second commutator group vanishes.

Trace identities. *Due to the definition of characters, in computation and even in proofs it is often useful to know certain trace identities which hold in $SL_2(\mathbb{C})$. The most important are briefly stated here, where capital letters denote elements of $SL_2(\mathbb{C})$.*

$$\operatorname{tr} A^{-1} = \operatorname{tr} A \tag{4.1}$$

$$\operatorname{tr}(B^{-1}AB) = \operatorname{tr} A \tag{4.2}$$

$$\operatorname{tr} A \operatorname{tr} B = \operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) \tag{4.3}$$

$$\begin{aligned} \operatorname{tr}(ABC) = {} & \operatorname{tr} A \operatorname{tr}(BC) + \operatorname{tr} B \operatorname{tr}(AC) + \operatorname{tr}(C) \operatorname{tr}(AB) \\ & - \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C - \operatorname{tr}(ACB) \end{aligned} \tag{4.4}$$

Note that the second identity is equivalent to the identity $\operatorname{tr} AB = \operatorname{tr} BA$, *and that the third implies* $\operatorname{tr}(A^2) = (\operatorname{tr} A)^2 - 2$.

Exercise 14. *Prove the trace identities. You may wish to use the Cayley–Hamilton theorem.*

Proposition 14. [14] *Suppose that Γ is generated by γ_1 and γ_2, and let ρ be a representation of Γ into $SL_2(\mathbb{C})$. Then ρ is reducible if and only if* $\operatorname{tr} \rho([\gamma_1, \gamma_2]) = 2$.

In particular, ρ is reducible if and only if $x^2+y^2+z^2-xyz = 4$, where $x = \operatorname{tr} \rho(\gamma_1)$, $y = \operatorname{tr} \rho(\gamma_2)$, and $z = \operatorname{tr} \rho(\gamma_1\gamma_2)$.

Proof. We first observe that using the trace identity (4.4), we have

$$\begin{aligned} \operatorname{tr} \rho([\gamma_1, \gamma_2]) &= \operatorname{tr} \rho(\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2) \\ &= (\operatorname{tr} \rho(\gamma_1))^2 + (\operatorname{tr} \rho(\gamma_2))^2 + (\operatorname{tr} \rho(\gamma_1\gamma_2))^2 \\ &\quad - \operatorname{tr} \rho(\gamma_1) \operatorname{tr} \rho(\gamma_2) \operatorname{tr} \rho(\gamma_1\gamma_2) - 2 \\ &= x^2 + y^2 + z^2 - xyz - 2. \end{aligned}$$

Thus $\operatorname{tr} \rho([\gamma_1, \gamma_2]) = 2$ is equivalent to the fact that $x^2 + y^2 + z^2 - xyz = 4$. The statement of the proposition is therefore clearly true if $\rho(\Gamma)$ is abelian. Hence assume that $\rho(\Gamma)$ is nonabelian. If ρ is reducible, then $\operatorname{tr} \rho([\gamma_1, \gamma_2]) = 2$ by Lemma 13.

Suppose now that $\operatorname{tr} \rho([\gamma_1, \gamma_2]) = 2$. We may assume without loss of generality that Γ is free on the generators γ_1 and γ_2. Let U denote the subgroup generated by $u = \gamma_1$ and $v = \gamma_2\gamma_1^{-1}\gamma_2^{-1}$. Then the character of a representation ρ of U into

$SL_2(\mathbb{C})$ is determined by its values at u, v and uv. We have $\operatorname{tr}\rho(u) = \operatorname{tr}\rho(v)$ and $\operatorname{tr}\rho(uv) = 2$.

Now consider the representation σ defined by $\sigma(u) = \rho(u)$ and $\sigma(v) = \rho(u)^{-1}$. Clearly, σ is a well defined homomorphism with cyclic image. Hence σ is reducible. Further $\operatorname{tr}\sigma(u) = \operatorname{tr}\rho(u) = \operatorname{tr}\rho(v) = \operatorname{tr}\sigma(v)$ and $\operatorname{tr}\sigma(uv) = \operatorname{tr}\rho(uv)$. It follows that σ and ρ have the same character. Thus by Lemma 13 $\rho|U$ is reducible. It follows that the images of the elements $u = \gamma_1$ and $uv = [\gamma_1, \gamma_2]$ of U have a common eigenvector.

Similarly, it follows that $\rho(\gamma_2)$ and $\rho([\gamma_1, \gamma_2])$ have a common eigenvector. Since $\rho([\gamma_1, \gamma_2])$ is nontrivial with trace 2, it has an unique 1-dimensional invariant subspace. Hence, $\rho(\gamma_1)$ and $\rho(\gamma_2)$ have an common invariant subspace and ρ is reducible. $\square$

Exercise 15. *Let* Γ *be a knot group and* χ *be the character of an abelian representation. Show that* $\dim \mathrm{t}^{-1}(\chi) = 2$.

4.7. **Computing character varieties.** In this subsection a cross–section for the quotient map from the representation space to the character variety is defined in the case of groups generated by two elements.

Let Γ be an arbitrary finitely generated group. It follows from Lemma 13, that the set $\mathfrak{Red}(\Gamma)$ consisting of reducible representations is a subvariety of $\mathfrak{R}(\Gamma)$. Let $\mathfrak{R}^i(\Gamma)$ denote the closure of the set of irreducible representations. According to [14], the images $\mathfrak{X}^r(\Gamma) = \mathrm{t}(\mathfrak{Red}(\Gamma))$ and $\mathfrak{X}^i(\Gamma) = \mathrm{t}(\mathfrak{R}^i(\Gamma))$ are closed algebraic sets. Then $\mathfrak{X}^r(\Gamma) \cup \mathfrak{X}^i(\Gamma) = \mathfrak{X}(\Gamma)$, and the union may or may not be disjoint. Note that $\mathfrak{X}^r(\Gamma)$ is completely determined by the abelianisation of Γ, since the character of any reducible non–abelian representation is also the character of an abelian representation. It follows from Lemma 13 that fibres of $\mathrm{t} : \mathfrak{R}^i(\Gamma) \to \mathfrak{X}(\Gamma)$ have dimension three.

Suppose that Γ is a 2–generator group with presentation $\langle \gamma, \delta \mid r_i \rangle$. Let ρ be an irreducible representation in $\mathfrak{R}(\Gamma)$. There are four choices of bases $\{b_1, b_2\}$ for $\mathbb{C}^2$ with respect to which ρ has the form:

$$\rho(\gamma) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\delta) = \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix}. \tag{4.5}$$

These bases can be obtained by choosing b_1' invariant under $\rho(\gamma)$, b_2' invariant under $\rho(\delta)$, and then adjusting by a matrix which is diagonal with respect to $\{b_1', b_2'\}$. Thus, any irreducible representation in $\mathfrak{R}^i(\Gamma)$ is conjugate to a representation in the subvariety $\mathfrak{C}(\Gamma) \subseteq \mathfrak{R}^i(\Gamma)$ defined by two equations which specify that the lower left entry in the image of γ and the upper right entry in the image of δ are equal to zero, and an additional equation which specifies that the upper right entry in the image of γ equals one. It follows from the construction that the restriction $\mathrm{t} : \mathfrak{C}(\Gamma) \to \mathfrak{X}(\Gamma)$ is generically 4–to–1, and corresponds to the action of the Kleinian four group on the set of possible bases for the normal form (4.5). The involutions $(s, t, u) \to (s^{-1}, t^{-1}, u)$ and $(s, t, u) \to (s, t^{-1}, u + (s - s^{-1})(t - t^{-1}))$ generate this group.

$\mathfrak{C}(\Gamma)$ may be thought of as a variety in $(\mathbb{C} - \{0\})^2 \times \mathbb{C}$, and the intersection of $\mathfrak{C}(\Gamma)$ with $\mathfrak{Red}(\Gamma)$ corresponds to the intersection with the hyperplane $\{u = 0\}$. For any reducible non–abelian representation $\sigma \in \mathfrak{R}(\Gamma)$, there is a representation $\rho \in \mathfrak{C}(\Gamma)$ with $u = 0$ such that $\chi_\sigma = \chi_\rho$. Moreover, σ is conjugate to a representation in $\mathfrak{C}(\Gamma)$ unless $\sigma(\delta)$ is a non–trivial parabolic.

Consider the "conjugation" map $c : \mathfrak{C}(\Gamma) \times SL_2(\mathbb{C}) \to \mathfrak{R}(\Gamma)$ defined by $c(\rho, X) = X^{-1}\rho X$. This is a regular map, and we have $\overline{c(\mathfrak{C}(\Gamma) \times SL_2(\mathbb{C}))} = \mathfrak{R}^i(\Gamma)$. Furthermore, if $V \subset \mathfrak{C}(\Gamma)$ is an irreducible component, then $\overline{c(V)} \subset \mathfrak{R}(\Gamma)$ is irreducible. It is convenient to work with $\mathfrak{C}(\Gamma) \subset \mathfrak{R}^i(\Gamma)$ in some applications of Culler–Shalen theory, and we therefore summarise its properties:

Lemma 15. [32] *Let $\Gamma = \langle \gamma, \delta \mid r_i(\gamma, \delta) = 1 \rangle$ be a 2–generator group. The variety $\mathfrak{C}(\Gamma)$ defined in $(\mathbb{C} - \{0\})^2 \times \mathbb{C}$ by (4.5) and the polynomial equations arising from $r_i(\rho(\gamma), \rho(\delta)) = E$ defines a 4–to–1 (possibly branched) cover of $\mathfrak{X}^i(\Gamma)$.*

We remark that $\mathfrak{C}(\Gamma)$ is defined up to polynomial isomorphism once an unordered generating set has been chosen.

4.8. **The Character Variety of the free group of rank two.** Let F_2 be the free group of rank 2 on the generators γ and δ and let $\rho : F_2 \to SL_2(\mathbb{C})$ be a representation. We know that $\mathfrak{X}(F_2)$ is the locus of the points $(\operatorname{tr}\rho(\gamma), \operatorname{tr}\rho(\delta), \operatorname{tr}\rho(\gamma\delta)) \in \mathbb{C}^3$ as ρ ranges over $\mathfrak{R}(F_2)$.

Proposition 16. [2] $\mathfrak{X}(F_2) = \mathbb{C}^3$

Proof. The variety $\mathfrak{R}(F_2)$ is parameterised by points in a subvariety of $\mathbb{C}^8$, more precisely, if ρ is a representation and

$$\rho(\gamma) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad \rho(\delta) = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

we then can identify ρ with the point

$$(\rho(\gamma), \rho(\delta)) = (a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2).$$

This in fact shows that $\mathfrak{R}(F_2) \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. We now consider the map

$$\varphi : \mathfrak{R}(F_2) \to \mathbb{C}^3 \quad \text{defined by} \quad \rho \to (\operatorname{tr}\rho(\gamma), \operatorname{tr}\rho(\delta), \operatorname{tr}\rho(\gamma\delta)).$$

It can be verified by direct computation that φ is a morphism $\mathbb{C}^8 \to \mathbb{C}^3$, and the fact we want to establish is that it is surjective.

Let $(x, y, z) \in \mathbb{C}^3$ and consider the quadratic equations

$$s^2 - xs + 1 = 0 \quad \text{and} \quad t^2 - yt + 1 = 0$$

for s and t. We have

$$s + s^{-1} = x \quad \text{and} \quad t + t^{-1} = y.$$

Put

$$G = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix},$$

with u still to be determined. We have $\operatorname{tr} G = x$, $\operatorname{tr} D = y$ and $\det G = 1 = \det D$. Computing the product, we get

$$GD = \begin{pmatrix} st + u & t^{-1} \\ s^{-1}u & s^{-1}t^{-1} \end{pmatrix}.$$

Observing that $\operatorname{tr} GD = st + u + s^{-1}t^{-1}$, we put $u = z - st - s^{-1}t^{-1}$ and get $\operatorname{tr} GD = z$. Let ρ be the representation defined by $\rho(\gamma) = G$ and $\rho(\delta) = D$, then ρ corresponds to the point $(x, y, z) \in \mathbb{C}^3$. $\square$

4.9. **Notation.** If Γ is the fundamental group of a topological space M, then we also write $\mathfrak{R}(M)$ and $\mathfrak{X}(M)$ instead of $\mathfrak{R}(\Gamma)$ and $\mathfrak{X}(\Gamma)$ respectively.

Example (m137). *Let N be the manifold* m137 *in the cusped census of* `SnapPea`*. It is hyperbolic, one–cusped and of volume approximately* 3.6638*. We can obtain N by* 0 *Dehn surgery on either component of the link 7_1^2 in S^3, which implies that N is the complement of a knot in $S^2 \times S^1$. In figure 15, we see a thrice punctured disc bounded by one of the link components. If we perform* 0 *surgery on this component, we obtain a thrice punctured sphere S in N. We may think of this sphere as the intersection of $S^2 \times z$ with N in $S^2 \times S^1$. The surface S will play a role towards the end of these notes.*

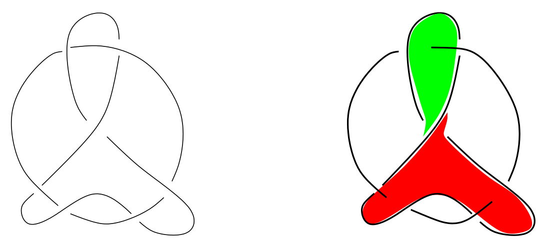


FIGURE 15. The link 7_1^2 and the thrice punctured disc

`SnapPea` *computes the fundamental group and peripheral system as follows:*

$$\pi_1(N) =< a, b \mid a^3b^2a^{-1}b^{-3}a^{-1}b^2 >, \quad \{\mathcal{M}, \mathcal{L}\} = \{a^{-1}b^{-1}, a^{-1}b^2a^4b^2\}.$$

Note that the meridian is nullhomologous. We may change the presentation into a more convenient form, where the meridian is one of the generators:

$$\pi_1(N) =< \mathcal{M}, b \mid b^{-1}\mathcal{M}^{-1}b^{-1}\mathcal{M}^{-1}b^2\mathcal{M} = \mathcal{M}b^{-2}\mathcal{M}^{-1}b^2 >,$$
$$\{\mathcal{M}, \mathcal{L}\} = \{\mathcal{M}, b^2\mathcal{M}^{-1}b^{-3}\mathcal{M}^{-1}b^2\}.$$

It turns out that there are no reducible metabelian representations, so the component containing abelian representations – which is isomorphic to $SL_2(\mathbb{C})$, generated by the image of b – is disjoint from any component containing an irreducible representation. In fact, there is only one such component, and we compute $\mathfrak{C}(N)$ as follows:

$$\rho(\mathcal{M}) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \qquad \textit{and} \qquad \rho(b) = \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix},$$

where

$$y = -\frac{1 - m^3 + x^2 - mx^2 + m^2x^2 - m^3x^2 + m^4x^2 - mx^4 + m^4x^4}{m(1 + m + m^2)x(1 + x^2)}$$

and m and x are subject to the equation

$$\begin{aligned}0 = f(m, x) = (1 - 2m^3 + m^6)(1 + x^8)& \\ +(3 - m + m^2 - 6m^3 + m^4 - m^5 + 3m^6)(x^2 + x^6)& \\ + (4 - 2m + 2m^2 - 9m^3 + 2m^4 - 2m^5 + 4m^6)x^4&\end{aligned}$$

This parameterisation is in fact a $4 : 1$ cover of the $SL_2(\mathbb{C})$–character variety, where the covering corresponds to quotiening by the Kleinian four group generated by $(m, x) \to (m^{-1}, x^{-1})$ and $(m, x) \to (m^{-1}, x)$.

