Towards an intersection homology theory for real algebraic varieties

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1. Standard intersection homology

1.1. Homology of smooth manifolds

X nice compact finite-dimensional space

Homology: groups $H_i(X,A)$ in degree $i \geq 0$, with coefficients in an abelian group A.

- Functorial with respect to continuous mappings.
- M compact d-dimensional *manifold*, k a field: nondegenerate *intersection product*

$$H_i(M,k) \times H_{d-i}(M,k) \to k$$

Remark. If X not compact: 2 kinds of homology (compact supports and closed supports), one dual to the other for manifolds.

Same for intersection homology of singular noncompact spaces (for simplicity we restrict discussion to compact spaces).

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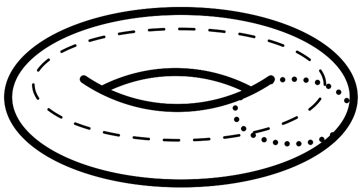
Example. For $k = \mathbf{Z}/2$:
$H_i(X,\mathbf{Z}/2) = \frac{\text{“}\{\text{subspaces of dim. } i \text{ without boundary}\}\text{”}}{\text{“}\{\text{boundaries of subspaces of dim. } i+1\}\text{”}}$

For $X =$ torus: $H_i(X,\mathbf{Z}/2) =$

$$\begin{cases} \mathbf{Z}/2 & \text{for } i=0\text{: any two points connected by line segment} \\ \mathbf{Z}/2 \times \mathbf{Z}/2 & \text{for } i=1\text{:} \\ \mathbf{Z}/2 & \text{for } i=2\text{: generated by whole space} \end{cases}$$



Intersection product “=” taking actual intersections:

$$H_0(X,\mathbf{Z}/2) \times H_2(X,\mathbf{Z}/2) \to \mathbf{Z}/2$$
$$H_1(X,\mathbf{Z}/2) \times H_1(X,\mathbf{Z}/2) \to \mathbf{Z}/2$$

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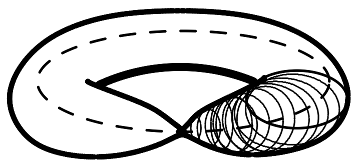
1.1. Homology of singular spaces

Example. X the crust of a croissant.



$$H_i(X,\mathbf{Z}/2) = \begin{cases} \mathbf{Z}/2 & \text{for } i=0 \\ \mathbf{Z}/2 & \text{for } i=1\text{: vertical loop is now boundary} \\ \mathbf{Z}/2 & \text{for } i=2 \end{cases}$$

Problem with

$$H_1(X,\mathbf{Z}/2) \times H_1(X,\mathbf{Z}/2) \xrightarrow{?} \mathbf{Z}/2$$

- No well-defined intersection product

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1.3. Intersection Homology

(due to M. Goresky & R. MacPherson, +/- 1980)

Stratification:

$$X = U \cup \bigcup_i S_i$$

U a manifold that open and dense in X, and $\Sigma = \bigcup_i S_i$ the 'singular locus' cut into locally closed submanifolds S_i, each of even codimension (+ some extra conditions).

Example. Whitney stratification of a complex algebraic variety.

Definition. Intersection homology $IH_i(X,A)$ as ordinary homology, but *intersection condition*: For each $V \subset X$ representing a class or giving a boundary:

$$\operatorname{codim}_V(V \cap S) > 1/2 \operatorname{codim}_X(S)$$

for each stratum $S \subset \Sigma$.

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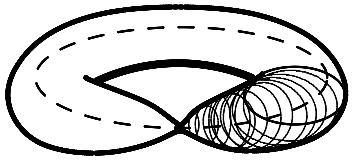
(Intersection condition: $\operatorname{codim}_V(V \cap S) > 1/2 \operatorname{codim}_X(S)$)

Example.



$IH_i(X, \mathbf{Z}/2) =$

$$\begin{cases} \mathbf{Z}/2 & \text{for } i = 0 \\ 0 & \text{for } i = 1\text{: vertical loop is boundary; horizontal loop not admissible} \\ \mathbf{Z}/2 & \text{for } i = 2 \end{cases}$$

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Intersection product

k a field, X as above of dimension d

$$IH_i(X,k) \times IH_{d-i}(X,k) \to k$$

nondegenerate.

Small resolutions

Definition. A resolution of singularities $\tilde{X} \to X$ of a complex projective variety X is a *small resolution* if for each $r > 0$ the locus

$$\{x \in X : \ \dim f^{-1}(x) \geq r\}$$

has codimension $> 2r$.

