General Weak Limit for Kähler-Ricci Flow

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Abstract

Consider the Kähler-Ricci flow with finite time singularities over any closed Kähler manifold. We prove the existence of the flow limit in the sense of current towards the time of singularity. This answers affirmatively a problem raised by Tian in [23] on the uniqueness of the weak limit from sequential convergence construction. The notion of minimal singularity introduced by Demailly in the study of positive current comes up naturally. We also provide some discussion on the infinite time singularity case for comparison. The consideration can be applied to more flexible evolution equation of Kähler-Ricci flow type for any cohomology class. The study is related to general conjectures on the singularities of Kähler-Ricci flows.

Key Words: Kähler-Ricci flow, minimal singularity, Kähler manifold.

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1 Introduction

Let X be a closed Kähler manifold with $\dim_{\mathbb{C}} X = n \geqslant 2$. We consider the following Kähler-Ricci flow over X,

$$\frac{\partial \widetilde{\omega}(t)}{\partial t} = -\mathrm{Ric}\left(\widetilde{\omega}(t)\right) - \widetilde{\omega}(t), \qquad \widetilde{\omega}(0) = \omega_0. \tag{1.1}$$

where ω_0 is the initial Kähler metric over X. This setting of Kähler-Ricci flow is general in the sense that we make no assumption on either the first Chern class of X, $c_1(X)$, or the initial Kähler class $[\omega_0]$ (not necessarily rational).

The tremendous efforts and great successes in the study of this flow over the last decade are motivated by all kinds of intentions and visions from algebraic geometry, geometric analysis and several complex variables. The most prominent one would be Tian's Program, i.e., the Geometric Analytic Minimal Model Program. Simply speaking, the static equation for this flow (1.1) is $\mathrm{Ric}\big(\widetilde{\omega}(\infty)\big) = -\widetilde{\omega}(\infty)$, which is in principle the desirable equation to study in

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search of the (singular and generalized) Kähler-Einstein with K_X being almost positive (as in [24], [19], [18] and so on).

This general Kähler-Ricci flow can still be reduced to a scalar evolution equation for the metric potential as described below.

Set $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ with

$$\omega_\infty = -\mathrm{Ric}(\Omega) := \sqrt{-1}\partial\bar{\partial}\log\frac{\Omega}{V_E}$$

for some smooth volume form Ω where V_E is the Euclidean volume form with respect to a local holomorphic coordinate chart $\{z_1, \cdots, z_n\}$. By the classic result in Kähler geometry, for a Kähler metric ω, the form $\mathrm{Ric}(\omega^n)$ using the volume form ω^n is equal to $\mathrm{Ric}(\omega)$, the Ricci form of ω. It's known that $[\omega_t]$ captures the cohomology information of the flow metric $\widetilde{\omega}(t)$, and so by $\partial\bar{\partial}$-Lemma, $\widetilde{\omega}(t) = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ with u satisfying

$$\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \qquad u(\cdot, 0) = 0, \tag{1.2}$$

and this evolution equation is equivalent to (1.1).

By the optimal existence result in [24], the smooth (metric) solution $\widetilde{\omega}_t$ exists as long as the class

$$[\omega_t] = -c_1(X) + e^{-t}\big([\omega_0] + c_1(X)\big) = e^{-t}[\omega_0] + (1 - e^{-t})\big(-c_1(X)\big)$$

remains in the Kähler cone of X, i.e., the open convex cone in the cohomology space $H^2(X;\mathbb{R}) \cap H^{1,1}(X;\mathbb{C})$ consisting of all Kähler classes of X. Thus if we define *the time of singularity*

$$T := \sup\{t \,|\, [\omega_t] \text{ Kähler}\} \in (0, \infty],$$

the (classic) solution for the flow exists in $[0, T)$. For convenience, we apply the convention of $e^{-\infty} = 0$.

If $K_X = -c_1(X)$ is Kähler, then $T = \infty$ and it's known that this flow always converges smoothly to the Kähler-Einstein metric for any ω_0 (in any Kähler class). This is the non-degenerate case in [24] or more explicitly in [31].