Note that if $f(m, x) = 0$, then $f(m^{-1}, x) = f(m, x^{-1}) = f(m^{-1}, x^{-1}) = 0$. This implies that we can write f as a polynomial function in $x + x^{-1}$ and $m + m^{-1}$. The character variety is then defined by $s = \operatorname{tr} \rho(\mathcal{M})$ and $t = \operatorname{tr} \rho(b)$, whilst $\operatorname{tr} \rho(\mathcal{M}b)$ is some function in these variables. We obtain:

$$1 = (4 + 4s - s^2 - s^3)t^2 - (2 + 3s - s^3)t^4,$$

and we have already observed that the line $s = 2$ parameterises the abelian representations. We may also verify that there is no point of intersection between these two components.

Exercise 16 (m004)**.** *In the cusped census of* `SnapPea`*, the complement of the figure eight knot $\mathcal{F}$ is the manifold* m004*. You can use this to get a presentation of the fundamental group, or you can compute a Wirtinger presentation from the knot projection in Figure 8. Compute the representation variety up to conjugacy, and hence give a defining equation for its character variety. Show that the component $\mathfrak{X}_0(\mathcal{F})$ containing irreducible representations is birationally equivalent to a torus with two punctures.*

4.10. **Tautological representation.** Let Γ be a finitely generated group, and let V be an irreducible subvariety of $\mathfrak{X}(\Gamma)$. By [14], there is an irreducible subvariety $R_V \subset \mathfrak{R}(\Gamma)$ such that $\mathrm{t}(R_V) = V$. The function field $F = \mathbb{C}(R_V)$ contains $K =$

$\mathbb{C}(V)$. We now obtain the *tautological representation* $\mathcal{P} : \Gamma \to SL_2(F)$ defined by

$$\mathcal{P}(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where the identity } \rho(\gamma) = \begin{pmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{pmatrix} \text{ for all } \rho \in R_V$$

determines the functions $a, b, c, d \in F$. One can think of this construction as restricting the coordinate functions to R_V.

For each $\gamma \in \Gamma$ define $I_\gamma = \operatorname{tr} \mathcal{P}(\gamma) \in K \subset F$. It follows from the definition of the tautological representation that $I_\gamma(\rho) = \operatorname{tr} \rho(\gamma) \in \mathbb{C}$ for all $\rho \in R_V$, and hence we have a function $I_\gamma : R_V \to \mathbb{C}$. Since $I_\gamma \in K$, it may also be thought of as a function on V.

More generally, for each $\gamma \in \Gamma$, we define a function $I_\gamma : \mathfrak{R}(\Gamma) \to \mathbb{C}$ by $I_\gamma(\rho) = \operatorname{tr} \rho(\gamma)$. Then I_γ is an element in the coordinate ring $\mathbb{C}[\mathfrak{R}(\Gamma)]$.

4.11. **Projective representations.** There is also a notion of character variety arising from representations into $PSL_2(\mathbb{C})$, and the relevant objects are denoted by placing a bar over the previous notation. The natural map $\mathrm{q} : \mathfrak{X}(\Gamma) \to \overline{\mathfrak{X}}(\Gamma)$ is finite–to–one, but in general not onto. It is the quotient map corresponding to the $H^1(\Gamma; \mathbb{Z}_2)$–action on $\mathfrak{X}(\Gamma)$, where $H^1(\Gamma; \mathbb{Z}_2) = \operatorname{Hom}(\Gamma, \mathbb{Z}_2)$. This action is not free in general.

Consider for example the representation of $\mathbb{Z} \oplus \mathbb{Z}$ into $PSL_2(\mathbb{C})$ generated by the images of

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In $PSL_2(\mathbb{C})$, this is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, but any lift to $SL_2(\mathbb{C})$ is isomorphic to the quaternion group $\mathbb{Q}_8$. In general, central extensions of $PSL_2(\mathbb{C})$ and Γ by $\mathbb{Z}_2$ must be studied in order to decide whether a representation into $PSL_2(\mathbb{C})$ lifts to a representation into $SL_2(\mathbb{C})$. In [5], Boyer and Zhang give examples of (non–hyperbolic) 3–manifolds M where $\dim_{\mathbb{C}} \mathfrak{X}(M) = 0$, but $\dim_{\mathbb{C}} \overline{\mathfrak{X}}(M) = 1$.

Exercise 17. *Let $\mathfrak{k}$ be a knot in S^3. Show that every representation of $\pi_1(S^3 - \mathfrak{k})$ into $PSL_2(\mathbb{C})$ lifts to a representation into $SL_2(\mathbb{C})$.*

As with the $SL_2(\mathbb{C})$–character variety, there is a surjective "quotient" map $\bar{\mathrm{t}} : \overline{\mathfrak{R}}(\Gamma) \to \overline{\mathfrak{X}}(\Gamma)$, which is constant on conjugacy classes, and with the property that if $\bar{\rho}$ is an irreducible representation, then $\bar{\mathrm{t}}^{-1}(\bar{\mathrm{t}}(\bar{\rho}))$ is the orbit of $\bar{\rho}$ under conjugation.

Again, a representation is irreducible if it is not conjugate to a representation by upper triangular matrices.

A statement similar to the description of $SL_2(\mathbb{C})$–characters is true. Let $\mathfrak{F}_n$ be the free group on $\xi_1, \ldots, \xi_n$, and let $y_1, \ldots, y_m$ be the $\frac{n(n^2+5)}{6}$ elements of $\mathfrak{F}_n$ corresponding to the single generators and ordered double and triple products thereof.

Lemma 17. [5] *Suppose that* Γ *is generated by* $\gamma_1, \ldots, \gamma_n$ *and that* $\overline{\rho}, \overline{\rho}' \in \overline{\mathfrak{R}}(\Gamma)$. *Choose matrices* $A_1, \ldots, A_n, B_1, \ldots, B_n \in SL_2(\mathbb{C})$ *satisfying* $\overline{\rho}(\gamma_i) = \pm A_i$ *and* $\overline{\rho}'(\gamma_i) = \pm B_i$ *for each* i. *Define* $\rho, \rho' \in \mathfrak{R}(\mathfrak{F}_n)$ *by requiring that* $\rho(\xi_i) = A_i$ *and* $\rho'(\xi_i) = B_i$ *for each* $i \in \{1, \ldots, n\}$. *Then* $\chi_{\overline{\rho}} = \chi_{\overline{\rho}'}$ *if and only if there is a homomorphism* $\epsilon \in \operatorname{Hom}(\mathfrak{F}_n, \{\pm 1\})$ *for which* $\operatorname{tr} \rho'(y_j) = \epsilon(y_j) \operatorname{tr} \rho(y_j)$ *for each* $j \in \{1, \ldots, m\}$.

Boyer and Zhang introduce the tautological representation associated to an irreducible component of $\overline{\mathfrak{R}}(\Gamma)$ in [5], which is analogous to the $SL_2(\mathbb{C})$–version, and will be denoted by $\overline{\mathcal{P}}$.

Example (m137). *We continue the previous example. The involutions* $x \to -x$ *and therefore* $t \to -t$ *correspond to the central extension since there is a unique homomorphism onto* $\mathbb{Z}_2$. *Substituting* $x^2 = z$ *in* f *therefore parametrises a* $4:1$ *cover of the* $PSL_2(\mathbb{C})$*–character variety. The latter is given as:*

$$\overline{\mathfrak{X}}_0(N) = \{(s,t) \mid 1 = (4 + 4s - s^2 - s^3)t - (2 + 3s - s^3)t^2\},$$

along with the component $\{s = 2\}$.

Exercise 18 (m004). *We continue the study of the figure eight knot complement. Determine the isomorphism type of the reducible metabelian representation corresponding to the intersection of* $X_0(\mathcal{F})$ *with the set of reducible representations. Also determine its isomorphism type if we descend to* $PSL_2(\mathbb{C})$. *What geometric structure does it correspond to?*

4.12. **Dehn surgery component.** If a 3–manifold M admits a complete hyperbolic structure of finite volume, then there is a discrete and faithful representation $\pi_1(M) \to PSL_2(\mathbb{C})$. This representation is necessarily irreducible, as hyperbolic geometry otherwise implies that M has infinite volume.

If M is not compact, then a *compact core* of M is a compact manifold $\overline{M}$ such that M is homeomorphic to the interior of $\overline{M}$. We will rely heavily on the following result:

Theorem 18 (Thurston)**.** [31, 27] *Let M be a complete hyperbolic 3–manifold of finite volume with h cusps, and let $\overline{\rho}_0 : \pi_1(M) \to PSL_2(\mathbb{C})$ be a discrete and faithful representation associated to the complete hyperbolic structure. Then $\overline{\rho}_0$ admits a lift ρ_0 into $SL_2(\mathbb{C})$ which is still discrete and faithful. The (unique) irreducible component X_0 in the $SL_2(\mathbb{C})$–character variety containing the character χ_0 of ρ_0 has (complex) dimension h.*

Furthermore, if $T_1, \dots, T_h$ are the boundary tori of a compact core of M, and if γ_i is a non–trivial element in $\pi_1(M)$ which is carried by T_i, then $\chi_0(\gamma_i) = \pm 2$ and χ_0 is an isolated point of the set

$$X^* = \{\chi \in X_0 \mid I_{\gamma_1}^2 = \dots = I_{\gamma_h}^2 = 4\}.$$

The respective irreducible components containing the so–called *complete representations* ρ_0 and $\overline{\rho}_0$ are denoted by $\mathfrak{R}_0(M)$ and $\overline{\mathfrak{R}}_0(M)$ respectively. In particular, $\mathrm{t}(\mathfrak{R}_0) = \mathfrak{X}_0$ and $\overline{\mathrm{t}}(\overline{\mathfrak{R}}_0) = \overline{\mathfrak{X}}_0$ are called the respective *Dehn surgery components* of the character varieties of M, since the holonomy representations of hyperbolic manifolds or orbifolds obtained by performing high order Dehn surgeries on M are near $\overline{\rho}_0$ (see [30]).

Example (m137)**.** *We have already noted that the only reducible representations are abelian, with trace of $\mathcal{M}$ equal to two, and that there is no point of intersection with the component containing irreducible representations. The discrete and faithful representation (unique up to conjugacy, complex conjugation and the sign of x) corresponds therefore to solutions of $f(-1, x) = 0$. We have*

$$f(-1, x) = (2 - 2x + 5x^2 - 2x^3 + 2x^4)(2 + 2x + 5x^2 + 2x^3 + 2x^4),$$

so up to everything we have listed above, we get one solution for x. Conversely, all solutions of the above equation parameterise a discrete and faithful representation, as given below:

$$\rho_0(\mathcal{M}) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad \rho_0(\mathcal{L}) = \pm \begin{pmatrix} 1 & \frac{1}{4}(1 \pm 9i) \\ 0 & 1 \end{pmatrix}, \qquad \rho_0(b) = \begin{pmatrix} x & 0 \\ 2 & x^{-1} \end{pmatrix},$$

where the sign and the complex conjugate of $\rho_0(\mathcal{L})$ *depend on the choice of* x*. Note that* $\operatorname{tr}\rho_0(b) \equiv i \mod \mathbb{Q}$ *for each of these solutions, which is confirmed by the invariant trace field computed by* `snap` *as* z^2+1*.*

Deformations of the hyperbolic structure can now be studied through the Dehn surgery component $\mathfrak{X}_0$.The introduction focused on the effect of the deformation on a boundary torus of the manifold. In the case of one cusp, there is strong evidence that this is a good approach:

Theorem 19. [16] *Let* M *be a finite–volume hyperbolic 3–manifold with one cusp. Let* X_0 *be a component of the* $SL_2(\mathbb{C})$*–character variety of* M *which contains the character of a discrete and faithful representation. The inclusion* $i : \partial M \to M$ *induces a regular map* $i_* : X_0 \to X(\partial M)$*. This map has degree onto its image at most* $|H^1(M, \mathbb{Z}_2)|/2$*, where* $|\cdot|$ *denotes the number of elements. In particular, if* $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$ *then* i_* *is a birational isomorphism onto its image.*

The proof uses facts about deformations of hyperbolic structures to show that the map between the respective $PSL_2(\mathbb{C})$–character varieties is of degree one, and hence a birational isomorphism. In the case of $SL_2(\mathbb{C})$–character varieties, we have such a birational isomorphism in particular for all knot complements in homology 3–spheres.

5. Surfaces and Ideal Points

This section describes how Culler and Shalen associate essential surfaces to ideal points of curves in the character variety. The ingredients are as follows:

1. A curve in $X(M)$ yields a field F with a discrete valuation at each ideal point and a representation $\mathcal{P} : \pi_1(M) \to SL_2(F)$.
2. $SL_2(F)$ acts on a tree $\mathcal{T}_v$ and using $\mathcal{P}$ we can pull back to an action of $\pi_1(M)$ which is non–trivial and without inversions.
3. A non–trivial action without inversions on a trees gives essential surfaces.

We have already seen the third step, so let us look at the preceding steps continuing in reversed order.

Reference. [27]*, Sections 5.4-5.6 and 3.6-3.9.*

5.1. **Serre's tree for $SL_2(F)$.** Let F be a field and F^* be its multiplicative group. A map

$$v : F \to \mathbb{Z} \cup \{\infty\}$$

is called a *discrete valuation* if

1. $v(0) = \infty$ where $z + \infty = \infty = \infty + z$ for $z \in \mathbb{Z}$,
2. $v : F^* \to \mathbb{Z}$ is an epimorphism onto $\mathbb{Z}$ as an additive group, i.e. $v(ab) = v(a) + v(b)$,
3. $v(a + b) \geq min\{v(a), v(b)\}$ where $\infty \geq z$ for all $z \in \mathbb{Z}$.