Then

$$IH_i(X,A) = H_i(\tilde{X},A)$$

Remark. In general, algebraic singularities do *not* admit small resolutions.

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2. Why standard intersection homology does not work for real algebraic varieties

For complex projective algebraic varieties we have intersection homology with coefficients in any field k. Most important properties:

- A non-degenerate intersection product
- An isomorphism to the homology of any small resolution
- Only dependent on the homeomorphism type

For projective real algebraic varieties (algebraic subsets of $\mathbf{P}^N(\mathbf{R})$) the standard construction does not work

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- Smooth real algebraic varieties need not be orientable, so in general the intersection product can only exist for 2-torsion coefficients.

  Example. $\mathbf{P}^2(\mathbf{R})$

- Real algebraic varieties do not admit even-dimensional stratifications in general.

  Example. $\{(u:v:x:y:z)\colon\ x^2+y^2+z^2=u^2\}\subset\mathbf{P}^4(\mathbf{R})\}$ is a threefold with an isolated singularity at the origin.

- Homeomorphic real algebraic varieties can have small resolutions that have different homology.

  Example. $\{y^2=x(x^2+z^2)\}\cup\{y^2=-x(z+x)\}\subset\mathbf{P}^2(\mathbf{R})$ is homeomorphic to $y^2=x^2(x+z)$.

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Hence in general we can hope for an intersection homology for real algebraic varieties with coefficients in $\mathbf{Z}/2$ and:

- A non-degenerate intersection product
- An isomorphism to the homology of any small resolution

This is what Goresky and MacPherson asked in Borel's seminar in 1984.

At that time it was not known whether different small resolutions of a real algebraic variety have the same homology.

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3. Complexification

Definition. A *projective algebraic variety X defined over* $\mathbf{R}$ is the set of zeroes in $\mathbf{P}^N(\mathbf{C})$ of a system of homogeneous polynomial equations with coefficients in $\mathbf{R}$.

- Real points: $X(\mathbf{R})\subset\mathbf{P}^N(\mathbf{R})$
- complex points: $X(\mathbf{C})\subset\mathbf{P}^N(\mathbf{C})$.

Complex conjugation on $\mathbf{P}^N(\mathbf{C})$ induces an involution

$$\sigma\colon\ X(\mathbf{C})\to X(\mathbf{C}).$$

(hence an action of $G=\mathbf{Z}/2$) and the real points are the fixed points:

$$X(\mathbf{C})^G=X(\mathbf{R}).$$

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Main idea

The intersection homology for $X(\mathbf{R})$ should come from the intersection homology of $X(\mathbf{C})$.

$$X(\mathbf{C})\xrightarrow{\text{take } G\text{-invariants}}X(\mathbf{R})$$
$$IH_*(X(\mathbf{C}))\xrightarrow{\text{take } G\text{-invariants}}IH_*(X(\mathbf{R}))$$

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3.1 Homology and G-invariants

In general

$$H_*(X(\mathbf{R}),\mathbf{Z}/2) \neq H_*(X(\mathbf{C}),\mathbf{Z}/2)^G.$$

More sophisticated:

For a G-module M consider not just M^G but all higher derived functors associated to $(-)^G$. Get *cohomology of G with coefficients in M*:

$$H^i(G,M) = \begin{cases} M^G & \text{for } i=0 \\ M^G/(1+\sigma)M & \text{for } i>0 \end{cases}$$

(periodicity of the cohomology of $G=\mathbf{Z}/2$)

Also: cohomology of G with coefficients in a *complex* of G-modules $\mathcal{C}$, rather than a single G-module M.

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Take a G-equivariant chain complex $\mathcal{C}_*(X(\mathbf{C}),\mathbf{Z}/2)$ that computes homology of $X(\mathbf{C})$, and put

$$H_i(X(\mathbf{C});G,\mathbf{Z}/2) := H^{-i}(G,\mathcal{C}_*(X(\mathbf{C}),\mathbf{Z}/2))$$

then

$$H_i(X(\mathbf{C});G,\mathbf{Z}/2) = H_{i-1}(X(\mathbf{C});G,\mathbf{Z}/2)$$

for all $i<0$.