Otherwise, there has to be (metric) singularities developed for the flow metric $\widetilde{\omega}(t)$ as $t \to T$ because $[\omega_T]$ is on the boundary of the Kähler cone for X (and no longer Kähler). The corresponding cases of $T < \infty$ and $T = \infty$ are naturally called the finite and infinite time singularities respectively.

Of course, the analysis of various metric singularities when approaching the time of singularity is crucial in understanding and dealing with the singularities for applications, for example, in order to carry out Tian's Program. For this purpose, it's useful to justify the existence and uniqueness of the limit, which provides the sole object for further discussion of regularity. For the uniqueness concern, the consideration in the weak sense is even more favorable.

In general, as shown in Tian's survey [23], we always have a sequential weak limit for $\widetilde{\omega}(t)$ as $t \to T$ in the weak (i.e., current) sense. Let's point out that the argument in [23] also works when $T = \infty$ for our version of the Kähler-Ricci flow because ω_t is uniformly controlled as form even if $T = \infty$. Normalization of the metric potential u is performed in that work to achieve the sequential limit, and it's conjectured there that the sequential limit is actually unique, i.e., independent of the sequence chosen.

This is the original motivation of the current work. We attack this problem by studying the metric potential u itself along the flow without normalization. If the flow limit exists, then the sequential limit is unique for sure.

In this work, we focus on the finite time singularity case, i.e., $T < \infty$, where we have the complete answer in the following theorem.

Theorem 1.1. *Over a closed Kähler manifold X, if the flow (1.1) develops finite time singularities, i.e., the time of singularity $T < \infty$, then we have $\Phi = \omega_T + \sqrt{-1}\partial\bar{\partial}u(T)$ with $u(T) \in PSH_{\omega_T}(X)$ and $\widetilde{\omega}(t) \to \Phi$ in the weak sense as $t \to T$. In fact, for some positive constant C, $u + Ce^{-\frac{t}{2}}$ decreases to $u(T) + Ce^{-\frac{T}{2}}$ as $t \to T$, and $u(T)$ is of minimal singularities in $PSH_{\omega_T}(X)$.*

The situation for the infinite time singularity case is different in general as illustrated in Subsection 3.2. However, the above conclusion is true for the global volume non-collapsed case, even if $T = \infty$.

Proposition 1.2. *Over a closed Kähler manifold X, consider the Kähler-Ricci flow (1.1) with the time of singularity $T \in (0, \infty]$. If $[\omega_T]^n > 0$, then we have $\Phi = \omega_T + \sqrt{-1}\partial\bar{\partial}u(T)$ with $u(T) \in PSH_{\omega_T}(X)$ and $\widetilde{\omega}(t) \to \widetilde{\omega}(T)$ in the weak sense as $t \to T$. In fact, for some positive constant C, $u + Ce^{-\frac{t}{2}}$ decreases to $u(T) + Ce^{-\frac{T}{2}}$ as $t \to T$, and $u(T)$ is of minimal singularities in $PSH_{\omega_T}(X)$.*

The conclusion in Proposition 1.2 for the case of $T < \infty$ is weaker than the result in [3]. Assuming $T = \infty$, as shown in [27], the converse (or inverse) statement of Proposition 1.2 is true, i.e. the convergence of unnormalized metric potential implying the non-collapsing of global volume. Let's clarify that it is possible to have weak convergence of metric (form) for the infinite time collapsed singularity, although the metric potential has to be normalized in this setting for its own convergence.

In this case, the metric potential actually stays uniformly bounded along the flow (and so is the limit $u(\infty)$) by the result in [5], [9] and [30]. The limit $u(\infty)$ is actually continuous by the result in [30] and [7].

The organization of this work is as follows. In Section 2, we provide some general computations useful later and discuss some interesting special cases including Proposition 1.2. In Section 3, we prove Theorem 1.1. In Section 4, we provide very brief but sufficient discussion on the generalization of Section 3 in the study of any cohomology class and elaborate on the relation with the notion of minimal singularity introduced by Demailly. Section 5 is about the relation between our consideration and general conjectures on the singularities for the Kähler-Ricci flow.