We then define the *valuation ring* $\mathcal{O} = \{a \in F \mid v(a) \geq 0\}$.

Exercise 19. *$\mathcal{O}$ is a subring of F. Moreover, if $0 \neq a \in F$, then either $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$. Furthermore, $\mathcal{O}$ is a principal ideal domain and the non–units of $\mathcal{O}$ form a maximal ideal $\mathcal{M}$. This maximal ideal is generated by a uniformiser π, i.e. an element with $v(\pi) = 1$.*

Example. *Let us define the p–adic valuation on $\mathbb{Q}$ for a prime p. For any non–zero integer $z \in \mathbb{Z}$, let $v_p(z)$ be the exponent of p in a prime factorisation of $|z|$, and let $v_p(0) = \infty$. Thus, $v_p(z) \geq 0$ for all $z \in \mathbb{Z}$. We now extend the valuation to $\mathbb{Q}$ by setting $v_p(\frac{x}{y}) = v_p(x) - v_p(y)$. It can be verified that this valuation satisfies all the above properties. What are the elements of the valuation ring?*

Exercise 20. *Note that if we apply the p–adic valuations to a fixed integer varying p, we recover the prime factorisation of its absolute value. Use this fact to show that the k–th root of an integer is either irrational or an integer.*

Let V be a 2–dimensional left vector space over F, so it is a left $\mathcal{O}$–module. An $\mathcal{O}$–lattice L of V is any $\mathcal{O}$–sub-module of the form

$$L = \mathcal{O}x + \mathcal{O}y =< x, y >,$$

where $x, y \in V$ are linearly independent over F. The group F^* acts by left multiplication on the set of these $\mathcal{O}$–lattices:

$$aL = \mathcal{O}ax + \mathcal{O}ay,$$

and orbits of this action give equivalence classes,

$$L_1 \sim L_2 \iff \exists a \in F \text{ such that } aL_1 = L_2.$$

Denote the equivalence class of L by $\Lambda = [L]$.

Lemma 20. *Given L_1, L_2 there is $m \in \mathbb{Z}$ such that $\pi^m L_2 \subset L_1$.*

Proof. Pick bases $< x_i, y_i >$ of L_i. Since they span V as a vector space respectively, there are $\alpha, \beta \in F$ such that $x_2 = \alpha x_1 + \beta y_1$. Since $x_2 \neq 0$, we have $m_0 = \min\{v(\alpha), v(\beta)\} \in \mathbb{Z}$. For all $m \geq m_0$ we have $v(\pi^m \alpha) = m + v(\alpha) \geq 0$. Hence $\pi^m \alpha \in \mathcal{O}$. For the same reason, we have $\pi^m \beta \in \mathcal{O}$, and hence $\pi^m x_2 \in \mathcal{O}x_1 + \mathcal{O}y_1 = L_1$. If we take m sufficiently large then by the same argument $\pi^m y_2 \in L_1$ and hence $\pi^m L_2 \in L_1$. □

The "basis theorem" for submodules of finitely generated free modules over principal ideal domains asserts that if $\pi^m L_2 \subset L_1$, then there is a basis $\{x, y\}$ for L_1 such that $\{\pi^f x, \pi^g y\}$ is a basis for $\pi^m L_2$ for some $f, g \in \mathbb{Z}$.

Lemma 21. *$|f - g|$ is well defined for the equivalence classes Λ_1, Λ_2.*

Exercise 21. *Prove the above lemma.*

We can now define a distance function $d(\Lambda_1, \Lambda_2) = |f - g|$ and call Λ_1 and Λ_2 adjacent if $d(\Lambda_1, \Lambda_2) = 1$. Note that if $d(\Lambda_1, \Lambda_2) = 1$, we can find representatives and a basis such that $L_1 =< x, y >$ and $L_2 =< \pi x, y >$. Hence $\pi L_1 \subsetneqq L_2 \subsetneqq L_1$.

We construct a graph $\mathcal{T}_v = (F, v)$ from the following sets of vertices and edges:

$$\mathcal{V} = \{\Lambda \mid \Lambda = [L],\ L \text{ is an } \mathcal{O}\text{–lattice in } V\}$$

$$\mathcal{E} = \{(\Lambda_1, \Lambda_2) \mid \Lambda_1, \Lambda_2 \text{ are adjacent}\}$$

We give orientations on edges by putting $(\Lambda_2, \Lambda_1) = \overline{(\Lambda_1, \Lambda_2)}$, and claim that $\mathcal{T}_v$ is a tree.

Exercise 22. *Show that $\mathcal{T}_v$ is a tree. Connectedness is quite easy, then show inductively that every closed path contains a backtracking, and hence that $\mathcal{T}_v$ is simply connected.*

The group $GL(V)$ acts on lattices by

$$AL = \mathcal{O}Ax + \mathcal{O}Ay,$$

where $A \in GL(V)$ and $L = \mathcal{O}x + \mathcal{O}y$, and this action on lattices is well–defined for equivalence classes of lattices, hence giving an action of $GL(V)$ on $\mathcal{T}_v$.

Furthermore, $GL(V)$ acts by isometries: if $d(\Lambda_1, \Lambda_2) = n$, then there are lattices and bases $L_1 =< x, y >$ and $L_2 =< \pi^n x, y >$. Hence $AL_1 =< Ax, Ay >$ and $AL_2 =< \pi^n Ax, Ay >$. So $d(A\Lambda_1, A\Lambda_2) = n$.

Let us think of V as F^2 and restrict the action to $SL_2(F)$. We have:

Lemma 22. [25, 27]

1. *$SL_2(F)$ acts on $\mathcal{T}_v$ simplicially, without inversions.*
2. *For any vertex Λ of $\mathcal{T}_v$, $Stab(\Lambda)$ is conjugate to $SL_2(\mathcal{O})$.*

Exercise 23. *Show that these properties hold. The second part can be obtained by going through the following four steps:*

- $Stab(\Lambda) = Stab(L)$
- $Stab(AL) = AStab(L)A^{-1}$
- $Stab(\mathcal{O}^2) = SL_2(\mathcal{O})$
- $\forall L \quad \exists A$ *such that* $A\mathcal{O}^2 = L$

This is everything we need for the moment – we may want to obtain more information about the action later.

5.2. **Some algebraic geometry.** Two varieties V, W are *birationally equivalent* if there are rational maps $\varphi : V \to W$ and $\psi : W \to V$ such that $\varphi(V)$ is dense in W, $\psi(W)$ is dense in V, and $\varphi \circ \psi = 1$ as well as $\psi \circ \varphi = 1$ where defined.

The map $J : \mathbb{C}^m \to \mathbb{C}P^m$ defined by $J(z_1, \ldots, z_m) = [1, z_1, \ldots, z_m]$ is a diffeomorphism, and if $V \subset \mathbb{C}^m$ is a variety, then $\overline{J(V)}$ is termed a *projective completion* of V. In case that V is a 1–dimensional irreducible variety, there is a unique non–singular projective variety $\tilde{V}$ which is birationally equivalent to $\overline{J(V)}$. $\tilde{V}$ is called the *smooth projective completion* of V, and the *ideal points* of $\tilde{V}$ are the points of $\tilde{V}$ corresponding to $\overline{J(V)} - J(V)$ under the birational equivalence. Moreover, the function fields of V and $\tilde{V}$ are isomorphic.

5.3. **Ideal points and valuations.** Let $C \subset \mathfrak{X}(M)$ be a curve, i.e. a 1–dimensional irreducible subvariety, and denote its smooth projective completion by $\tilde{C}$. We will refer to the ideal points of $\tilde{C}$ also as ideal points of C. The function fields of C and $\tilde{C}$ are isomorphic. Denote them by K. Any ideal point ξ of $\tilde{C}$ determines a (normalised, discrete, rank 1) valuation ord_ξ of K, by

$$ord_\xi(f) = \begin{cases} k & \text{if } f \text{ has a zero of order } k \text{ at } \xi \\ \infty & \text{if } f = 0 \\ -k & \text{if } f \text{ has a pole of order } k \text{ at } \xi \end{cases}$$

Note that $ord_\xi(z) = 0$ for all non-zero constant functions $z \in \mathbb{C}$. In the language of algebraic geometry, the valuation ring $\{f \in K \mid ord_\xi(f) \geq 0\}$ of ord_ξ is the local ring at ξ.

5.4. **The action.** Let $C \subset \mathfrak{X}(M)$ be a curve. We want a representation into SL_2 of a field with a discrete valuation. A good candidate is the *tautological representation* $\mathcal{P} : \pi_1(M) \to SL_2(F)$ of Subsection 4.10. Recall that here, $F = \mathbb{C}(R_C)$ is the function field of an irreducible subvariety $R_C \subset \mathfrak{R}(M)$ with the property that $t(R_C) = C$, and F is a finitely generated extension of $K = \mathbb{C}(C)$. In order to get Serre's tree, we need a suitable valuation on F. To this end, we use the following extension theorem for valuations:

Lemma 23 (1.1 in [1])**.** *Let F be a finitely generated extension of a field K and let $w : K^* \to \mathbb{Z}$ be a valuation of K. Then there exist an integer $d > 0$ and a valuation $v : F^* \to \mathbb{Z}$ such that $v|_{K^*} = dw$.*

Note that $\mathcal{O}_w \subset \mathcal{O}_v$ and $\mathcal{O}_w^* \subset \mathcal{O}_v^*$. In particular, the uniformizer in $\mathcal{O}_w$ is a d-th power of a uniformiser in $\mathcal{O}_v$.

The condition of the theorem is satisfied since:

$$\mathbb{C}^{4n} \supset \mathbb{C}[z_i]/J = K \supset F = \mathbb{C}[y_i]/J \subset \mathbb{C}^m \text{ where } m = n + \binom{n}{2} + \binom{n}{3},$$

where the y_i are contained in $span\{z_i\}$, and we can think of them as trace functions of generators and ordered double and triple products of generators.

So if we apply the lemma to F, K and the valuation ord_x on K, we obtain a valuation v_x on F. We fix our ideal point and write $v_x = v$.

With the valuation v, we can associate a tree $\mathcal{T}_v$ on which $SL_2(F)$ acts in the way described earlier. The tautological representation $\mathcal{P} : \pi_1(M) \to SL_2(F)$ can be used to pull the action of $SL_2(F)$ back to an action of $\pi_1(M)$ on $\mathcal{T}_v$, and we write $\mathcal{P}(\gamma) \cdot \Lambda = \gamma\Lambda$.

This action is simplicial and without inversions by the way $SL_2(F)$ acts. Is it non–trivial?

5.5. **Properties of the action.** For each $\gamma \in \pi_1(M)$ we have the function $I_\gamma : R_C \to \mathbb{C}$, defined by $I_\gamma = \operatorname{tr} \mathcal{P}(\gamma) \in K \subset F$. Since $I_\gamma \in K$, we may think of it as a function on C. At an ideal point ξ, it has a well–defined value $I_\gamma(\xi) \in \mathbb{C} \cup \{\infty\}$.

Lemma 24. [14] *For any $\gamma \in \pi_1(M)$ the following are equivalent:*

1. *$I_\gamma(\xi) \in \mathbb{C}$, i.e. I_γ does not have a pole at ξ.*
2. *Some vertex of $\mathcal{T}_v$ is fixed by γ, where $v|_{K^*} = d \cdot ord_\xi$.*

Proof. Note that $I_\gamma(x) \in \mathbb{C} \Leftrightarrow v(I_\gamma) \geq 0 \Leftrightarrow w(I_\gamma) \geq 0 \Leftrightarrow I_\gamma = \operatorname{tr} \mathcal{P}(\gamma) \in \mathcal{O}$.

($2 \Rightarrow 1$) If a vertex is fixed, then $\mathcal{P}(\gamma)$ is in the stabilizer of the vertex, hence conjugate to an element in $SL_2(\mathcal{O})$. This gives $\operatorname{tr} \mathcal{P}(\gamma) \in \mathcal{O}$.

($1 \Rightarrow 2$) Suppose $\operatorname{tr} \mathcal{P}(\gamma) \in \mathcal{O}$. If $\mathcal{P}(\gamma) = \pm E$, then γ acts trivially on the tree. Hence assume that $\mathcal{P}(\gamma) \neq \pm E$. Since $\mathcal{P}(\gamma)$ is in the special linear group, there exists a vector $e \in K^2$ such that e and $\mathcal{P}(\gamma)e =: f$ are linearly independent. With respect to the basis $\{e, f\}$, $\mathcal{P}(\gamma)$ has the form $\begin{pmatrix} 0 & c \\ 1 & d \end{pmatrix} = A^{-1}\mathcal{P}(\gamma)A$ for some $A \in GL_2(F)$. Now $1 = \det \mathcal{P}(\gamma) = -c$ and $d = \operatorname{tr} \mathcal{P}(\gamma) \in \mathcal{O}$. Thus $\mathcal{P}(\gamma) \in ASL_2(\mathcal{O})A^{-1}$ and fixes a vertex. $\square$

The previous lemma can be generalised as follows:

Lemma 25. *A finitely generated subgroup Γ_1 of $\pi_1(M)$ fixes a vertex if and only if I_γ takes a finite value at x for all $\gamma \in \Gamma_1$.*

Proof. The "only if" follows directly from the first property in Lemma 24, the "if" follows from the first property along with Lemma 2. □

Recall that the action of $\pi_1(M)$ on a tree is called *non–trivial* if $\pi_1(M)$ is not contained in the stabiliser of a vertex.

Proposition 26. [14] *The action of $\pi_1(M)$ on $\mathcal{T}_v$ is non–trivial.*

Proof. Assume that the action is trivial, then $I_\gamma(x) \in \mathbb{C}$ for all $\gamma \in \pi_1(M)$. But each of our coordinate functions is in particular of the form I_γ for some $\gamma \in \pi_1(M)$ and at least one of them has to have a pole at an ideal point. This creates a contradiction. □

Thus, associated to each ideal point of the character variety, there is a non–trivial action without inversions on a tree, and hence an essential surface. So whenever there is a curve in $\mathfrak{X}(M)$ we find an essential surface in M. We have now covered most of the basic background one needs in order to look at applications.

6. The Weak Neuwirth Conjecture

Let M be a manifold with boundary a single torus. We call the unoriented isotopy class of a non–trivial simple closed curve in ∂M its slope.

