On Degenerate Monge-Ampère Equations over Closed Kähler Manifolds

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1 Introduction

Monge-Ampère equation has been a classic problem in analysis for a long time. Its complex version, complex Monge-Ampère equation, has drawn a lot of intentions from an even wider group of mathematicians since the breakthrough work of S. -T. Yau almost thirty years ago. The solvability of this equation over a closed manifold for a Kähler class, which was proved in [Ya], allows a lot of applications in algebraic and differential geometry. Later, by using a completely different method (as in [Ko1]), S. Kolodziej was able to achieve a priori L^∞ estimate and also continuity of the solution with much less regularity assumption on the measure. In those works, the class is always Kähler. We can now generalize the results to certain cases when the class is no longer Kähler. That's the degeneracy of the equation as in the title of this note. The degeneracy allowed is of quite some interest in algebraic geometry (in the study of minimal model problem, for example).

The main goal of this note is to prove the following theorem which is an improved version of what is stated in [TiZh].

Theorem 1.1. *Let X be a closed Kähler manifold with (complex) dimension $n \geqslant 2$. Suppose we have a holomorphic map $F : X \to \mathbb{CP}^N$ with the image $F(X)$ of the same dimension as X. Let ω_M be any Kähler form over some neighbourhood of $F(X)$ in $\mathbb{CP}^N$. For the following equation of Monge-Ampère type:*

$$(F^*\omega_M + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega, \tag{1.1}$$

where Ω is a fixed smooth (non-degenerate) volume form over X and f is a nonnegative function in $L^p(X)$ for some $p > 1$ with the correct total integral over X, i.e. $\int_X f\Omega = \int_X (F^*\omega_M)^n$, we have the following:

(1) (A priori estimate) If u is a weak solution in $PSH_{F^*\omega_M}(X) \cap L^\infty(X)$ of the equation with the normalization $\sup_X u = 0$, then there is a constant C such that $\|u\|_{L^\infty} \leq C\|f\|^n_{L^p}$ where C only depends on F, ω_M and p;

(2) (Existence of a bounded solution) There exists a bounded (weak) solution for this equation;

(3) (Continuity and uniqueness of bounded solution) If F is locally birational, any bounded solution is actually the unique continuous solution.

The lower index "M" of ω_M is the initial letter of "*model*" since ω_M can be naturally understood as the model metric of the degenerate metric which we are interested in with the degeneration information being hidden in the map F. And by Sard's Theorem, this quantity $\int_X (F^*\omega_M)^n$ is clearly positive from the setting.

The improvements are in two places:

i) we need X to be closed Kähler instead of projective;

ii) in statement (3) about continuity, the assumption is weakened a lot since we no longer require the image $F(X)$ to have an orbifold structure.

It might be worth taking a little time to clarify some terminology appearing in the statement.

First, u being a weak solution means both sides of the equation are equal as (Borel) measures. The meaning of the right hand side of the eqaution as a measure is quite classic with u being bounded as explained in [BeTa].

In definition of $L^p(X)$ space, we choose Ω as the volume form to integrate over X. Clearly different choices of smooth volume forms would end up with equivalent L^p-norms and the same L^p space.

In (3), "locally birational" means that for a small enough neighbourhood U of any point on $F(X)$, each component of $F^{-1}(U)$ would be birational to U (under F). Clearly it would be the case if F is birational itself and in fact this is the case with most geometric interest as the author see it now.

If F is an embedding, this theorem is proved in [Ko1] for even more general function f. Actually, he only needs $F^*\omega_M$ to represent a Kähler class.

To prove the main theorem above, we would generalize the original argument there which makes use of the results in [BeTa] and many other works in pluripotential theory.

Remark 1.2. *The discussion in this short note is supposed to be fairly concise. This is achieved by omitting some of the details of the proofs and also taking shortcuts which might make the idea of the argument less shown. To complete the argument, we'll frequently refer the reader to the works of Kolodziej's [Ko1], [Ko2]. For further details and related discussions, see [Zh].*

In this note, plurisubharmonic sometimes means plurisubharmonic with respect to some background form. Hopefully, it'll be clear from the context.

Acknowledgment 1.3. *For X projective, the main results of this paper except this version of the continuity result as in (3) of Theorem 1.1 were announced and discussed in my previous preprint with Professor Tian. They were also presented in a talk at Imperial College in November, 2005. The general continuity* *was proved soon after in January, 2006 after a few discussions with Professors S. Kolodziej and H. Rossi on approximating plurisubharmonic functions over singular spaces. I would like to thank them both for very useful discussions. The current results were presented in several talks afterwards. A new result in the recent preprint by Blocki and Kolodziej allows the current generalization to a closed Kähler manifold X. I really appreciate their informing me about this result. I would also like to thank my advisor, Professor Tian, for bringing to my attention this basic question on the complex Monge-Ampère equation. Later, we were informed that a similar boundedness result has been obtained by Philippe Eyssidieux, Vincent Guedj and Ahmed Zeriahi in the recent preprint [EyGuZe].*

2 Idea of Generalizing Kolodziej's Argument and Preparation

The (degenerate Monge-Ampère) equation we are considering is the following:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega$$

over X, where $\omega_\infty = F^*\omega_M$ and Ω is a volume form over X. This lower index ∞ illustrates the point of view that this degenerate case most naturally arises as the limit of non-degenerate case which has been used in [TiZh]. Our goal is to find a bounded (and even continuous) solution and get some properties for it, for example, uniqueness.