Notations: as usual, we use C to stand for a positive constant, possibly different at places. Also, $f \sim g$ means $\lim_{t\to T} \frac{f}{g} = 1$.

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2 The Plan and Special Cases

In this section, we list the discussion for some special cases. Although in Section 3 we take care of the general case altogether, they serve very well as motivations for this topic. Before heading into them, we provide general discussion for the Kähler-Ricci flow, leading to the plan of proving the main result, Theorem 1.1.

The following computations are already quite standard as in [24] for example. The Laplacian, denoted by Δ, in the following is always with respect to the evolving metric along the flow, $\widetilde{\omega}(t)$.

We start with the t-derivative of (1.2),

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle \widetilde{\omega}(t), \omega_0 - \omega_\infty \rangle - \frac{\partial u}{\partial t}. \tag{2.1}$$

Notation: the expression $\langle \omega, \alpha \rangle$ stands for the trace of the real smooth closed $(1,1)$-form α with respect to the Kähler metric ω. Clearly, $\langle \omega, \alpha \rangle = (\omega, \alpha)_\omega$ where $(\cdot, \cdot)_\omega$ is the hermitian inner product with respect to ω. This notation is frequently applied and comes from the calculation, $\langle \omega, \alpha \rangle = \frac{n\omega^{n-1}\wedge\alpha}{\omega^n}$.

Then we transform (2.1) into the following two equations,

$$\frac{\partial}{\partial t}\left(e^t \frac{\partial u}{\partial t}\right) = \Delta\left(e^t \frac{\partial u}{\partial t}\right) - \langle \widetilde{\omega}(t), \omega_0 - \omega_\infty \rangle, \tag{2.2}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) = \Delta\left(\frac{\partial u}{\partial t} + u\right) - n + \langle \widetilde{\omega}(t), \omega_\infty \rangle. \tag{2.3}$$

The difference of these two equations above is

$$\frac{\partial}{\partial t}\left((1-e^t)\frac{\partial u}{\partial t} + u\right) = \Delta\left((1-e^t)\frac{\partial u}{\partial t} + u\right) - n + \langle \widetilde{\omega}(t), \omega_0 \rangle. \tag{2.4}$$

In light of $\langle \widetilde{\omega}(t), \omega_0 \rangle > 0$, by the standard Maximum Principle argument, we have from (2.4)

$$(1 - e^t)\frac{\partial u}{\partial t} + u + nt \geqslant 0.$$

Meanwhile, there is a uniform upper bound for the metric potential u by applying Maximum Principle to (1.2) directly. So the above control gives us

$$\frac{\partial u}{\partial t} \leqslant \frac{u + nt}{e^t - 1} \leqslant \frac{nt + C}{e^t - 1}, \tag{2.5}$$

which is the essential decreasing (i.e., decreasing up to an exponentially decaying term) of u along the flow. Notice that this control only depends on the upper bound of u.

The t-derivative of (2.1) is

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) = \Delta\left(\frac{\partial^2 u}{\partial t^2}\right) + e^{-t}\langle \widetilde{\omega}(t), \omega_0 - \omega_\infty \rangle - \frac{\partial^2 u}{\partial t^2} - \left|\frac{\partial \widetilde{\omega}(t)}{\partial t}\right|^2_{\widetilde{\omega}(t)}. \tag{2.6}$$

Take the sum of (2.1) and (2.6) to get

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - \left|\frac{\partial \widetilde{\omega}(t)}{\partial t}\right|^2_{\widetilde{\omega}(t)}.$$

Applying Maximum Principle to it, one has

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \leqslant Ce^{-t},$$

which implies the essential decreasing of the "logarithm" of the volume form, $\widetilde{\omega}(t)^n = e^{\frac{\partial u}{\partial t} + u}\Omega$, along the flow, i.e.,