Weak Neuwith Conjecture. *Let M be a compact, orientable, irreducible 3–manifold whose boundary is a single torus. Then either*

1. *M is a solid torus, or*
2. *M contains an essential separating annulus, or*
3. *M contains and essential non–separating torus, or*
4. *M has at least two boundary slopes.*

Proof. ([27], Section 6) We shall deal with the fourth statement in the equivalent form: If s is any slope, there is an essential surface which has non–empty boundary and has a boundary slope different from s.

6.1. **The case when M is hyperbolic.** The proof is easiest if the interior of M has a finite volume hyperbolic structure. Let s be represented by a curve $\tilde{\gamma}$ on ∂M and denote the corresponding element in the fundamental group of M by γ. Consider a component X_0 containing a discrete and faithful character. We know X_0 is a curve on which the function I_γ is non–constant. So it must have a pole at an ideal point x of X_0. So we get a tree and an action and a surface S. We need to show that the surface has non–empty boundary with slope different from s.

So assume that S is closed or that its boundary slope is s. Then $\tilde{\gamma}$ is isotopic to a simple closed curve in $M - S$, and hence $\gamma \in \mathrm{im}(\pi_1(M_i) \to \pi_1(M))$ for some component M_i of $M - S$. But this implies that γ fixes a vertex and hence I_γ does not have a pole at x, contradicting our choice of x.

6.2. **The other cases.** In order to deal with the other cases, we need Thurston's Hyperbolisation Theorem:

Thurston's Hyperbolisation Theorem. *If M is a compact, connected, orientable irreducible 3–manifold whose boundary consists of one or more tori, then its interior admits a hyperbolic structure of finite volume unless M either contains an essential torus or it is a Seifert fibered space.*

We are given a compact, connected, orientable irreducible 3–manifold with one torus component in its boundary. Split M along some maximal disjoint system of essential tori, then by the above, each resulting component is either Seifert fibered or hyperbolic. Let M_0 be the component containing the boundary component T_1 of M, and let the other components of ∂M_0 be the tori $T_2, \ldots, T_n$. We may assume that $M_0 \ncong T \times [0,1]$, since otherwise a system of fewer tori would have sufficed. If M_0 is a Seifert fibered space, then either it is a solid torus, and the first case of the theorem holds, or M_0 contains a separating annulus. If the annulus separates M, we are in the second case, and if the annulus does not separate M, we are in the third case. These "topological cases" are fun to go through in more detail, but don't involve the techniques we wish to talk about. So let us now assume that the interior of M_0 admits a hyperbolic structure of finite volume. Since there are n cusps, we know that an irreducible component X_0 containing a discrete and faithful character has dimension n.

Lemma 27. *There is a curve $Y_0 \subset X_0$ such that*

1. $\forall i = 2, \ldots, n$ $\forall \gamma \in \operatorname{im}(\pi_1(T_i) \to \pi_1(M_0)) =: \Gamma_i$ *the functions* $I_\gamma|_{Y_0}$ *are constant equal to* ± 2, *and*
2. *for all non–trivial* $\gamma \in \operatorname{im}(\pi_1(T_1) \to \pi_1(M_0)) =: \Gamma_1$ *the functions* $I_\gamma|_{Y_0}$ *are non–constant.*

Proof. Indeed, we define the set $Y_0 = X_0 \cap \{I^2_{\gamma_2} = \ldots = I^2_{\gamma_n} = 4\}$ with fixed non–trivial elements $\gamma_i \in \Gamma_i$. Then $\dim Y_0 \geq 1$. Also, $\chi_0 \in Y_0$, and since χ_0 was isolated in X^*, we have $\dim Y_0 = 1$. This proves the second claim.

The first claim is true since any element which commutes with a parabolic element is again parabolic, and the images of the γ_i are generically non–trivial since χ_0 is faithful. □

We proceed as in the case of one cusp. Fix an element γ in $\pi_1(M_0)$ which is carried by a non–trivial curve on T_1. By the second assertion in our lemma, $I_\gamma|_{Y_0}$ is non–constant. So the function has a pole at some ideal point of Y_0, and we get an action and a dual surface S. As before, some boundary component of S lies on T_1 and has slope different from the slope of γ. But we are not done yet since S could have other boundary components elsewhere. If we can choose S to be disjoint from $T_2, \ldots, T_n$, then since $S \subset M_0$ is essential and $\partial S \subset \partial M$, we have an essential surface in M with a slope other than the one of γ. This shows that we are in the theorem's fourth case.

The first item of the above lemma implies that $I_\gamma \in \mathbb{C}$ for all $\gamma \in \Gamma_i$ where $i = 2, \dots, n$. Thus, each group Γ_i fixes a vertex in the tree. What we have to show follows now from the following more general set–up:

Proposition 28. *Assume that the fundamental group of an orientable 3–manifold acts non–trivially, simplicially and without inversions on a tree. Let K_i be disjoint connected subcomplexes in ∂M such that each $\operatorname{im}(\pi_1(K_i) \to \pi_1(M))$ stabilises a vertex in the tree and that $\ker(\pi_1(K_i) \to \pi_1(M)) = 1$. Then there is a dual essential surface which is disjoint from each K_i.*

The assumption on the kernels merely simplifies notation. That is, if the kernels were non–trivial, we can obtain the elements which act by considering the quotients $\operatorname{im} / \ker$.

Exercise 24. *Prove Proposition 28.*

This completes the proof of the weak Neuwirth conjecture. □

6.3. **Boundary slopes of associated surfaces.** The existence result of the previous section leads to strong information in the case that the boundary of M consists of tori, in particular when there is just a single torus,

Let M be a compact, orientable, irreducible 3–manifold, and T be a torus component of ∂M. The relationship between boundary slopes of essential surfaces in M and sequences of representations of $\pi_1(M)$ be described as a consequence of Proposition 28. A sequence $\{\rho_n\}$ of representations on a curve C in $\mathfrak{R}(M)$ is said to *blow up*, if there is an element $\gamma \in \pi_1(M)$ such that $I_\gamma(\rho_n) \to \infty$ as $n \to \infty$. Let S be an essential surface associated to the corresponding ideal point.

Corollary 29. *Let $\partial M = T^2$ and $\alpha \in \operatorname{im}(\pi_1(\partial M) \to \pi_1(M))$ such that $I_\alpha(x) \in \mathbb{C}$ for some ideal point x of a curve in $\mathfrak{X}(M)$. Then either*

1. *$I_\beta(x) \in \mathbb{C}$ for all $\beta \in \operatorname{im}(\pi_1(\partial M) \to \pi_1(M))$ and there is a closed incompressible surface in M, or*
2. *α determines a boundary slope of M.*

Proof. We know that α stabilises a vertex in T. Thus, there is an associated essential surface S disjoint from any curve C on ∂M representing α. If $\partial S \neq \emptyset$ then C is a boundary slope. Otherwise $\mathrm{im}(\pi_1(\partial M) \to \pi_1(M))$ stabilises a vertex and the first part is true. $\square$

We may paraphrase this as follows:

1. If there is an element γ in $\mathrm{im}(\pi_1(T) \to \pi_1(M))$ such that $I_\gamma(\rho_n) \to \infty$, then up to inversion there is a unique element $\beta \in \mathrm{im}(\pi_1(T) \to \pi_1(M))$ such that $\{I_\beta(\rho_n)\}$ is bounded. Then β is parallel to the boundary components of S on T.
2. If $\{I_\gamma(\rho_n)\}$ is bounded for all $\gamma \in \mathrm{im}(\pi_1(T) \to \pi_1(M))$, then S may be chosen to be disjoint from T.

This will be the key to the relationship between boundary slopes and the A–polynomial in the following section.

7. Boundary slopes and the A–polynomial

Throughout this section, let M be a 3–manifold with boundary consisting of a single torus, and fix a generating set $\{\mathcal{M}, \mathcal{L}\}$ for the fundamental group of ∂M. The elements $\mathcal{M}$ and $\mathcal{L}$ will be referred to as meridian and longitude respectively.

7.1. **A–polynomial.** The A–polynomial was introduced in [7], and we define it following [9]. Let $\mathfrak{R}_U(M)$ be the subvariety of $\mathfrak{R}(M)$ defined by two equations which specify that the lower left entries in $\rho(\mathcal{M})$ and $\rho(\mathcal{L})$ are equal to zero. Any representation in $\mathfrak{R}(M)$ is conjugate to a representation in $\mathfrak{R}_U(M)$ since two commuting matrices in $SL_2(\mathbb{C})$ have a common invariant subspace. We define an *eigenvalue map* from $\mathfrak{R}_U(M)$ to $(\mathbb{C}-\{0\})^2$ by taking an element ρ of $\mathfrak{R}_U(M)$ to the upper left entries of $\rho(\mathcal{M})$ and $\rho(\mathcal{L})$. Taking the closure of the image of this map and discarding zero–dimensional components, one obtains the *eigenvalue variety*, which is necessarily defined by a principal ideal. A generator for the radical of this ideal is called the *A–polynomial*. After fixing a basis for the boundary torus, the A-polynomial is well defined up to multiplication by units in $\mathbb{C}[l^{\pm 1}, m^{\pm 1}]$, and it follows from [7] and [11] that the constant multiple can be chosen such that the coefficients are all integers with greatest common divisor equal to one. For a generalisation of the A–polynomial to multi–cusped 3–manifolds, see [33].

Exercise 25. *If E is an irreducible component of the eigenvalue variety, show that $(l, m) \in E$ if and only if $(l^{-1}, m^{-1}) \in E$. Hence for each factor A_E of the A—polynomial, we have $A_E(l, m) = 0$ if and only if $A_E(l^{-1}, m^{-1}) = 0$.*

The curve defined by the A–polynomial is a useful parameterisation of Dehn surgery coefficients due to Theorem 19. Note that there is a map from the eigenvalue variety to $\mathfrak{X}(\partial M)$ which generically has degree two. This can be interpreted as a distinction between the manifolds $M(p, q)$ and $M(-p, -q)$, which is geometrically sensible. Depending on the desired applications and filling curves, one may want to change from a fixed basis (or framing) for the boundary torus to another.

Exercise 26. *How does a change of framing for the boundary torus affect the A—polynomial? Write down explicit formulae for $(\mathcal{M}, \mathcal{L}) \to (\mathcal{M}', \mathcal{L}')$ and $A(l, m) \to A(l', m')$.*

One generally needs a computer to calculate the A–polynomial, and some computational techniques will be mentioned shortly. It is shown in [7] that if $\mathfrak{k}$ is a non–trivial (p, q)–torus knot, then $A_{\mathfrak{k}}(l, m)$ is divisible by $lm^{pq} + 1$ (where we follow the convention that if M is the complement of a knot $\mathfrak{k}$ in S^3, then $\{\mathcal{M}, \mathcal{L}\}$ is a standard peripheral system).

Proposition 30. [33] *Let $\mathfrak{k}$ be a non–trivial (p, q)–torus knot. If $p = 2$ or $q = 2$, then $A_{\mathfrak{k}}(l, m) = (l - 1)(lm^{pq} + 1)$, otherwise $A_{\mathfrak{k}}(l, m) = (l - 1)(lm^{pq} + 1)(lm^{pq} - 1)$.*

Proof. The fundamental group of $S^3 - \mathfrak{k}$ is presented by $\Gamma = \langle u, v \mid u^p = v^q \rangle$, and a standard peripheral system is given by $\mathcal{M} = u^n v^m$ and $\mathcal{L} = u^p \mathcal{M}^{-pq}$, where $mp + nq = 1$. These facts can be found in [6] on page 45. The factor $l - 1$ arises from reducible representations, and all factors arising from components containing irreducible representations are to be determined.

The element u^p is in the centre of Γ, since it is identical to v^q and hence commutes with both generators. Thus the image of u^p is in the centre of $\rho(\Gamma)$. Since two commuting elements have a common eigenvector, $\rho(u)^p$ has a common eigenvector with $\rho(u)$ and $\rho(v)$ respectively. If the representation is irreducible, these eigenvectors have to be distinct. Thus, after conjugation it may be assumed that the generators map to upper and lower triangular matrices respectively, and the central element is represented by a diagonal matrix. Direct matrix computations show that the commutativity with either of the generators gives $\rho(u)^p = \pm E$.

Assume that $\rho(u)^p = -E$. The relation $\rho(\mathcal{L}) = -\rho(\mathcal{M})^{-pq}$, which is equivalent to $\rho(\mathcal{L}\mathcal{M}^{pq}) = -E$, then implies the equation $lm^{pq} = -1$. This is the curve obtained in [7] by sending u and v to non-commuting elements of $SL_2(\mathbb{C})$ of order exactly $2p$ and $2q$ respectively.

If the image of $\rho(u)^p$ is trivial, then $\rho(v)^q = \rho(u)^p = E$. If p or q equals 2, this implies that the image of one of the generators is $\pm E$. But this yields that $\rho(\Gamma)$ is abelian, and therefore contradicts the irreducibility assumption. This completes the proof of the first assertion.

Now assume that neither of p and q is equal to two. As above, curves of irreducible representations are obtained by sending u and v to non-commuting elements of $SL_2(\mathbb{C})$ of order exactly p and q respectively. Any of these curves yields a component of the eigenvalue variety defined by the equation $lm^{pq} = 1$, and this finishes the proof of the proposition. In fact, with a more precise analysis one could count the number of 1–dimensional curves in the character variety of a torus knot. □

Exercise 27. *How many curves are in the character variety of the (p,q)–torus knot?*

7.2. **$\overline{\mathbf{A}}$–polynomial.** Analogous to the $SL_2(\mathbb{C})$–eigenvalue variety, there is an eigenvalue variety associated to the $PSL_2(\mathbb{C})$–character variety. This is the variety containing points corresponding to pairs of squares of eigenvalues of the meridian and longitude (in the case where the boundary of M consists of a single torus). A generator for its defining (principal) ideal is the $\overline{A}$–polynomial, and its variables have the unfortunate names L and M. Thus, $\overline{A}_M(L,M)$ is the $\overline{A}$–polynomial associated to the manifold M.

Proposition 31. [33] *Let $\mathfrak{k}$ be a non–trivial (p,q)–torus knot. Then $\overline{A}_{\mathfrak{k}}(L,M) = (L-1)(LM^{pq}-1)$.*

Proof. It follows from Exercise 17 that all $PSL_2(\mathbb{C})$–representations of a knot group lift to $SL_2(\mathbb{C})$. Thus, all factors of the A–polynomial arise from factors of the $\overline{A}$–polynomial and vice versa.