2.1 Idea of Generalization

In our method, the solution is to be obtained by taking the limit of solutions for a family of approximation equations. Of course we want the solvability of the approximation equations to be known. In our case, the results in [Ko1] should be sufficient to guarantee this.

In order to get a limit, we need the a priori (L^∞) bound for those approximate solutions as in [Ko1]. In the original argument there, it is the generalized right hand side that is treated. We can deal with the right hand of the equation above just as well. The main difficulty now is of course the degenaracy of ω_∞ as a Kähler metric over X.

We have a natural family of approximation equations as follows:

$$(\omega_\infty + \epsilon\phi + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = C_\epsilon f\Omega$$

where $\phi > 0$ is a closed smooth $(1,1)$-form and $\int_X C_\epsilon f\Omega = \int_X (\omega_\infty + \epsilon\phi)^n$ for $\epsilon \in (0,1]$. Obviously, we have $C_\epsilon \in (1, C]$. In the following, C will always denote a uniform constant, and it may change from line to line.

From Kolodziej's result, we have no trouble to find a unique continuous solution for each of these equations after requiring the normalization $\sup_X u_\epsilon = 0$.

Now we just have to prove that u_ϵ's are uniformly bounded from below.

The difficulty appears at this time since $\omega_\infty + \epsilon\phi$ is not uniformly positive for $\epsilon \in (0, 1]$, i.e., if we consider the local potentials in coordinate balls, they will no longer be uniformly convex. Thus no matter how good the choice is, we do not have a uniform growth of the potential when moving away from the origin of the coordinate ball which is very crucial for the original argument in [Ko1].

This one blow seems to completely destroy Kolodziej's argument. As we see it now, the main reason is that the picture of a coordinate (Euclidean) ball in X is a little too local.

The most important observation is that if we can choose a domain V which has those degenerate directions of ω_∞ going around inside, then we can still have the uniform convex local potential, i.e., the values for the very outside part are greater than those of the very inside part by a uniform positive constant. In the case of ω_∞ being a genuine Kähler metric, this picture can be achieved in local Euclidean disks sitting inside the manifold X just as what's used in [Ko1].

More precisely, if we take a ball in $\mathbb{CP}^N$ which covers part of $F(X)$, then the preimage of that ball in X would be the domain V mentioned above. The potential of ω_∞ in V be convex in the sense above from the positivity of ω_M. Furthermore, we can see that the domain V is hyperconvex in the usual sense which means we can have a continuous exhaustion of the domain, and there are actually a lot of nice plurisubharmonic functions over V which can be got by pulling back classic functions defined over the ball in $\mathbb{CP}^N$.

There seems to be another problem since the metric ϕ, which is used to perturb the equation, may (should) not have a global potential in the domain V. But we can deal with this by considering plurisubharmonic functions in V with respect to $\omega_\infty + \epsilon\phi$ for each $\epsilon \in (0, 1]$. In fact we can include the case of $\epsilon = 0$ in all the argument. The important thing is that our argument is uniform for all such ϵ's.

Remark 2.1. *For the a priori estimate in* (1) *of the main theorem, we only need to work with* ω_∞ *(i.e.,* $\epsilon = 0$*). But we will need the estimate uniformly for all* $\omega_\infty + \epsilon\phi$ *for* $\epsilon \in (0, 1]$ *in order to prove the existence result in* (2) *of the main theorem. The above just says we can treat them together.*

We should point out that a global argument will be used below which might apparently hide the above idea of considering a generalized domain V. Indeed the punchline is still to study such a domain V. It should be quite natural that after getting all the necessary information for the local domains, we can patch them up to get for the whole of X just as in [Ko2].

2.2 Preparation

Many classic results in pluripotential theory are quite local, for example, weak convergence results, and so can be used in our situation automatically. Many definitions also have their natural versions for the domain V with background metric which can't be reduced to potential level globally in V, for example, relative capacity. Since it is of the most importance for us in this work, let's give the definition below in the case when $V = X$, which is the version that we are going to use.

Definition 2.2. *Suppose ω is a (smooth) nonnegative $(1,1)$-form. For any (Borel) subset K of X, we define the relative capacity of K with respect to ω as follows:*

$$Cap_\omega(K) = \sup\{\int_K (\omega + \sqrt{-1}\partial\bar{\partial}v)^n | v \in PSH_\omega(X), \quad -1 \leqslant v \leqslant 0\}.$$

We require ω to be nonnegative so that $PSH_\omega(X)$ won't be empty. We also point out that it is usually sufficient to consider only sets K which are compact in order to study any set by approximation.