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) \leqslant Ce^{-t}.$$

From this inequality, we can see

$$\frac{\partial u}{\partial t} \leqslant (C + Ct)e^{-t} \leqslant Ce^{-\frac{t}{2}},$$

which also follows from (2.5). Thus, for some $C > 0$, $u + Ce^{-\frac{t}{2}}$ is decreasing along the flow. By the well known property of plurisubharmonic function, as long as this term (or equivalently u) doesn't converge to $-\infty$ uniformly as $t \to T \in (0, \infty]$, u would converge to some $u(T) \in PSH_{\omega_T}(X)$, and so $\Phi = \omega_T + \sqrt{-1}\partial\bar{\partial}u(T)$ is the flow limit for $\widetilde{\omega}(t)$ as $t \to T$ in the weak sense. Hence, in order to prove the existence of the flow limit in the weak sense for Theorem 1.1 and Proposition 1.2, we just need to

exclude the possibility of $u \to -\infty$ uniformly as $t \to T$

and this is the plan for us. In short, we don't normalize u as in [23].

Next, we briefly recall the notion of minimal singularity introduced by Demailly in the study of positive $(1,1)$-current.

Definition 2.1. *Consider a smooth real closed $(1,1)$-form α over X with*

$$PSH_\alpha(X) := \{w \in L^1(X) \,|\, \alpha + \sqrt{-1}\partial\bar{\partial}w \geqslant 0 \text{ in the sense of distribution}\} \neq \varnothing.$$

Then $u \in PSH_\alpha(X)$ is of minimal singularities if for any $v \in PSH_\alpha(X)$, $u \geqslant v - C$ for C depending on v.

Any element in $PSH_\alpha(X)$ is well known to be bounded from above. So the "singularity" is used to describe where and how the function approaches $-\infty$. Obviously, if $u \in PSH_\alpha(X)$ is bounded, then it's of minimal singularities.

Remark 2.2. *Since $|u|^p\widetilde{\omega}(t)^n = |u|^p e^{\frac{\partial u}{\partial t}+u}\Omega \leqslant C\Omega$ by the known upper bounds of $\frac{\partial u}{\partial t}$ and u, the weak convergence of all wedge powers $\widetilde{\omega}(t)^k \to \Phi^k$ as $t \to T$ is also available by the discussion in [12] as long as we have the existence of the metric potential weak limit $u(T) \in PSH_{\omega_T}(X)$ for the metric potential u.*

2.1 Cases from algebraic geometry

In the consideration motivated by algebraic geometry interest, $[\omega_0]$ (or at least $[\omega_T]$) is rational, and the following result is the core of this subsection.

Claim: if $T < \infty$ and $[\omega_T]$ is has a smooth real $(1,1)$ non-negative form (i.e., with non-negative eigenvalues when considered as a hermitian matrix) as a representative, then the conclusion of Theorem 1.1 holds.

This is because we actually have $u \geqslant -C$ in this case by applying Maximum Principle argument directly on the flow. It has already been shown in [24] and the argument is included here for later convenience. We start with

$$\frac{\partial}{\partial t}\left((1-e^{t-T})\frac{\partial u}{\partial t} + u\right) = \Delta\left((1-e^{t-T})\frac{\partial u}{\partial t} + u\right) - n + \langle\widetilde{\omega}(t), \omega_T\rangle \tag{2.7}$$

which is a proper linear combination of (2.2) and (2.3).

The assumption of the claim gives an $f \in C^\infty(X)$ such that $\omega_T + \sqrt{-1}\partial\bar{\partial}f \geqslant 0$. We then modify the above equation as follows.

$$\begin{aligned}\frac{\partial}{\partial t}\left((1-e^{t-T})\frac{\partial u}{\partial t} + u - f\right) = \Delta&\left((1-e^{t-T})\frac{\partial u}{\partial t} + u - f\right) \\ &- n + \langle\widetilde{\omega}(t), \omega_T + \sqrt{-1}\partial\bar{\partial}f\rangle.\end{aligned}$$

Applying Maximum Principle and noticing $T < \infty$, one has

$$(1-e^{t-T})\frac{\partial u}{\partial t} + u - f \geqslant -C.$$

Since $u \leqslant C$ and $\frac{\partial u}{\partial t} \leqslant C$, we conclude that for $t \in [0, T)$,

$$u \geqslant -C, \qquad \frac{\partial u}{\partial t} \geqslant -\frac{C}{1 - e^{t-T}} \sim -\frac{C}{T - t}.$$

In fact, one can get the lower bound for u more directly using (1.2) as observed in [21]. After a proper choice of Ω when coming up with (1.2), we can make sure $\omega_T \geqslant 0$, and so $\omega_t \geqslant C(T - t)\omega_0$ for $t \in [0, T)$. Then by applying Maximum Principle to (1.2), we have

$$\frac{d \min_{X \times \{t\}} u}{dt} \geqslant n \log(T - t) - C - \min_{X \times \{t\}} u,$$

from which the lower bound of u easily follows.