The eigenvalue of the meridian appears with even powers in the (unfactorised) A–polynomial of $\mathfrak{k}$, and the sign of the longitude's eigenvalue is uniquely determined since $\mathcal{L}$ is null–homologous. Thus, $A_{\mathfrak{k}}(l,m) = A_{\mathfrak{k}}(l,-m)$, but $A_{\mathfrak{k}}(l,m) = 0$ does not imply $A_{\mathfrak{k}}(-l,m) = 0$ in general. However, since $\overline{A}_{\mathfrak{k}}(l^2,m^2) = \overline{A}_{\mathfrak{k}}((-l)^2,m^2)$, it follows that the variety defined by $\overline{A}_{\mathfrak{k}}(l^2,m^2) = 0$ is the locus of $A_{\mathfrak{k}}(l,m)A_{\mathfrak{k}}(-l,m) = 0$. Expanding $A_{\mathfrak{k}}(l,m)A_{\mathfrak{k}}(-l,m) = 0$, substituting $l^2 = L$ and $m^2 = M$ and deleting repeated factors therefore gives the result as stated. □

7.3. **Elimination theory.** Calculations of the A–polynomial (and also of character varieties) often use the elimination and extension theorems as found in [12]. Given two polynomials $f_1, f_2 \in \mathbb{C}[x_1, \ldots, x_m]$ of positive degree in x_1, let $g_1, g_2 \in \mathbb{C}[x_2, \ldots, x_m]$ such that $f_i = g_i x_1^{n_i} +$ terms in which x_1 has degree lower than n_i. The resultant $Res(f_1, f_2, x_1)$ (whose definition we omit) is an element of $\mathbb{C}[x_2, \ldots, x_m]$. If this resultant vanishes at $P = (p_2, \ldots, p_m)$ and not both of g_1 and g_2 vanish at P, then there is $p_1 \in \mathbb{C}$ such that f_1 and f_2 vanish at $(p_1, \ldots, p_m)$.

This fact can iteratively be applied to collections of polynomials, and to eliminate more than one variable.

Many computer programs, like `Mathematica` for example, have a function of the name "resultant", which takes as its argument two polynomials and a variable which one wishes to eliminate, e.g. $\mathtt{Resultant}[\mathtt{f}_1, \mathtt{f}_2, \mathtt{z}]$. If one has more than one polynomial, one needs to take resultants of all (unordered) pairs. A resultant is basically a determinant, and since we are only interested in radical ideals, one often deletes repeated factors from the resultant.

Another approach to elimination is the use of Göbner bases, which is well explained in [12].

Example (m137)**.** *The A–polynomial of the manifold N is easily obtained from our computations above. In the above parameterisation $\mathfrak{C}(N)$, the meridian is upper triangular, so we only need to determine the the upper left entry of $\rho(\mathcal{L})$, which is a rational function in m and x, and then to eliminate the variable x. This can be done using resultants as described above. We get*

$$
\begin{aligned}
(m-1)(m^4 + 2m^5 + 3m^6 + m^7 - m^8 - 3m^9 - 2m^{10} - m^{11})& \\
+l^2(-1 - 3m - 2m^2 - m^3 + 2m^4 + 4m^5 + m^6& \\
+ 4m^7 + m^8 + 4m^9 + 2m^{10} - m^{11} - 2m^{12} - 3m^{13} - m^{14})& \\
+l^4(-m^3 - 2m^4 - 3m^5 - m^6 + m^7 + 3m^8 + 2m^9 + m^{10}),&
\end{aligned}
$$

where the line $(m-1)$ corresponds to the abelian representations. As it should be, the two components have no finite point of intersection. The Newton polygon of the second factor is given in Figure 16, and its boundary slopes are -2, $3/2$ and ∞.

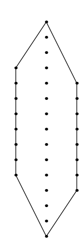


FIGURE 16. The Newton polygon associated to $m137$

We will need the following facts in order to prove the main result of this section.

7.4. **Newton polytopes.** Let $f \in \mathbb{C}[X_1^{\pm 1}, ..., X_m^{\pm 1}]$ be given as an expression $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha X^\alpha$. Then the *Newton polytope* of f is the convex hull of $s(f) = \{\alpha \in \mathbb{Z}^m \mid a_\alpha \neq 0\}$. Thus, if $s(f) = \{\alpha_1, \ldots, \alpha_k\}$, then

$$Newt(f) = Conv(s(f)) = \left\{ \sum_{i=1}^{k} \lambda_i \alpha_i \mid \lambda \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}. \tag{7.1}$$

Note that $Newt(0) = \emptyset$, and since $V(0) = (\mathbb{C} - \{0\})^m$, we have $V(0)_\infty = S^{m-1}$. For arbitrary subsets A, B of $\mathbb{R}^m$, let $A + B = \{a + b \mid a \in A, b \in B\}$.

Exercise 28. [12]

1. *Let* $f, g \in \mathbb{C}[X^\pm]$ *such that* $fg \neq 0$. *Then* $Newt(fg) = Newt(f) + Newt(g)$.
2. $Newt(f + g) \subseteq Conv(s(f) \cup s(g)) = Conv(Newt(f) \cup Newt(g))$.

7.5. **Spherical duals.** The set of outward pointing normal vectors can be described using *spherical duals*. The *spherical dual* of a bounded convex polytope P in $\mathbb{R}^m$ is the set of vectors ξ of length 1 such that the supremum $\sup_{\alpha \in P} \alpha \cdot \xi$ is achieved for more than one α, and it is denoted by $Sph(P)$. Geometrically, $Sph(f)$ consists of all outward unit normal vectors to the support planes of P which meet P in more than one point. If P is the Newton polytope of a non–zero polynomial f, then the spherical dual of $Newt(f)$ is also denoted by $Sph(f)$.

Spherical duals are easy to visualise in low dimensions. Given a convex polygon P in $\mathbb{R}^2$, its spherical dual is the collection of points on the unit circle defined by outward pointing unit normal vectors to edges of P. Given a convex polyhedron in $\mathbb{R}^3$, we obtain the vertices of its spherical dual again as points on the unit sphere S^2 arising from outward pointing unit normal vectors to faces. We then join two of these points along the shorter geodesic arc in S^2 if the corresponding faces have a common edge. This gives a finite graph in S^2.

Exercise 29. [33] *Let* $f, g \in \mathbb{C}[X^\pm]$. *If* $fg \neq 0$, *then* $Sph(fg) = Sph(f) \cup Sph(g)$.

7.6. **Slopes are slopes.** We are now in a position to prove the "boundary slopes" theorem which was stated in the introduction.

Theorem 32. [7] *Let M be a compact, orientable, irreducible 3–manifold with boundary consisting of a single torus. The slopes of edges of the Newton Polygon of the A–polynomial are the boundary slopes of essential surfaces in the knot complement.*

Proof. Since the boundary of M is a torus, we choose a basis $\{\mathcal{M}, \mathcal{L}\}$ for this torus. Assume that $C \subset \mathfrak{X}(M)$ is a curve with the property that the associated component E in the eigenvalue variety is a curve. Let the factor of the A–polynomial defining E be $A_E(l, m) = \sum_{(i,j)\in\mathcal{A}} c(i, j)m^i l^j$ where $\mathcal{A} \subset \mathbb{Z}^2$ is a finite set and $c(i, j) \neq 0$.

We have a component R_C of $\mathfrak{R}_U(M)$ such that $\mathrm{t}(R_C) = C$, and a tautological representation $\mathcal{P} : \pi_1(M) \to SL_2(F)$ where F is a finitely generated extension of $\mathbb{C}(C)$. We may assume that $\mathcal{P}$ is upper triangular on the meridian $\mathcal{M}$. Then F is also a finitely generated extension of $\mathbb{C}(E)$, since we obtain a regular map $R_C \to E$ by restriction to the upper left entries m of $\mathcal{P}(\mathcal{M})$ and l of $\mathcal{P}(\mathcal{L})$.

Let x be an ideal point of E. We consider the discrete valuation ord_x of $\mathbb{C}(E)$. By the extension lemma, this extends to a valuation of F. Since x is an ideal point, at least one of the coordinate functions m or l approaches 0 or ∞, and hence at least one of the functions $m + m^{-1}$ or $l + l^{-1}$ blows up. (Note that the argument given here also works if we regard E as a variety in $\mathbb{C}^2$ since eigenvalues on E come in pairs (m, l) and (m^{-1}, l^{-1}).)

Assume that $ord_x(m) = -a$ and $ord_x(l) = -b$, where $a, b \in \mathbb{Z}$ are co-prime and at least one of them is not zero since x is an ideal point and the valuation is normalised, discrete, rank 1. Then the eigenvalue of the element $\mathcal{M}^{-b}\mathcal{L}^a$ has valuation $ord_x(m^{-b}l^a) = ab - ba = 0$. This implies that $I_{\mathcal{M}^{-b}\mathcal{L}^a}$ stays finite at x, and the results of Subsection 6.3 imply that $\mathcal{M}^{-b}\mathcal{L}^a$ hence determines a boundary slope of M, which we also write as $-b/a$.

To know all detected boundary slopes, we therefore need to determine (up to scaling) the possible valuations v which arise at ideal points of components of the eigenvalue variety. Assume that v is such a valuation with $v(m) = -a, v(l) = -b$. For simplicity, write $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$. We can compute the valuation of a monomial using the Euclidean dot product: $v(m^i l^j) = -\xi \cdot \begin{pmatrix} i \\ j \end{pmatrix}$.

Now $A_E(l,m) = \sum c(i,j)m^i l^j$ vanishes on the curve defined by it if and only if $v(A_E) = v(0) = \infty$. This is the case only if the monomials that blow up fastest cancel, i.e. only if the minimum of the terms $v(m^i l^j) = -\xi \cdot \begin{pmatrix} i \\ j \end{pmatrix}$, $(i,j) \in \mathcal{A}$, is assumed at least twice. Conversely, there is a result due to Bergman [3], which asserts that if this minimum is attained at least twice, then one can define a valuation v which satisfies $v(A_E) = \infty$ and $v(m) = -a$ and $v(l) = -b$.

Thus, we have a valuation v with $v(A_E) = \infty$ if and only if the maximum of the terms $v(m^i l^j) = \xi \cdot \begin{pmatrix} i \\ j \end{pmatrix}$, $(i,j) \in \mathcal{A}$, is assumed at least twice. This is the case if and only if ξ is an outward pointing normal vector to a support plane of the Newton polygon of A_E, hence if and only if $-b/a$ is the slope of a side of the Newton polygon.

Putting everything together, we see that if $-b/a$ is the slope of a side of the Newton polygon of A_E, then $-b/a$ is the slope of an essential surface in M. The result now follows from Exercise 29, which implies that the set of boundary slopes of the Newton polygon of a polynomial is the union of the boundary slopes of the Newton polygons of its irreducible factors. $\square$

8. Representations and the Alexander polynomial

This section does not contain an application of Culler–Shalen theory, but has a common theme with the previous and the next section: eigenvalues. Following [7], we interpret non–abelian reducible representations of knot groups as representations into the affine group of the complex plane, and obtain a relationship to zeros of the Alexander polynomial.

8.1. **Alexander module.** Let $M = S^3 - \nu(\mathfrak{k})$. Then there is a covering $p : C_\infty \to M$, such that $p_*(\pi_1(C_\infty)) = \Gamma'$, where Γ' denotes the commutator subgroup of Γ, and the group of covering transformations is infinite cyclic. This covering is called the *infinite cyclic covering*. The induced homomorphism p_* is injective, we have $\pi_1(C_\infty) \cong \Gamma'$, and the first homology group of C_∞, $H_1(C_\infty)$, is isomorphic to Γ'/Γ''.

Recall that Γ is a semidirect product $\Gamma = \mathfrak{Z} \ltimes \Gamma'$, where $\mathfrak{Z} \cong \Gamma/\Gamma'$ is infinite cyclic by Proposition 11. Choose a meridian t of Γ as a representative of a coset of $\mathfrak{Z}$. Since $\mathfrak{Z}$ acts by conjugation on the commutator subgroup, and since commutator subgroups are characteristic, this induces an action on Γ'/Γ'':

$$g \to t^{-1}gt \quad \text{for } g \in \Gamma'/\Gamma''.$$

We denote this action by $g \to tg$. This induced action is independent of the choice of the representative t in $t\Gamma'$, since any other representative which is a pullback of the generating element under $\Gamma \to \mathfrak{Z}$ gives an action which differs by an inner automorphism of Γ. Furthermore, we can regard Γ'/Γ'' as a finitely presented $\mathbb{Z}(t)$-module, $\mathbb{Z}(t) = \mathbb{Z}[t, 1/t]$, which is called the *Alexander module* $M(t)$ of the knot group Γ.

In other words, the Alexander module is $H_1(C_\infty)$ and the group of covering transformations induces the module operation. The elements $\zeta \in \mathbb{Z}(t)$ such that $\zeta(h) = 0$ for all $h \in H_1(C_\infty)$ form an ideal I. It can be shown that this ideal is principal. The defining polynomial of this ideal is denoted by $\Delta_{\mathfrak{k}}$ and called the *Alexander polynomial* of $\mathfrak{k}$. The Alexander polynomial is therefore a general relation in $H_1(C_\infty)$, and it is defined up to units, i.e. up to $\pm t^n$.

Take a $(m \times n)$ presentation matrix $A(t)$ of the Alexander module, which we shall refer to as an *Alexander matrix*. We can calculate the Alexander polynomial algebraically as the greatest common divisor of the elements of the ideal generated by the $(n \times n)$ minors of $A(t)$. This is invariant under Tietze transformations.

Consider the epimorphism $\varphi : \Gamma \to \mathbb{Z}$ given by $\varphi(g) = \mathrm{lk}(g, \mathfrak{k})$. Choose a presentation of $\Gamma =< g_0, \dots, g_n \mid r_1, \dots, r_n >$, which has one generator more than relations, and perform basis transformations such that we have a presentation where $\varphi(g_0) = 1$ and $\varphi(g_i) = 0$ for all $i \in \{1, \dots, n\}$. Then $g_i \in \Gamma'$ for $i \in \{1, \dots, n\}$, and $\Gamma' =< g_i \mid r_1, \dots, r_n >$ is a presentation of Γ' as an $\mathbb{Z}(t)$–module. Passing to $H_1(C_\infty)$, we have to abelianise Γ', and the Alexander polynomial follows from the relations.