Remark 2.3. *Classic results in pluripotential theory can be found in works such as [Le], [BeTa]. More recent works as [De] and [Ko2] might also be convenient as references.*

The only thing which is not so trivially adjusted to our situation might be the comparison principle which is so important for this area and also has a global feature. There are several ways to deal with this situation. One of them seems to be the easiest to describe and in a sense it minimizes the modification of Kolodziej's argument for our case. For these reasons, we will use it in this note. The other methods are also of interest and will be discussed in [Zh].

Now let's state the version of the comparison principle we are going to use later.

Proposition 2.4. *For X as above, suppose $u,\ v \in PSH_\omega(X) \cap L^\infty(X)$ where ω is a smooth nonnegative closed $(1,1)$-form, then*

$$\int_{\{v<u\}} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n \leqslant \int_{\{v<u\}} (\omega + \sqrt{-1}\partial\bar{\partial}v)^n.$$

This version is slightly different from other more classic versions because X may not be projective, ω may not be positive and the functions may not be continuous. The brief description of justification is as follows.

We still just need a decreasing approximation for any bounded plurisubharmonic function by smooth plurisubharmonic functions according to the argument in [BeTa]. This is not as easy as in Euclidean space where convolution is available. And the possible loss of projectivity of X makes it difficult to use some other classic results.

But according to the recent result of Blocki and Kolodziej in [BlKo], we can have a decreasing smooth approximation for bounded plurisubharmonic functions over X. The approximation result needs the background form to be positive (i.e., a Kähler metric), but clearly nonnegative form (as ω_∞ for us) is acceptable when it comes down to comparison principle by simple approximation argument. This is also why we can now have X to be just Kähler instead of projective as stated in [TiZh]. Notice that we do need X to be Kähler in order to allow this approximation and justify the comparison principle.

The next few sections will be devoted to prove each of the statements in the main theorem.

3 A priori L^∞ Estimate

3.1 Bound Relative Capacity by Measure

In the following, ω is a (smooth) nonnegative closed $(1,1)$-form. Keep in mind that ω will stand for $\omega_\infty + \epsilon\phi$ for any $\epsilon \in [0,1]$. The constants do not depend on ϵ.

For $u,\ v \in PSH_\omega(X) \cap L^\infty(X)$ with $U(s) := \{u - s < v\} \neq \varnothing$ for $s \in [S, S+D]$. Also assume v is valued in $[0, C]$. Then $\forall w \in PSH_\omega(X)$ valued in $[-1, 0]$, for any $t \geqslant 0$, since $0 \leqslant t + Ct + tw - tv \leqslant t + Ct$, we have:

$$U(s) \subset V(s) = \{u - s - t - Ct < tw + (1-t)v\} \subset U(s + t + Ct).$$

So we have for $0 < t \leqslant 1$:

$$\begin{aligned}
\int_{U(s)} (\omega + \sqrt{-1}\partial\bar{\partial}w)^n &= t^{-n} \int_{U(s)} \big(t\omega + \sqrt{-1}\partial\bar{\partial}(tw)\big)^n \\
&\leqslant t^{-n} \int_{U(s)} \big(t\omega + \sqrt{-1}\partial\bar{\partial}(tw) + (1-t)\omega + \sqrt{-1}\partial\bar{\partial}((1-t)v)\big)^n \\
&= t^{-n} \int_{U(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(tw + (1-t)v)\big)^n \\
&\leqslant t^{-n} \int_{V(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(tw + (1-t)v)\big)^n \\
&\leqslant t^{-n} \int_{V(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(u - s - t - Ct)\big)^n \\
&\leqslant t^{-n} \int_{U(s+t+Ct)} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n.
\end{aligned}$$

The comparison principle is applied to get the second to the last inequality. All the other steps are rather trivial from the setting.

Thus from the definition of Cap_ω, we can conclude

$$t^n \cdot Cap_\omega(U(s)) \leqslant \int_{U(s+t+Ct)} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n$$

for $t \in (0, min(1, \frac{S+D-s}{1+C})]$. Of course, for our purpose, it is always safe to assume $\frac{S+D-s}{1+C} < 1$. In fact, let's take $D < 1$.

Now we can rewrite this inequality as:

$$t^n \cdot Cap_\omega(U(s)) \leqslant (1+C)^n \int_{U(s+t)} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n$$

for $t \in (0, S+D-s]$ by rescaling t.

Intuitively, the constant D can be seen as the gap where the values of u can stretch over.