Homological fixed point theorem

$$H_{-1}(X(\mathbf{C});G,\mathbf{Z}/2) = \bigoplus_{i=0}^{\dim(X)} H_i(X(\mathbf{R}),\mathbf{Z}/2).$$

Remark. $H_{-1}(X(\mathbf{C});G,\mathbf{Q})=0$

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4. Defining ungraded IH via the complexification

Definition. Take a G-equivariant chain complex $\mathcal{IC}_*(X(\mathbf{C}),\mathbf{Z}/2)$ that computes the intersection homology of $X(\mathbf{C})$, Put:

$$IH_i(X(\mathbf{C});G,\mathbf{Z}/2) := H^{-i}(G,\mathcal{IC}_*(X(\mathbf{C}),\mathbf{Z}/2))$$
$$IH_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) := IH_{-1}(X(\mathbf{C});G,\mathbf{Z}/2)$$

Observe: No grading on $IH_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2)$.

Remark. Sheaf-theoretic definition:

$$IH_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) := H^{\infty}(X(\mathbf{C});G,\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2))$$

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Intersection product

- Non-degenerate intersection product for $IH_*(X(\mathbf{C}),\mathbf{Z}/2)$
- Duality in cohomology of G

This gives a nondegenerate intersection pairing

$$IH_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) \times IH_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) \to \mathbf{Z}/2$$

Small resolutions

- $IH_*(X(\mathbf{C}),\mathbf{Z}/2) \simeq H_*(\tilde{X}(\mathbf{C}),\mathbf{Z}/2)$ for a small resolution $\tilde{X} \to X$
- Homological fixed point theorem

This gives

$$IH_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2) \simeq H_*(\tilde{X}(\mathbf{R}),\mathbf{Z}/2)$$

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5. Looking for a grading

How to give $IH_{\circledast}(X(\mathbf{C}),\mathbf{Z}/2)$ a grading?

First: how to recover the grading on

$$H_{-\infty}(X(\mathbf{C});G,\mathbf{Z}/2)=H_*(X(\mathbf{R}),\mathbf{Z}/2)$$

Sheaf-theory:

$$H_{-i}(X(\mathbf{C});G,\mathbf{Z}/2)=H^i(X(\mathbf{C});G,\mathfrak{D}_{X(\mathbf{C})})$$

For $i>0$:

$$H^i(X(\mathbf{C});G,\mathfrak{D}_{X(\mathbf{C})})=H^i(X(\mathbf{R}),R\pi_*^G\mathfrak{D}_{X(\mathbf{C})}|_{X(\mathbf{R})})$$

The complex $R\pi_*^G\mathfrak{D}_{X(\mathbf{C})}|_{X(\mathbf{R})}$ splits into an infinite direct sum of shifted copies of $\mathfrak{D}_{X(\mathbf{R})}$.

Taking the cohomology sheaf with respect to the top perverse t-structure on the derived category of sheaves on $X(\mathbf{R})$ gives a single copy of $\mathfrak{D}_{X(\mathbf{R})}$, hence a grading on

$$H_{-\infty}(X(\mathbf{C});G,\mathbf{Z}/2)=H_*(X(\mathbf{R}),\mathbf{Z}/2).$$

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Similarly, we can consider the almost periodic complex of sheaves

$$R\pi_*^G\,\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2)|_{X(\mathbf{R})}$$

and try to split it.

Problems

- At this stage I don't know whether the complex splits in any way.
- A perverse t-structure does not seem to give the right cohomology sheaves in general.

When X admits a small resolution these problems disappear.

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When X admits a small resolution:

- Any of the two middle perverse t-structures on $X(\mathbf{R})$ gives the correct splitting of

  $$R\pi_*^G\,\mathbf{IC}(X(\mathbf{C}),\mathbf{Z}/2)|_{X(\mathbf{R})}$$

  The corresponding grading on $IH_{\circledast}(X(\mathbf{R}),\mathbf{Z}/2)$ coincides with the grading on the homology of $\tilde{X}(\mathbf{R})$.

Corollary. *If $\tilde{X}\to X$ and $\tilde{X}'\to X$ are two small resolutions, then for all $i\geq 0$*

$$H_i(\tilde{X}(\mathbf{R}),\mathbf{Z}/2)=H_i(\tilde{X}'(\mathbf{R}),\mathbf{Z}/2).$$

This was an open question since Goresky's and MacPherson's question – Independently, B. Totaro and A. Parusinski have found different proofs for this result.

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Conclusion

- The techniques of equivariant (co)homology give an ungraded intersection homology theory for real algebraic varieties with the right formal properties.
- This homology theory does admit a natural grading whenever there is a small resolution.
- At this stage it is not clear whether it is possible to obtain a natural grading in general.

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