8.2. **Metabelian representations.** Any abelian homomorphic image of a knot group is cyclic. Apart from abelian representations there are other reducible representations which have to be metabelian by the condition on the trace of elements in the commutator group in Lemma 13:

Exercise 30. *Show that a non-abelian reducible representation of a finitely generated group* Γ *into* $SL_2(\mathbb{C})$ *is metabelian, i.e. that the image of* $\Gamma'' = [\Gamma', \Gamma']$ *is trivial.*

Assume that $\rho : \Gamma \to H$ is a surjective homomorphism, where H is a metabelian group (not necessarily a subgroup of $SL_2(\mathbb{C})$). Since $\Gamma = \mathfrak{Z} \ltimes \Gamma'$, we have

$$H = \rho(\Gamma) = \rho(\mathfrak{Z}) \ltimes \rho(\Gamma') = \rho(\mathfrak{Z}) \ltimes H',$$

since the commutator subgroup is characteristic. Assume that $\rho(\mathfrak{Z})$ is infinite cyclic. Then H' can be considered as a $\mathbb{Z}(t)$–module. Since H is metabelian, H' is abelian, and we have the following unique factorisation:

$$\begin{array}{ccc}
\mathfrak{Z} \ltimes \Gamma' & \xrightarrow{\varphi} & \mathfrak{Z} \ltimes H' \\
\downarrow & & \uparrow \\
\mathfrak{Z} \ltimes \Gamma'/\Gamma'' & = & \mathfrak{Z} \ltimes H_1(C_\infty)
\end{array}$$

Thus, we factor through the first homology group of the infinite cyclic covering, which is the Alexander module, and the operation is defined by the operation of the semidirect product. This shows that any metabelian representation of a knot group Γ factors through $\mathfrak{G} \to \mathfrak{Z} \ltimes H_1(C_\infty)$.

Let $\Gamma =< g_0, \dots, g_n \mid r_0, \dots, r_n >$ be a Wirtinger presentation. We have already observed that $\mathrm{lk}(g_i, \mathfrak{k}) = 1$ for all $i \in \{0, \dots, n\}$. We now change the basis to the set $\{g_0, h_1, \dots, h_n\}$ where $h_i = g_i g_0^{-1}$ for $i \in \{1, \dots, n\}$. Thus $\mathrm{lk}(h_i, \mathfrak{k}) = 0$ and hence $h_i \in \Gamma'$.

The relations are of one of the following forms:

$$\text{(W1)} \qquad r_j = g_j g_i^{-1} g_k^{-1} g_i,$$

$$\text{(W2)} \qquad r_j = g_j g_i g_k^{-1} g_i^{-1}.$$

Using the equivalent presentation, we get

$$\begin{aligned}
\text{(W1')} \qquad r_j &= h_j h_i^{-1} g_0^{-1} h_k^{-1} h_i g_0 \\
&= h_j h_i^{-1} (g_0^{-1} h_k^{-1} g_0)(g_0^{-1} h_i g_0), \\
\text{(W2')} \qquad r_j &= h_j g_0 h_i h_k^{-1} g_0^{-1} h_i^{-1} \\
&= h_j (g_0 h_i g_0^{-1})(g_0 h_k^{-1} g_0^{-1}) h_i^{-1}.
\end{aligned}$$

As described above, we get any metabelian representation via factorisation through $H_1(C_\infty)$. Writing $H_1(C_\infty)$ additively, conjugation by g_0 becomes multiplication by t, and we have the relations:

$$\text{(A1)} \qquad r_j = h_j - h_i - t h_k + t h_i,$$

$$\text{(A2)} \qquad r_j = h_j + \frac{1}{t} h_i - \frac{1}{t} h_k - h_i.$$

The matrix corresponding to this system of equations is the Alexander matrix, and its determinant is the Alexander polynomial. In order to find metabelian representations into $SL_2(\mathbb{C})$, we will now take a small detour.

8.3. **Affine representations.** Consider a metabelian representation ρ of the knot group Γ into the affine group $Af(\mathbb{C})$ of the complex plane. Let $\rho(\Gamma) = H$ and assume that H' is nontrivial. We may conjugate this representation such that the fixed point of $\rho(g_0)$ is zero, if we assume that the derivative of $\rho(g_0)$ is not equal to one. Otherwise the representation would be abelian, which contradicts our choice. Thus, $\rho(g_0) = f_0(z) = tz$ for some complex number t. Since all meridians are conjugate, we have $\rho(g_i) = f_i(z) = tz + h_i$ for all $i \in \{1, \dots, n\}$, i.e. the images of meridians have the same derivative.

Setting the image of an relator equal to the identity, gives homogeneous linear equations in $h_1, \ldots, h_n$. It can easily be verified that this system of equations is identical to the system of equations given by (A1) and (A2). Starting with the relations (W1) and (W2), we get:

$$z = z + h_j - h_i - th_k + th_i, \tag{A1'}$$

$$z = z + h_j + \frac{1}{t}h_i - \frac{1}{t}h_k - h_i. \tag{A2'}$$

Clearly, ρ gives a nonabelian representation if and only if some of the h_i are nonzero. This is the case if and only if the system of equations has a nonzero solution. Note that if all $h_i = 0$, then H' is trivial, and H abelian.

Therefore, there exists a nonabelian affine representation of Γ for which the image of each meridian has derivative t if and only if t is a zero of the Alexander polynomial.

Since we are interested in representations into $SL_2(\mathbb{C})$, consider the composite mapping:

$$f_i(z) = tz + h_i \xrightarrow{\iota} \begin{pmatrix} t & h_i \\ 0 & 1 \end{pmatrix} \xrightarrow{\mu} \begin{pmatrix} \sqrt{t}^{-1} & 0 \\ 0 & \sqrt{t}^{-1} \end{pmatrix} \begin{pmatrix} t & h_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{t} & h_i\sqrt{t}^{-1} \\ 0 & \sqrt{t}^{-1} \end{pmatrix}.$$

Here, we have first mapped affine transformations into $GL_2(\mathbb{C})$. Left multiplication is a bijective mapping of a group, which in this case yields elements in $SL_2(\mathbb{C})$. We claim that this composite mapping gives us a representation of Γ into $SL_2(\mathbb{C})$. For this, we perform the following matrix computation:

$$\begin{pmatrix} \sqrt{t} & h_i\sqrt{t} \\ 0 & \sqrt{t}^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{t} & h_j\sqrt{t} \\ 0 & \sqrt{t}^{-1} \end{pmatrix} = \begin{pmatrix} t & h_j + t^{-1}h_i \\ 0 & t^{-1} \end{pmatrix}$$

Thus, the composite mapping respects the composition of affine transformations, and only the identity $f(z) = z$ is contained in its kernel, so $\mu\iota$ is a monomorphism. This yields a representation of Γ into $SL_2(\mathbb{C})$. The commutator of two elements is upper triangular with entries equal to one on the diagonal. Thus, the representation is reducible by Lemma 13.

This shows that there is a reducible representation of Γ into $SL_2(\mathbb{C})$, which has non-abelian image and sends a meridian to an element with eigenvalue $\pm\sqrt{t}$, if t is a root of $\Delta_{\mathfrak{k}}$.

8.4. **Conclusion.** We now wish to prove the following:

Proposition 33. [7] *The following are equivalent:*

1. *There exists a reducible representation of* $\Gamma(\mathfrak{k})$ *into* $SL_2(\mathbb{C})$ *which has non-abelian image and sends a meridian to an element with eigenvalue* m.
2. m^2 *is a root of* $\Delta_{\mathfrak{k}}$.

Proof. One direction has already been shown. Now assume that there exists a reducible representation of Γ in $SL_2(\mathbb{C})$ which has non-abelian image and sends a meridian to an element with eigenvalue m. We may assume that the meridians are upper triangular matrices, and that the upper left entry of some meridian is m. Consider the following mapping:

$$\begin{pmatrix} m & h \\ 0 & m^{-1} \end{pmatrix} \to m^2 z + hm.$$

This gives a homomorphism from the subgroup of all upper triangular matrices in $SL_2(\mathbb{C})$ to the group of affine motions of the complex plane. Since the group was supposed to be metabelian, this is the case if and only if m^2 is a zero of the Alexander polynomial. □

We conclude that the eigenvalues m of meridians in metabelian representations are invariants of the knot — however, this information is already contained in the Alexander polynomial.

Proposition 34. [7] *Suppose that* ρ *is a reducible representation of a knot group* $\Gamma(\mathfrak{k})$ *such that the character of* ρ *lies on a component of* $\mathfrak{X}(\Gamma)$ *which contains the character of an irreducible representation. If* γ *is a meridian of* Γ*, then* $\rho(\gamma)$ *has eigenvalue* m *where* m^2 *is a root of the Alexander polynomial of* $\mathfrak{k}$.

Proof. It is sufficient to show that there is a reducible non-abelian representation with the same character as ρ. We have $\dim t^{-1}(y) \geq 3$ for all points y of a component which contains the character of an irreducible representation. It is stated in Exercise 15 that the variety of abelian representations with a given character is 2–dimensional. Since the dimensions of points in the same component are to be equal, $\mathrm{t}^{-1}(\mathrm{t}\,\rho)$ contains a representation with nonabelian image. □

9. The Roots of Unity Phenomenon

Let us assume that a surface S is associated to an ideal point of the character variety of a manifold M. We know that the "character" at the ideal point takes finite values when restricted to components of $M - S$, and hence corresponds to representations of these pieces. However, since we are at an ideal point of the character variety, these representations cannot be glued together to form a representation of the manifold.

In the present section, we will deduce a little more information about the characters at infinity. It turns out that they are reducible on the associated surface. This observation leads to the roots of unity phenomenon, but also gives a necessary condition on a surface to be detected.

Reference. [7], *Section 5.*

9.1. **Reduced Surfaces.** Let M be a compact, orientable, irreducible 3–manifold with boundary consisting of a single torus. Assume that there is an ideal point ξ of a curve C in $\mathfrak{X}(M)$ with the property that $I_\gamma(\xi) = \infty$ for some $\gamma \in \mathrm{im}(\pi_1(\partial M) \to \pi_1(M))$. This ensures that an essential surface S which is associated to the action of $\pi_1(M)$ on $\mathcal{T}_v$ determined by $\mathcal{P}$ has non–empty boundary. The boundary components of S are a family of parallel simple closed curves on the torus ∂M, so they all lie in the same homotopy class. Let h be a representative of the particular homotopy class, and let $I_{\partial S}(\xi) = I_h(\xi)$. This is well–defined since the trace is invariant under conjugation and taking inverses.

A surface is called *reduced* if it has the minimal number of boundary components amongst all associated essential surfaces. Let $n(S)$ denote the greatest common divisor of the number of boundary components of the components of S.

Theorem 35. [7] *Let M be a compact, orientable, irreducible 3–manifold with boundary consisting of a single torus, and let ξ be an ideal point of a curve C in $\mathfrak{X}(M)$ with the property that $I_\gamma(\xi) = \infty$ for some $\gamma \in \mathrm{im}(\pi_1(\partial M) \to \pi_1(M))$.*

Let S be a surface associated to the action of $\pi_1(M)$ on $\mathcal{T}_v$ determined by $\mathcal{P}$. Then $I_{\partial S}(\xi) = \lambda + \lambda^{-1}$, where λ is a root of unity. Moreover, if S is reduced, then $\lambda^{n(S)} = 1$.

In order to prove this theorem, we will have to establish more facts about our action.

9.2. **Edges and eigenvalues.** Going back to what we have learnt about actions on trees: what does an element $A \in SL_2(F)$ look like if it fixes an edge (Λ_1, Λ_2)?

Exercise 31. *If $A \in Stab(\Lambda_1, \Lambda_2)$, then A is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, d \in \mathcal{O}$ and $c \in \pi\mathcal{O}$. Furthermore, any element in the commutator group of $Stab(\Lambda_1, \Lambda_2)$ has trace equal to $2 \mod (\pi)$. You can show this directly or using the induced action of A on the residue field $k = \mathcal{O}/(\pi)$.*

Now let $A \in SL_2(F)$ be any element in the stabilizer of our edge. We know that we can find representatives L_1 and L_2 such that the proper inclusions

$$\pi L_1 \subsetneqq L_2 \subsetneqq L_1$$

hold. Passing to the quotient space $L_1/\pi L_1$, which is a 2–dimensional k-vector space, the above inclusions give

$$0 \subsetneqq L_2/\pi L_1 \subsetneqq L_1/\pi L_1.$$

So $L_2/\pi L_1$ is a non–trivial proper subspace, hence a 1–dimensional subspace. The matrix A induces a linear transformation on the quotient space, which stabilises $L_2/\pi L_1$. Thus, it multiplies this line by an element $\lambda \in k = \mathcal{O}/(\pi)$, which we call an *eigenvalue associated to our fixed edge.*

Exercise 32. *The eigenvalue associated to an edge is independent upon the choice of lattice representatives.*

Note that the eigenvalue was only dependent upon the choice of orientation of the edge. We claim that

Lemma 36. *The eigenvalue of A associated to $\overline{(\Lambda_1, \Lambda_2)} = (\Lambda_2, \Lambda_1)$ is λ^{-1}.*

Proof. The above inclusions imply that

$$\pi L_2 \subsetneqq \pi L_1 \subsetneqq L_2.$$

So if $\overline{\lambda}$ is the eigenvalue associated to (Λ_2, Λ_1), then A induces multiplication by this value on $\pi L_1/\pi L_2 \cong L_1/L_2$. The latter suggests that we ought to look at the exact sequence

$$0 \to L_2/\pi L_1 \to L_1/\pi L_1 \to L_1/L_2 \to 0.$$

Thus, we have found the eigenvalues corresponding to two direct summands of $L_1/\pi L_1$, and hence $\lambda\overline{\lambda}$ is the image of $\det A$ in k. Since $A \in SL_2(F)$, this proves the claim. $\square$

Since the action is without inversions, A fixes the initial and terminal vertices of our edge, and hence $\operatorname{tr} A \in \mathcal{O}$. If λ is the eigenvalue associated to the edge, then $\lambda + \lambda^{-1}$ is the image of $\operatorname{tr} A$ under the quotient map.

9.3. **Comparable edges.** An *edge path* in T is a sequence $e_0, \dots, e_n$ of edges in T such that $i(e_j) = t(e_{j-1})$ for $j = 1, \dots, n$. The edge path is *reduced* if there is no backtracking, i.e. $e_j \neq \overline{e_{j-1}}$ for $j = 1, \dots, n$.

We can define a partial ordering on T by setting $e \leq f$ if there is a reduced edge path with $e_0 = e$ and $e_n = f$. Thus, for any pair of distinct edges, exactly one of the following relationships holds:

$$e < f, \quad \bar{e} < f, \quad f < e, \quad f < \bar{e}.$$

By the preceding section, if $A \in SL_2(F)$ fixes two edges, then the associated eigenvalues are either the same or inverses.

Exercise 33. *Assume A fixes the edges (Λ_1, Λ) and (Λ_2, Λ). If the eigenvalue associated to the first edge is λ, then the eigenvalue associated to the latter is λ^{-1}.*

Moreover, if A fixes the first and last edge of a reduced edge path, then it fixes each edge in between and the associated eigenvalues are all equal.