3.2 Bound Gap D by Capacity

We are still in the previous setting. Now assume that for any (Borel or compact) subset E of X, we have:

$$\int_E (\omega + \sqrt{-1}\partial\bar{\partial}u)^n \leqslant A \cdot \frac{Cap_\omega(E)}{Q\big(Cap_\omega(E)^{-\frac{1}{n}}\big)}$$

for some constant $A > 0$, where $Q(r)$ is an increasing function for positive r with positive value. From now on, this condition will be denoted by Condition (A).

The result to be proved in this subsection is as follows:

$$D \leqslant \kappa(Cap_\omega(U(S+D)))$$

for the following function

$$\kappa(r) = (1+C) \cdot C_n A^{\frac{1}{n}} \Big(\int_{r^{-\frac{1}{n}}}^{\infty} y^{-1}(Q(y))^{-\frac{1}{n}} dy + \big(Q(r^{-\frac{1}{n}})\big)^{-\frac{1}{n}} \Big),$$

where C_n is a positive constant only depending on n and the $1+C$ is from the rescaling at the end of the previous step.

The proof is a little technical but quite elementary in spirit. We will only briefly describe the idea below.

The previous part gives us an inequality "$Cap \leqslant measure$".

Condition (A) gives the other direction "$measure \leqslant Cap$".

We can then combine them to get some information about the length of the interval which comes from t in the inequality proved before. The assumption of nonemptiness of the sets is needed because we have to divide $Cap_\omega(U(\cdot))$ from both sides in order to get something purely for t.

Finally, we can sum all these small t's up to get control for D. [1]

[1] We need the trivial fact that nonemptiness, nonzero (Lebesgue) measure and nonzero capacity are equivalent for such sets $U(s)$ from the fundamental properties of plurisubharmonic functions.

Of course we'd better use a delicate way to carry out all these just in sight of the rather complicated final expression of the function κ. It has been done beautifully in [Ko1]. [2]

Let's emphasize that in the argument, we do not have a positive lower bound for the t's to be summed up, so it is important that the inequality proved in the previous part holds (uniformly) for all small enough $t > 0$.

3.3 Bound Capacity

Claim: For $u \in PSH_\omega(X) \cap L^\infty(X)$ and $u \leqslant 0$, if K is a compact set in X which can well be X itself, then there exists a positive constant C such that:

$$Cap_\omega(K \cap \{u < -j\}) \leqslant \frac{C\|u\|_{L^1(X)} + C}{j}.$$

This is aiming for a uniform upper bound for the relative capacity appearing on the right hand side of the inequality proved in the previous subsection.

Proof. For any $v \in PSH_\omega(X)$ and valued in $[-1, 0]$, consider any compact set $K' \subset K \cap \{u < -j\}$. Using CLN inequality [3] as in [Ko2], we get:

$$\begin{aligned} \int_{K'} (\omega + \sqrt{-1}\partial\bar{\partial}v)^n &\leqslant \frac{1}{j} \int_K |u|(\omega + \sqrt{-1}\partial\bar{\partial}v)^n \\ &\leqslant \frac{C\|u\|_{L^1(X)} + C}{j} \end{aligned}$$

where the constant C in the final expression comes from the (chosen) local potentials of ω.

From the definition of relative capacity, this would give the inequality we want.

□

Now we consider the L^1-norm for those approximation solutions u_ϵ (and also the solution u if it exists by assumption). The following is just the standard Green's function argument. Strictly speaking, the computation needs the function to be smooth, but we can achieve the final estimate by using approximation sequence given by the result in [BlKo] for our situation. So let's pretend that we have the necessary regularity in the following.

For fixed $\epsilon \in [0, 1]$, suppose $u_\epsilon(x) = 0$ and $C > G$ where G is the Green function for the metric $\omega_1 = \omega_\infty + \phi$.

[2] There is no constant $1 + C$ in that paper. But one can just add this constant into the computation trivially and get this result as stated in [Ko2].

[3] The global version of this inequality over X is quite easy to justify in light of the locality of the result.

Also since $\omega_\infty + \epsilon\phi + \sqrt{-1}\partial\bar{\partial}u_\epsilon \geqslant 0$, we have

$$\Delta_{\omega_1} u_\epsilon = \langle \omega_1, \sqrt{-1}\partial\bar{\partial}u_\epsilon \rangle \geqslant \langle \omega_1, -\omega_\infty - \epsilon\phi \rangle \geqslant -C$$

where C is uniform for $\epsilon \in [0,1]$.

Then we have:

$$\begin{aligned}
0 = u_\epsilon(x) &= \frac{1}{\int_X {\omega_1}^n} \int_X u_\epsilon {\omega_1}^n + \int_{y\in X} G(x,y)\Delta_{\omega_1} u_\epsilon \cdot {\omega_1}^n \\
&= \frac{1}{\int_X {\omega_1}^n} \int_X u_\epsilon {\omega_1}^n + \int_{y\in X} \big(G(x,y) - C\big)\Delta_{\omega_1} u_\epsilon \cdot {\omega_1}^n \\
&\leqslant \frac{1}{\int_X {\omega_1}^n} \int_X u_\epsilon {\omega_1}^n - C\int_{y\in X} \big(G(x,y) - C\big){\omega_1}^n \\
&\leqslant \frac{1}{\int_X {\omega_1}^n} \int_X u_\epsilon {\omega_1}^n + C.
\end{aligned}$$

This gives the uniform L^1 bound for u_ϵ's by noticing that they are all non-positive.

Hence we know *the set where u_ϵ has very negative value should have (uniformly) small relative capacity.*

3.4 Conclusion

Combining all the results above, if we assume Condition (A) for some function $Q(r)$ and set the function v at the beginning to be 0, we have:

$$D \leqslant \kappa(\frac{C}{D})$$

if $U(s) = \{u < -s\}$ nonempty for $s \in [-2D, -D]$ where C is a positive constant.

Furthermore, if we can choose the function $Q(r)$ to be $(1+r)^m$ for some $m > 0$ so that Condition (A) holds, this would imply that the function u only take values in a bounded interval since D can not be too large because as D goes to ∞, κ goes to 0. This D may not be smaller than 1, but since the existence of a big gap implies the existence of small ones, it's enough for our contradiction.

That's enough for the lower bound in light of the normalization $\sup_X u = 0$. The more explicit bound claimed in the theorem is not hard to get by carefully tracking down the relation [4]. Of course the constant A in Condition (A) is fairly involved here. Anyway, as the author sees it, the most important thing is the existence of such a bound.

3.5 Condition (A)

In this subsection, we justify Condition (A) under the measure assumption in the main theorem. This part is the essential generalization of Kolodziej's origi-

[4] The choice of the power for $\|f\|_{L^p}$ is actually very flexible as shown in [Zh].

nal argument.

In our case, $f \in L^p$ for some $p > 1$, which is the measure on the left hand side of Condition (A) from the equation we want to solve. For the approximation equations, the measures are different, but clearly we can bound the L^p-norm uniformly.

Applying Hölder inequality, it suffices to prove the following inequality:

$$\lambda(K) \leqslant C \cdot \big(Cap_\omega(K)(1 + Cap_\omega(K)^{-\frac{1}{n}})^{-m}\big)^q,$$

where λ is the smooth measure over X using the volume form Ω, i.e., $\lambda(K) = \int_K \Omega$, and q is the positive constant decided by $\frac{1}{p} + \frac{1}{q} = 1$. Obviously, it would be enough to prove:

$$\lambda(K) \leqslant C_l \cdot Cap_\omega(K)^l \qquad (A)$$

for l sufficiently large.

Of course we have $\lambda(K) < C$, and so in fact we can get for any nonnegative l if the above is true. In the following, we'll consider Condition (A) in this form.

For ω (uniformly) positive, this can be easily reduced to a Euclidean ball. As in [Ko2], using a classical measure theoretic result in [Ts], we have:

$$\lambda(K) \leqslant C \cdot \exp\big(-\frac{C}{Cap_\omega(K)^{\frac{1}{n}}}\big) \qquad (\star).$$

This is actually stronger than the version above after noticing small capacity situation is of the main interest.

In the following proof of Condition (A), the essential step is to prove the following inequality:

$$\lambda(K) \leqslant C_1 \cdot \delta^{N_1} + C_1 \cdot \delta^{-N_2} \exp\big(\frac{C_2}{\log\delta \cdot Cap_\omega(K)^{\frac{1}{n}}}\big) \qquad (B)$$

for sufficiently small $0 < \delta < \frac{1}{2}$. All the positive constants C_i's and N_i's do NOT depend on δ.

After proving this, by putting $\delta = Cap_\omega(K)^\beta$ for properly chosen $\beta > 0$, we can justify Condition (A) for any chosen l by noticing the dominance of exponential growth over polynomial growth.

It is easy to notice that we can have uniform constants for all ω's related once we get for ω_∞ from the favorable direction of the control we want. And we also only need to prove Condition (A) for sets close to the subvariety $\{\omega_\infty^n = 0\}$ in light of the results in [Ko2] by localizing the problem.

The rest of this section will be devoted to the proof of inequality (B). The following construction is of fundamental importance for this goal.

Let's start with a better description of the map $F : X \to F(X) \subset \mathbb{CP}^N$. For simplicity, we'll assume here that F provides a birational morphism between X and $F(X)$. This assumption will be removed at the end.

Using this assumption, we have subvarieties $Y \subset X$ and $Z \subset F(X)$ such that $X \setminus Y$ and $F(X) \setminus Z$ are isomorphic under F and $F(Y) = Z$. Clearly Z should contain the singular subvariety of $F(X)$. It's the situation near Y (or Z) that is of the main interest to us.