9.4. **Compatibly oriented curves.** Let us go back to our manifold, and fix orientations of M and S. Then $\partial\tilde{M}$ and $p^{-1}(\partial S)$ have induced orientations. Furthermore, the components of S and ∂S have induced transverse orientations in M and ∂M. To save us some time, assume that the boundary of M is incompressible. Then the components of $\partial\tilde{M}$ are planes and the components of $p^{-1}(\partial S)$ contained in a connected component of $\partial\tilde{M}$ form a family of parallel lines. We say that two components of $p^{-1}(\partial S)$ have *compatible orientations* if they project to homologous oriented simple closed curves on ∂M.

For each component $\tilde{C}$ of $p^{-1}(\partial S)$ there is a unique directed edge $e(\tilde{C})$ in T such that $\tilde{f}(\tilde{C})$ is the midpoint of this edge and such that the direction of $e(\tilde{C})$ pulls back to the transverse orientation of $\tilde{C}$.

Proposition 37. *Assume that S is reduced. If $\tilde{C}$ and $\tilde{C}'$ are components of $p^{-1}(\partial S)$ which lie in the same component of $\partial\tilde{M}$, then the orientations of $\tilde{C}$ and $\tilde{C}'$ are compatible if and only if the edges $e(\tilde{C})$ and $e(\tilde{C}')$ are comparable.*

Proof. Let us first show that $\tilde{f}(\tilde{C}) \neq \tilde{f}(\tilde{C}')$ if $\tilde{C}$ and $\tilde{C}'$ are adjacent. So suppose this is false. Then there is a strip $\tilde{A}$ in the plane spanned by $\tilde{C}$ and $\tilde{C}'$, and $\tilde{f}(\tilde{A} - (\tilde{C} \cup \tilde{C}'))$ lies in a connected component of $T - E$. Thus, we can deform $\tilde{f}$ in a neighbourhood of $\tilde{A}$ so that $\tilde{f} \cap E = \emptyset$, and we obtain a new dual surface with less boundary components as illustrated in figure 18. But this contradicts our assumption that S is reduced.

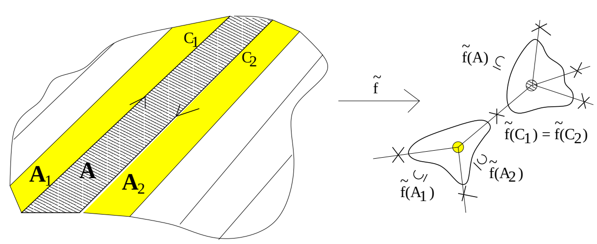


FIGURE 17. Surgery on the map $\tilde{f}$

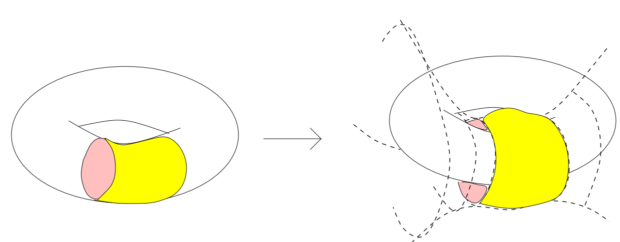


FIGURE 18. Surgery on the annulus

Now then $\tilde{f}(\tilde{C}) \neq \tilde{f}(\tilde{C}')$. We can find a finite sequence of lines $\tilde{C} = C_0, \ldots, C_n = \tilde{C}'$ such that C_i is adjacent to C_{i+1}. Then the images $\tilde{f}(C_i)$ are midpoints of edges in a reduced path $e_0, \ldots, e_n$, where e_i is equal to $e(C_i)$ or $\overline{e(C_i)}$. We know that e_0 and e_n are comparable. If the orientations of $\tilde{C}$ and $\tilde{C}'$ are compatible, then either $e_0 = e(\tilde{C})$ and $e_n = e(\tilde{C}')$ or $\overline{e_0} = e(\tilde{C})$ and $\overline{e_n} = e(\tilde{C}')$. $\square$

9.5. **Proof of Theorem 35.** Let S_0 be a component of S with non–empty boundary, and pick a component $\tilde{S}_0$ of $p^{-1}(S_0)$. The group $\pi_1(S_0)$ stabilises $\tilde{S}_0$ and hence an edge e in the tree. We assume that e is directed to match the transverse orientation of $\tilde{S}_0$.

A component of ∂S_0 determines a conjugacy class in $\pi_1(S_0)$, and hence a conjugacy class in $Stab(e)$. Choose a representative h of this class, and denote the eigenvalue of h associated to e by λ_C.

Now list all boundary components of S_0, say $C_1, \ldots, C_n$. For each component, let h_i be an element of $\pi_1(S_0)$ in the conjugacy class determined by C_i with the induced orientation. Since the image of $\tilde{S}_0$ was the midpoint of the edge e, each h_i is contained in $Stab(e)$ with associated eigenvalue $\lambda_i = \lambda_{C_i}$. Since S is a connected surface, the product $h_1 \cdots h_n$ is an element of the commutator group of $\pi_1(S_0)$ and hence of the commutator group of the stabilizer of an edge. By the results from one of the earlier sections, we know that $h_1 \cdots h_n$ has associated eigenvalue $\lambda_1 \cdots \lambda_n$ and that $\lambda_1 \cdots \lambda_n = 1 \mod \pi$.

We claim that $\lambda_i = \lambda_1^{\pm 1}$ for all i. Let $\tilde{C}_i$ be lifts of the boundary components of S_0 to a connected component U of $p^{-1}(\partial M)$ with corresponding lifts of $\tilde{S}_0$ for each i. We have $\tilde{f}(\tilde{C}_i) = e_i$ for some edges in T.

Now h_1 is a deck transformation of the covering of the torus ∂M by U and leaves the line $\tilde{C}_1$ invariant. It also preserves the whole family of lines $U \cap p^{-1}(\partial S)$, and hence in particular each $\tilde{C}_i$.

So for each i, either h_1 or h_1^{-1} is the generator of $Stab(C_i)$, depending upon whether the orientations of $\tilde{C}_1$ and $\tilde{C}_i$ are compatible. This implies that $\lambda_i = \lambda_1^{\pm 1}$ for all i. Putting $\lambda_1 = \lambda$, this gives $\lambda^m = 1$ in the residue field for some m. Now $\operatorname{tr} \partial S = \lambda + \lambda^{-1}$ and this proves the first part of the theorem.

Claim. *If S is a reduced surface and if C and C' are boundary components of S_0, then $\lambda_C = \lambda_{C'}$.*

Proof. By the above, we know that either h or h^{-1} is the generator of $Stab(C')$, depending upon whether the orientations of $\tilde{C}$ and $\tilde{C}'$ are compatible. Since S is reduced, we also know that the edges e and e' are comparable if and only if the orientations of $\tilde{C}$ and $\tilde{C}'$ are compatible.

Putting all of this together gives that if the orientations of $\tilde{C}$ and $\tilde{C}'$ are compatible, then h generates both stabilisers, and we have the same eigenvalue. If the orientations are not compatible, then the edges are not comparable, and $\lambda_{C'}^{-1}$ is the eigenvalue of h associated to e', and again $\lambda_C = \lambda_{C'}$. □

So if S is reduced, we have that all eigenvalues associated to the components of ∂S_0 are equal, which gives the equality $\lambda^n = 1$ in the residue field. Since S_0 was an arbitrary component of the surface S, we have proved Theorem 35.

9.6. **Non-trivial roots.** Given the main theorem in this section, we see that knowing the trace of the unique boundary slope can give us information about associated reduced surfaces. However, if the trace is just 2 or -2, we don't get much extra information. So let us call a root of unity non–trivial if it is not equal to ± 1. The first examples with non–trivial roots of unity associated to a boundary slope were found by Nathan Dunfield in [15], and the manifold $m137$ was one of them.

Example. *Going back to the tautological representation of* $m137$ *on page 39, it is not hard to work out that the ideal points of the character variety are encoded in the numerator of the expression given for the coordinate* y*. If we write the ideal points naively as tuples* (m, x)*, and let* ξ_k *denote a non–trivial* k*–th root of unity, we have the following cases and results:*

- $(1, 0)$: *The longitude has an eigenvalue equal to* 0.
- $(\xi_3, 0)$: *The longitude has an eigenvalue equal to* 0.
- $(0, \xi_6)$: *The element* $\mathcal{M}^2\mathcal{L}^{-1}$ *has eigenvalues equal to* 1.
- $(0, \xi_3)$: *The element* $\mathcal{M}^2\mathcal{L}^{-1}$ *has eigenvalues equal to* -1.
- $(0, \xi_4)$: *The element* $\mathcal{M}^3\mathcal{L}^2$ *has eigenvalues equal to* -1.

This behaviour can be observed as follows. Firstly, we use the A-polynomial. As m *tends to one of the third roots of unity, the eigenvalue of the longitude tends to zero, and we are done for the ideal points where* $x \to 0$*, since* $\mathcal{M}$ *is now a boundary slope.*

We then perform a basis transformation on the torus such that $\mathcal{L} \to \mathcal{M}^2\mathcal{L}^{-1}$*. As* $m \to 0$*, we obtain the two eigenvalues* 1 *and* -1 *for* $\mathcal{M}^2\mathcal{L}^{-1}$ *from above. We* *use our tautological representation, determine the upper left entry of* $\mathcal{M}^2\mathcal{L}^{-1}$*, and find that as* x *tends towards one of the third or sixth roots of unity, the limit is well defined and as stated above. As the approaches* $x \to \pm i$ *are well defined, as well, but give eigenvalues equal to* 0 *for* $\mathcal{M}^2\mathcal{L}^{-1}$*, the slope* -2 *is not associated to these ideal points.*

So if we have classified the ideal points correctly, the points $(0, \xi_4)$ *should correspond to the slope* $3/2$*. In order to determine the roots of unity associated to the slope, we may again perform basis transformations on the torus, which gives the stated eigenvalue of the element* $\mathcal{M}^3\mathcal{L}^2$*. We may use this information to verify in the tautological representation that* $(0, \xi_4)$ *are the corresponding points at infinity.*

Thus, we only find non–trivial roots associated to boundary slopes in the second item above. Applying the main result from this section, we deduce that reduced dual surfaces associated to the ideal points $(\xi_3, 0)$ *have* $3k$ *boundary components for some* k*. Thus, the thrice punctured sphere* S *from page 39 is a good candidate for such a surface. We need to show that it is associated to an ideal point. The following section establishes a method to tell whether a connected surface is associated to an ideal point.*

10. Detected surfaces

We conclude these notes by giving a method to decide whether a connected essential surface can be associated to an ideal point of the character variety, and illustrate this with the thrice punctured sphere in the manifold $m137$.

Reference. [32], *Sections 1 and 2.*

10.1. **Limiting character.** The *limiting character* at an ideal point ξ of a curve C in $\mathfrak{X}(M)$ is defined by the trace functions $I_\gamma(\xi) \in \mathbb{C} \cup \{\infty\}$ of Subsection 4.10. Let S be a surface associated to an ideal point of the character variety of a manifold M. Then the limiting character takes finite values when restricted to components of $M - S$, and hence corresponds to representations of these pieces. However, since we are at an ideal point of the character variety, these representations cannot be glued together to form a representation of the manifold.

We have the following information about the limiting representation of any component of an associated surface:

Lemma 38. [27] *The limiting representation of every component of an associated surface is reducible.*

Exercise 34. *Prove the previous lemma. You will need the statements of Lemma 13 and Exercise 31.*

10.2. **Associating surfaces.** Given an essential surface S in a 3–manifold M, how can we decide whether S is detected by (the action on a tree associated to) an ideal point of a curve in $\mathfrak{X}(M)$?

Denote the components of $M - S$ by $M_1, \ldots, M_k$ (where $M - S$ really stands for M minus an open collar neighbourhood of S). If S is detected by an ideal point, then the limiting character restricted to M_i is finite for all $i = 1, \ldots, k$. There is a natural map from $\mathfrak{X}(M)$ to the Cartesian product $\mathfrak{X}(M_1) \times \ldots \times \mathfrak{X}(M_k)$, by restricting to the respective subgroups. Splittings along S which are detected by ideal points of curves in $\mathfrak{X}(M)$ correspond to points $(\chi_1, \ldots, \chi_k)$ in the Cartesian product satisfying the following necessary conditions:

C1. $\chi_i \in \mathfrak{X}(M_i)$ is finite for each $i = 1, \ldots, k$.

C2. For each component of S, let $\varphi : S^+ \to S^-$ be the gluing map between its two copies arising from the splitting, and assume that $S^+ \subset \partial M_i$ and $S^- \subset$

∂M_j, where i and j are not necessarily distinct. Denote the homomorphism induced by φ on fundamental group by φ_*. Then for each $\gamma \in \mathrm{im}(\pi_1(S^+) \to \pi_1(M_i))$, $\chi_i(\gamma) = \chi_j(\varphi_*\gamma)$.

C3. For each $i = 1, \ldots, k$, the restriction of χ_i to any component of S in ∂M_i is reducible.

C4. There is an ideal point ξ of a curve C in $\mathfrak{X}(M)$ and a connected open neighbourhood U of ξ on C such that the image of U under the map to the Cartesian product contains an open neighbourhood of $(\chi_1, \ldots, \chi_k)$ on a curve in $\mathfrak{X}(M_1) \times \ldots \times \mathfrak{X}(M_k)$, but not $(\chi_1, \ldots, \chi_k)$ itself.

Note that C2 defines a subvariety of the Cartesian product containing the image of $\mathfrak{X}(M)$ under the restriction map. According to Lemma 38, C3 is necessary for a surface to be detected, and reduces the set of possible limiting characters to a subvariety. The last condition implies that at least one element of $\pi_1(M)$ has non–trivial translation length with respect to the action on Serre's tree, and the first condition implies that $\mathrm{im}(\pi_1(M_i) \to \pi_1(M))$ is contained in a vertex stabiliser for each $i = 1, \ldots, k$.

Lemma 39. [32] *Let S be a connected essential surface in a 3-manifold M. S is associated to an ideal point of the character variety of M if and only if there are points in the Cartesian product of the character varieties of the components of $M - S$ satisfying conditions C1–C4.*

Proof. It is clear that the conditions are necessary for any associated surface. We need to show that they are sufficient when S is connected and essential. Assume that S is non–separating. Let $A = \mathrm{im}(\pi_1(M - S) \to \pi_1(M))$, and denote the subgroups of A corresponding to the two copies of S in $\partial(M - S)$ by A_1 and A_2. Since $\pi_1(M)$ is an HNN–extension of A across A_1 and A_2, there is $t \in \pi_1(M)$ such that $t^{-1}A_1t = A_2$, and the subgroup generated by t and A is $\pi_1(M)$.