Now we use finitely many unit coordinate balls on X to cover Y. The union of the half-unit balls will be called V, which still covers Y by our choice. Then we take two finite sets of open subsets depending on $\delta > 0$ as follows:

$\{U_i\}$, $\{V_i\}$, with $i \in I$, finite coverings of $V \setminus W$, where W is the intersection of δ-neighbourhood of Y, which is a neighbourhood of Y correspondent to the intersection of balls of radius δ in $\mathbb{CP}^N$ covering Z, with $F(X)$, such that each pair $V_i \subset U_i$ is in one of the chosen unit coordinate balls. Moreover, $F(U_i)$ and $F(V_i)$ are the intersections of $F(X)$ with balls of sizes $\frac{1}{2}\delta^C$ and $\frac{1}{6}\delta^C$ where some fixed $C > 0$ are chosen to be big enough in order to justify the above construction.

Clearly the covering can be chosen so that $|I|$ is controlled by $C \cdot \delta^{-N_2}$.

For any compact set K in V, we have the following computation:

$$\begin{aligned}
\lambda(K) &\leqslant \lambda(W) + \sum_{i\in I} \lambda(K \cap \bar{V}_i) \\
&\leqslant C \cdot \delta^{N_1} + \sum_{i\in I} C \cdot \exp\big(-\frac{C}{Cap(K \cap \bar{V}_i, U_i)^{\frac{1}{n}}}\big) \\
&\leqslant C \cdot \delta^{N_1} + \sum_{i\in I} C \cdot \exp\big(\frac{C}{\log\delta \cdot Cap_{\omega_\infty}(K \cap \bar{V}_i)^{\frac{1}{n}}}\big) \\
&\leqslant C \cdot \delta^{N_1} + \sum_{i\in I} C \cdot \exp\big(\frac{C}{\log\delta \cdot Cap_{\omega_\infty}(K)^{\frac{1}{n}}}\big) \\
&\leqslant C \cdot \delta^{N_1} + C\delta^{-N_2} \cdot \exp\big(\frac{C}{\log\delta \cdot Cap_{\omega_\infty}(K)^{\frac{1}{n}}}\big).
\end{aligned}$$

That's just what we want. C_1 and C_2 are used in the original statement of (B) since the C's at different places have different affects on the magnitude of the final expression. Of course, the same C for each term in the big sum have to be really the same constant. In the following, we justify the computation above.

First the control of $\lambda(W)$ by $C \cdot \delta^{N_1}$ is quite clear from the map F where N_1 might be small but still positive. Then the only nontrivial steps are the second and third ones.

The term $Cap(K \cap \bar{V}_i, U_i)$ is the relative capacity of $K \cap \bar{V}_i$ with respect to U_i. The definition of it is classic as in [BeTa]. In comparison to Definition 2.2, we should use $PSH(U_i)$ instead of $PSH_\omega(X)$ with the corresponding measure to integrate over the set.

The second step is the direct application of the local version of $(\star)$ as proved in [Ko2], the classic result for domains in $\mathbb{C}^n$, since V_i and U_i are in one of the finitely many unit coordinate balls which can clearly be taken as the unit Euclidean ball in a uniform way, and we are using a smaller domain U_i instead of the big ball here which obviously increases the relative capacity.

The third step uses the following inequality:

$$Cap(K \cap \bar{V}_i, U_i) \leqslant C \cdot (-\log\delta)^n \cdot Cap_{\omega_\infty}(K \cap \bar{V}_i).$$

This result also has its primitive version in classical pluripotential theory for domains in $\mathbb{C}^n$. It is sufficient to prove a result about extending plurisubharmonic functions, which we give below.

Let v be any element in $PSH(U_i)$ with values in $[-1, 0]$. If we can "extend" this function to an element $-C\log\delta \cdot \tilde{v}$ where $\tilde{v}$ is plurisubharmonic with respect to ω_∞ valued in $[-1, 0]$ over X, and also make sure that the measures $(\sqrt{-1}\partial\bar{\partial}v)^n$ and $(-C\log\delta)^n(\omega_\infty + \sqrt{-1}\partial\bar{\partial}\tilde{v})^n$ are the same over $\bar{V}_i$, then this would clearly imply the inequality above from the definition of relative capacity.

The construction will be done mostly on $F(X)$. The function v can be considered over $F(U_i)$. We'll "extend" it to a neighbourhood $F(X) \setminus O_i$ in $\mathbb{CP}^N$ where O_i is a neighbourhood of $\bar{V}_i$ in U_i.

Let's first extend it locally in $\mathbb{CP}^N$. We can safely assume that the construction happens in (finite) half-unit Euclidean balls in $\mathbb{CP}^N$ which cover the variety Z and have ω_M defined on the corresponding unit balls. ω_M can be expressed at the level of potentials, and so the construction is merely about functions.

Consider the plurisubharmonic function function

$$h = \left(\log\left(\frac{36|z|^2}{\delta^{2C}}\right)\right)^+ - 2,$$

where the upper $+$ means taking maximum with 0, on the unit ball in $\mathbb{CP}^N$ but with the coordinate system z centered at the center of $F(V_i)$. It's easy to see that the pullback of this function, still denoted by h, is plurisubharmonic and $\max(h, v)$ on U_i is equal to v near $\bar{V}_i$ and equal to h near ∂U_i. So this function extends v to the preimage of the unit ball in $\mathbb{CP}^N$ while keeping the values near $\bar{V}_i$.

Now we want to extend further to the whole of X. We still work on $F(X) \subset \mathbb{CP}^N$. And it's only left to extend the function h for the remaining part where the value is less restrictive.

$|h|$ is bounded by $-C \cdot \log\delta$ in the unit ball. So we can have

$$\sqrt{-1}\partial\bar{\partial}h = -C \cdot \log\delta \cdot (\omega_M + \sqrt{-1}\partial\bar{\partial}H)$$

for H plurisubharmonic with respect to ω_M valued in $[-1, 0]$ in the unit ball [5]. Then using the same argument as in [Ko2], we can extend H to (uniformly bounded) $\tilde{H} \in PSH_{\omega_M}(O)$, where O is a neighbourhood of $F(X)$, using the positivity of ω_M. Finally we just take $\tilde{v} = F^*\tilde{H}$.

This ends the argument for the case when $F : X \to F(X)$ is a birational map.

Now we want to remove the birationality assumption. In fact, after removing proper subvarieties Y and $Z = F(Y)$ of X and $F(X)$ respectively, we can have $F : X \setminus Y \to F(X) \setminus Z$ be a finitely-sheeted covering map, since the map is clearly of full rank there and the finiteness of sheets can be seen by realizing that the preimage of any point in $F(X \setminus Z)$ should be a finite set of points.

Then it's easy to see that the argument before would still work in this situation. Clearly we can still have the construction before, and now the only difference is that the numbers of small pieces U_i and V_i need to be multiplied by (at most) the number of sheets, which won't affect the previous argument too much.

Hence we get the a priori L^∞ bound in general.

4 Existence of Bounded Solution

Now we discuss the existence of a bounded solution. As suggested in Section 2, approximation of the original equation in the main theorem is used. The (uniform) a priori estimate obtained in the previous section would give us the uniform L^∞ bound for the approximation solutions u_ϵ there.

Thus exactly the same argument of taking limits as used in [Ko1] can be applied for our case to get a bounded solution for the original equation in Theorem 1.1.

Remark 4.1. *Indeed, u_ϵ is essentially decreasing as $\epsilon \to 0$ which will make it easier to take the limit. More details about this and some uniqueness results for bounded solutions will be provided in [Zh].*

5 Continuity of Bounded Solution and Stablity Result

In fact, we can prove that a bounded solution for the original equation is actually continuous with just a little more assumption.

The argument is almost line by line the same as the original argument in [Ko1], using the corresponding L^∞ argument above.

[5] In order to do this, one might need the local potential of ω_M to be valued in a short interval. In fact, it's only necessary for H to be valued in an interval with uniform length.

We still just have to analyze the situation around a carefully chosen point. But now the point might be in the set $\{{\omega_\infty}^n = 0\}$. So in order to have convexity of the local potential of ω_∞, we have to consider the domain in X which is the preimage of a ball in $\mathbb{CP}^n$ under the map F. In other words, we do have to consider the domain V mentioned earlier.

For such a domain, where $\omega_\infty + \sqrt{-1}\partial\bar{\partial}u = \sqrt{-1}\partial\bar{\partial}U$ for a bounded plurisubharmonic function U, we don't have convolution which gives a decreasing smooth approximation of any plurisubharmonic function in Euclidean case.

That's where we'll use the additional assumption of F being locally birational and the classic result in [FoNa]. The reason for requiring this local birationality would be clear below.

Simply speaking, we want to be able to push forward the function U above to the singular domain in $F(X)$ in the most straightforward way (without averaging, etc.). The standard blowing-down picture would work. But it'll be OK if there are several conponents in X correspondent to the same singular domain in $F(X)$ as we only need to treat each one of those components. Notice that the most interesting case of F being birational to its image falls right into this category. In a sense, we just do not want any branching.

Then it's actually quite straightforward to see that the pushforward function F_*U is a weakly plurisubharmonic function in the classic sense as in [FoNa]. One needs to prove that for any holomorphic map from the unit disk in $\mathbb{C}$ to $F(X)$, the pullback function of F_*U is subharmonic. The idea is to reduce the proof for a holomorphic map with image contained in subvariety of $F(X)$ with decreasing dimension. More details can be found in [Zh].

Now the classic result in [FoNa], pointed out to me by Professor Kolodziej, tells us that we can (locally) extend F_*U to a plurisubharmonic function in a ball of $\mathbb{CP}^N$ which would be enough for us to go through local argument (for a properly chosen V). So now we can again use convolution to get the desired approximation for U.