Let ξ be the ideal point provided by C4, and denote Serre's tree associated to ξ by $\mathcal{T}_v$. C1 implies that the subgroup A stabilises a vertex Λ of $\mathcal{T}_v$, and hence condition S1 is satisfied.

Note that A is finitely generated. C4 yields that the action of $\pi_1(M)$ on $\mathcal{T}_v$ is non–trivial, and Lemma 2 implies that either t is loxodromic with respect to the action on $\mathcal{T}_v$ or there is $a \in A$ such that ta or at is loxodromic. In the first case, we keep A_1 and A_2 as they are; in the second case, we replace t by ta and A_2 by $a^{-1}A_2a$; and in the third case, we replace t by at and A_1 by aA_1a^{-1}. Thus, the generator t satisfies condition S4.

Since A stabilises Λ, $t^{-1}At$ stabilises $t^{-1}\Lambda$, and since t acts as a loxodromic, we have $t^{-1}\Lambda \neq \Lambda$. In particular, A_2 fixes these two distinct vertices, and hence the path $[\Lambda, t^{-1}\Lambda]$ pointwise, which implies that it is contained in an edge stabiliser. Thus, condition S2 is satisfied, and the lemma is proven in the case where S is connected, essential and non–separating, since condition S3 does not apply.

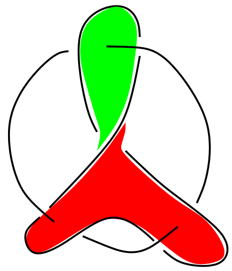
The proof for the separating case is similar, and will therefore be left as an exercise. $\square$

Exercise 35. *Complete the proof of the above lemma.*

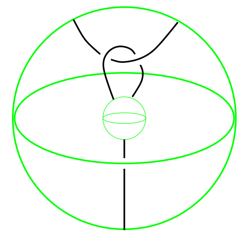
Note that the conditions are not sufficient when S has more than one component, since condition C4 does not rule out the possibility that the limiting character is finite on all components of $M - S'$ for a proper subsurface S' of S.

10.3. We now describe a reduced surface associated to an ideal point of the character variety of the manifold $m137$, which is still denoted by N, at which the non-trivial roots of Subsection 9.6 occur.

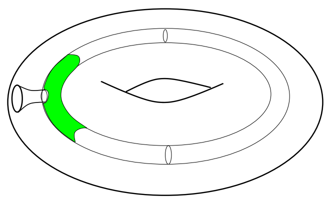
Recall that we obtain N by 0–surgery on either component of the link 7_1^2 in S^3, which implies that N can be viewed as the complement of a knot in $S^2 \times S^1$. The following is a discussion of Figure 19. In (a), we see a thrice punctured disc in the link complement, and we assume that the 0–surgery is performed on its boundary curve. The union of the punctured disc and a meridian disc of the added solid torus is a thrice punctured sphere S in N. We may think of S as the intersection of $S^2 \times z$ with N in $S^2 \times S^1$. Cutting N open along S results in the complement of three arcs in $S^2 \times I$, as shown in (b). The interior of $N - S$ is homeomorphic to the interior of an I–bundle over the once–punctured torus, the compact core of which is show in (c). A surface corresponding to one of the copies of S in the boundary of $N - S$ is shaded. The interior of $N - S$ is homeomorphic to the complement in S^3 of the trivalent graph of (d). For triangulations of and geometric structures on this space, see [4] as well as Figure 21.



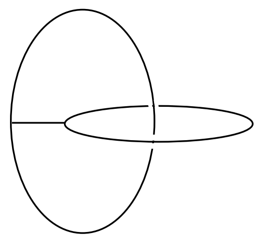
The link 7_1^2



$N - S$



$T_1 \times I$



Graph in S^3

FIGURE 19. The manifold $m137$

10.4. **Fundamental group.** We compute a Wirtinger presentation for the fundamental group of the complement of the link 7_1^2 with meridians oriented according to Figure 20. Denote the peripheral system by $\{\mathcal{M}_1, \mathcal{L}_1\} \cup \{\mathcal{M}_2, \mathcal{L}_2\}$. The fundamental group is generated by $\mathcal{M}_1 = \mathfrak{a}$ and $\mathcal{M}_2 = \mathfrak{b}$, and we obtain a single relator which is equivalent to either of the relations $[\mathcal{M}_i, \mathcal{L}_i] = 1$, where the longitudes are words in the meridians:

$$\mathcal{L}_1 = \mathfrak{a}^2\mathfrak{b}^{-1}\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{a}\mathfrak{b}\mathfrak{a}^{1-}\mathfrak{b}^{-1}\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}\mathfrak{b}^{-1},$$

$$\mathcal{L}_2 = \mathfrak{b}^2\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{a}\mathfrak{b}\mathfrak{a}\mathfrak{b}^{1-}\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{a}\mathfrak{b}\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{a}^{-1}.$$

Note that $\mathcal{M}_1$ is homologous to $\mathcal{L}_2$ and that $\mathcal{M}_2$ is homologous to $\mathcal{L}_1$. If we perform 0–surgery on the second cusp of 7_1^2, we have

$$\pi_1(N) = \langle \mathcal{M}_1, \mathcal{M}_2 \mid \mathcal{L}_2 = 1 \rangle.$$

A peripheral system of N is given by $\{\mathcal{M}_1, \mathcal{L}_1\}$, where $\mathcal{M}_1$ is nullhomologous. We now give a HNN–extension of $\pi_1(N)$ which corresponds to the splitting along S.

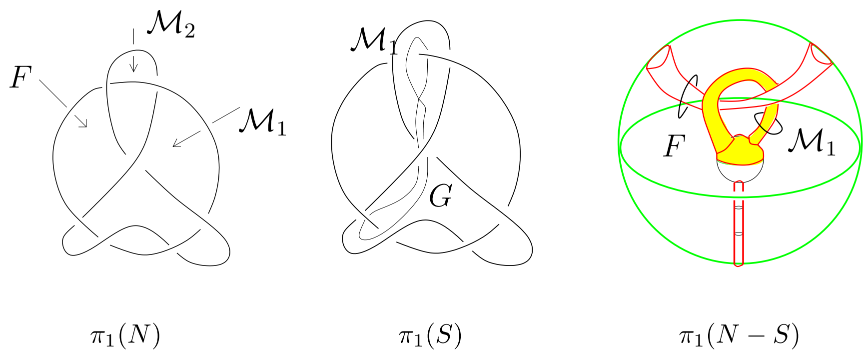


FIGURE 20. Generators for $m137$

It can be observed from Figure 20, that the elements $\mathcal{M}_1$ and F correspond to generators of $\mathrm{im}(\pi_1(N-S) \to \pi_1(N))$, and if S_- and S_+ are the two copies of S in $\partial(N-S)$, then $\mathcal{M}_1 \in \mathrm{im}(\pi_1(S_-) \to \pi_1(N))$ implies $F \in \mathrm{im}(\pi_1(S_+) \to \pi_1(N))$. Let

$$
\begin{aligned}
& F = \mathfrak{b}\mathfrak{a}\mathfrak{b}^{-1}, \qquad G = F^{-1}\mathfrak{a}^{-1}F\mathfrak{a}, \qquad H = F^{-1}\mathfrak{a}F\mathfrak{a}^{-1}, \\
\text{and} \quad & A = \langle \mathfrak{a}, F \rangle, \qquad A_1 = \langle \mathfrak{a}, G \rangle, \qquad A_2 = \langle F, H \rangle.
\end{aligned}
$$

Each of the groups A, A_1 and A_2 is free in two generators, and $\pi_1(N)$ is an HNN–extension of A across A_1 and A_2 by $\mathcal{M}_2 = \mathfrak{b}$ with the relations

$$
\mathfrak{b}\mathfrak{a}\mathfrak{b}^{-1} = F, \qquad \mathfrak{b}G\mathfrak{b}^{-1} = HF.
$$

This HNN–extension corresponds to the splitting of N along S.

10.5. **Representation spaces.** With the method of Subsection 4.7, irreducible representations of N can (up to conjugation and birational equivalence) be parameterised by:

$$\rho(\mathfrak{a}) = \begin{pmatrix} m & 0 \\ \frac{(1-m)(b^2(m^2-1)-m)}{b^2(m-1)+m^2+m^3} & m^{-1} \end{pmatrix}, \qquad \rho(F) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}$$

$$\rho(\mathfrak{b}) = \begin{pmatrix} \frac{b^2(1-m)+b^4(1-m)^2-m^2}{b(m-1)(b^2(m-1)+m^2+m^3)} & b \\ \frac{b(1-m)(b^2(m^2-1)-m)}{b^2(m-1)+m^2+m^3} & b(m^{-1}-m) \end{pmatrix},$$

subject to

$$\begin{aligned} 0 = {} & m^3(1+m+m^2) \\ & + b^2m(m^2-1)(1-m+m^2+2m^4+m^5) - b^4(m-1)^2(1+m+m^2). \end{aligned}$$

Thus, as $b \to 0$, we observe that either $m \to 0$ or m tends to a non–trivial third root of unity, which we denote by ζ_3. Since

$$\operatorname{tr}\rho(\mathfrak{b}) = \frac{-m^4 - b^2m^3(1-m)^2 + b^4(1-m)^2}{bm(m-1)(b^2(m-1)+m^2+m^3)},$$

it follows that as $b \to 0$ and $m \to \zeta_3$, we have $\operatorname{tr}\rho(\mathfrak{b}) \to \infty$, and an ideal point ξ of the character variety is approached. Moreover, the A–polynomial can be computed from the above, and we obtain:

$$\begin{aligned} 0 = {} & (m^{13} - l^4)(1-m)(1+m+m^2)^3 \\ & - l^2m^3(1+m^2)(1+3m+m^2-2m^3-3m^4-2m^5+2m^6 \\ & \qquad -2m^7-3m^8-2m^9+m^{10}+3m^{11}+m^{12}). \end{aligned}$$

As $m \to \zeta_3$, we have $l \to 0$, and hence $\operatorname{tr}\rho(\mathcal{L}_1) \to \infty$. Thus, $\mathcal{M}_1$ is a strongly detected boundary slope associated to ξ. The limiting representations of the images of A, A_1 and A_2 in $\pi_1(N)$ are determined by:

$$\begin{array}{llll} \operatorname{tr}\rho(\mathfrak{a}) \to -1, & \operatorname{tr}\rho(F) \to -1, & \operatorname{tr}\rho(F\mathfrak{a}) \to -\zeta_3, & \operatorname{tr}\rho[\mathfrak{a},F] \to -1, \\ \operatorname{tr}\rho(\mathfrak{a}) \to -1, & \operatorname{tr}\rho(G) \to -1, & \operatorname{tr}\rho(G\mathfrak{a}^{-1}) \to -1, & \operatorname{tr}\rho[\mathfrak{a},G] \to 2, \\ \operatorname{tr}\rho(F) \to -1, & \operatorname{tr}\rho(H) \to -1, & \operatorname{tr}\rho(FH) \to -1, & \operatorname{tr}\rho[F,H] \to 2. \end{array}$$

Thus, the limiting representation is irreducible on A, and reducible on A_1 and A_2. Since both $\operatorname{tr}\rho(\mathfrak{b})$ and $\operatorname{tr}\rho(\mathcal{L}_1)$ blow up, Lemma 39 implies that S is a reduced associated surface.

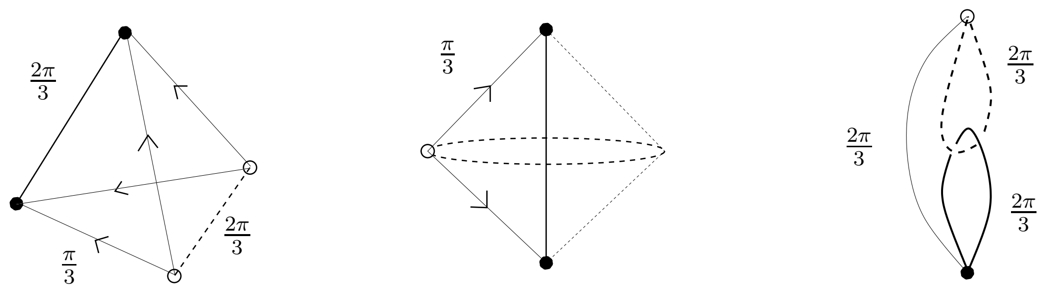


FIGURE 21. To obtain the limiting orbifold, first glue the faces meeting along one of the edges with cone angle $2\pi/3$ to obtain a spindle, and then identify the boundary discs of the spindle. The result is S^3 minus two points, with the labelled graph (minus its vertices) as the singular locus.

Exercise 36. *There are two thrice punctured discs in the complement of the link 7_1^2, one of which has boundary slopes determined by the pair $(\mathcal{L}_2, \mathcal{M}_1)$, the other by the pair $(\mathcal{L}_1, \mathcal{M}_2)$. Denote these discs by D_1 and D_2 respectively. We observed that the boundary components of each of these discs are homologous, and that* 0 *Dehn filling on the second cusp lets D_1 descend to a properly embedded surface in N, which we have associated to the slope* 0*.*

The disc D_2 descends to a surface in N which is not properly embedded since the boundary curve $\mathcal{M}_2$ is not peripheral. However, the slope ∞ determined by $\mathcal{L}_1$ is detected by the reducible representations of $\pi_1(N)$. Can you modify D_2 and detect the resulting (essential) surface?

10.6. **Degeneration of the hyperbolic structure.** Since the limiting eigenvalue is a third root of unity, the limiting representation may correspond to a geometric decomposition of the orbifold $N(3,0)$. Using `SnapPea`, we can observe the deformation of the ideal triangulation of N as we approach $N(3,0)$. The manifold $m137$ has a triangulation with four tetrahedra, and as we approach $N(3,0)$ through positively oriented triangulations determined by the surgery coefficients $(p,0)$ with $p \geq 3$, three tetrahedra degenerate, whilst the remaining tetrahedron has limiting shape parameter $\frac{1}{2} + \frac{\sqrt{3}}{6}i$. In fact, $N(3,0)$ contains a Euclidean $(3,3,3)$–turnover whose complement is a hyperbolic 3–orbifold of volume approximately 0.68 which can be triangulated by a single ideal tetrahedron as shown in Figure 21. Note that the link of each vertex of the tetrahedron is a Euclidean $(3,3,3)$–turnover.

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