Thus we can justify continuity for bounded solutions as in [Ko1]. The punchline is as follows. Suppose $\{U_j\}$ is the sequence of smooth plurisubarmonic functions constructed above which are defined on a neighbourhood slightly larger than V, which pointwisely decreases to U. Then by the construction in [Ko1], which is very local and can be easily adjusted to our case, we can prove that the sets $\{U + c < U_j\}$ are nonempty and relatively compact [6] inside V for all $c \in (0, a)$ for $a > 0$ and $j > j_0$.

The argument for L^∞ estimate before then gives

$$\frac{a}{2} \leqslant \kappa(Cap(\{U + \frac{a}{2} < U_j\}, V)).$$

[6] For the relative compactness of the sets, strictly speaking, we have to use another function which is constructed from U linearly instead of U itself in the definition of these sets as in [Ko1]. It's a little too tedious to describe the details here.

Remark 5.1. *Let's point out that we have to use a local argument to get this inequality in which a local notion of relative capacity is used. For this, the justification of comparison principle would be different from the global case since there is now the boundary to consider. Notice that the locality of the approximation above using the result in [FoNa] seems to make it difficult to get the approximations for two functions for the same domain which is needed for proving comparison principle.*

We can deal with this problem by using another method mentioned in the discussion on the comparison principle. There we could use a different definition of relative capacity where only continuous functions are considered in taking the supremum. Then together with the approximation we have for U, we can go through the local argument for the sets $\{U + c \leqslant U_j\}$.

In fact the point of view we are using in this note still works after small modification as shown in [Zh]. The details which will appear in [Zh].

We also notice that the relative capacity of the set $\{U + \frac{a}{2} < U_j\}$ would go to 0 as $j \to \infty$. This can be justified by the decreasing convergence and $\frac{a}{2} > 0$. At last, we can draw the contradiction by letting j tend to ∞ in the equality above because the right hand side is going to 0.

Remark 5.2. *The continuity of the solution can be achieved without the additional assumption if we have a decreasing approximation of the solution by functions in $PSH_{\omega_\infty}(X) \cap C^\infty(X)$. There are all kinds of results in this direction, but somehow the author feels semi-positivity of ω_∞ won't be sufficient for applying them.*

Finally, let's point out that the stability result for bounded (hence continuous) solutions in [Ko2] also hold in the current situation. The original proof there works for us without any essential change. So we have the uniqueness of such solutions.

We would like to mention that the stability argument in [Ko2] can almost be directly used to consider merely bounded solutions. This is not pointless here since our continuity result needs a little bit more assumption than the boundedness result as for now. But there is an inequality used in that argument which bounds the measures of mixed terms from two plurisubharmonic functions by the measures from each of them. This inequality seems to be hard to prove for merely bounded functions. More discussions for this can also be found in [Zh].

6 Application

The most useful application of the results above would be the L^∞ estimate. Combining it with other estimates from PDE methods (for example, maximum principle), we can further understand the solution.

Let's illustrate this using the application in Kähler-Ricci flow. This has been discussed in [TiZh] which focuses more on the maximum principle argument.

Consider the following Kähler-Ricci flow over X.

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t, \qquad \tilde{\omega}_0 = \omega_0, \tag{6.1}$$

where ω_0 is any given Kähler metric and $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω, i.e., in the complex coordinates, $\mathrm{Ric}(\omega) = \sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j$ where $(R_{i\bar{j}})$ is the Ricci tensor of ω.

Let $\omega_\infty = -\mathrm{Ric}(\Omega)$ for a volume form Ω. Here Ω can be taken as a function locally on X using a local volume form coming from complex coordinates. The ambiguity would not matter after taking "Ric" as "$-\sqrt{-1}\partial\bar{\partial}$". Set $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ and $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$, we can reformulate (6.1) at the level of potentials as:

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t} + u}\Omega.$$

This is now a degenerate Monge-Ampère equation in the sense that ω_t will tend to a degenerate metric ω_∞ as $t \to \infty$.

We have seen in [TiZh] that the right hand side has a uniform L^p bound for all t with any $1 < p \leqslant \infty$.

Now suppose $[\omega_\infty] = K_X$ is nef. and big. Hence it would also be semi-ample by a classic algebraic geometry result as in [Ka], and so fall right into the setting of the main theorem.

Note that, in general, the term $\omega_0 - \omega_\infty$ appearing in ω_t may not be strictly positive. By combining with the degenerate lower bound of u in [TiZh], we can still have uniform L^∞ estimate for $u(t, \cdot)$ with $t \in [0, \infty)$ simply by using part of ω_∞ in the front to dominate the second term since in fact, from our main argument, we know that a positive perturbation won't affect our estimate at all (even if it's not so small).

Actually, the uniqueness result in [TiZh] allows us to only consider the case when $\omega_0 > \omega_\infty$. But the above discussion makes the situation more satisfying and illustrates the flexibility of our main argument in this note.

Anyway, this would give us the boundedness of the limit and the continuity follows as well since we can choose the map F to be birational to its image for this case.

There would be generalizations and more applications in [Zh].

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