Together with the essential decreasing of u, we have the limit of the metric potential u as a flow weak limit as $t \to T$, $u(T) \in PSH_{\omega_T}(X) \cap L^\infty(X)$. Thus in this case, by a well known result in pluripotential theory as in [1], one has the flow weak convergence as $t \to T$, $\widetilde{\omega}^k(t) \to \Phi^k = \left(\omega_T + \sqrt{-1}\partial\bar{\partial}u(T)\right)^k$ for $k = 1, \cdots, n$. This $u(T)$ is bounded and so of minimal singularities. Hence Theorem 1.1 holds in this case.

Let's point out that if $[\omega_0]$ is a rational class (and so X is projective by Kodaira Embedding Theorem), the non-negativity of $[\omega_T]$ is available by the classic Rationality Theorem and the Base-Point-Free Theorem from algebraic geometry (for example in [14] and [13]). So from the algebraic geometry point of view, the result in this work is certainly true. Of course, our goal is to remove the assumption of $[\omega_0]$ being rational and even X being projective.

Remark 2.3. *It remains interesting to see whether the limit of u is continuous, especially for the collapsed case, i.e., when $[\omega_T]^n = 0$. In the algebraic geometry case, the continuity in the global volume noncollapsed case is known by the result in [30] and also [7].*

The above discussion can be generalized to the case when $[\omega_T] - D$ is non-negative with an effective $\mathbb{R}$-divisor D. For simplicity of notations, we assume that D is an effective $\mathbb{Z}$-divisor, which can then be identified as a holomorphic line bundle with a defining section σ such that $D = \{\sigma = 0\}$ and a hermitian metric $\|\cdot\|$. We get this information involved in the estimation before as follows.

$$\begin{aligned}\frac{\partial}{\partial t}\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u - \log\|\sigma\|^2\right) &= \Delta\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u - \log\|\sigma\|^2\right) - n \\ &\quad + \langle \widetilde{\omega}(t), \omega_T + \sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2 \rangle\end{aligned}$$

Since $[\omega_T] - D$ has a smooth non-negative representative, by choosing $\|\cdot\|$ properly, we have over $X \setminus D$,

$$\omega_T + \sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2 \geqslant 0.$$

As $T<\infty$, by Maximum Principle with the minimum clearly achieved in $X\setminus D$, we arrive at

$$(1-e^{t-T})\frac{\partial u}{\partial t}+u-\log\|\sigma\|^2\geqslant -C.$$

Since $u\leqslant C$ and $\frac{\partial u}{\partial t}\leqslant C$, we conclude that

$$u\geqslant \log\|\sigma\|^2-C,\quad \frac{\partial u}{\partial t}\geqslant \frac{C\log\|\sigma\|^2}{1-e^{t-T}}\sim\frac{C\log\|\sigma\|^2}{T-t}.$$

This (degenerate) lower bound for u is enough to guarantee the existence of its limit $u(T)\in PSH_{\omega_T}$ and the weak convergence of $\widetilde{\omega}(t)$ to $\Phi=\omega_T+\sqrt{-1}\partial\bar{\partial}u(T)$ as $t\to T$. The limit $u(T)$ is of minimal singularities by the general argument in Section 3.

Remark 2.4. *The case of $[\omega_T]=D$, an effective $\mathbb{R}$-divisor, is a special case in the above consideration. Furthermore, this generalization would be more interesting in Section 4 when studying a general nef. class by the more general evolution equation of Kähler-Ricci flow type.*

2.2 Global volume non-collapsed case

Obviously, it is always the case that $[\omega_T]^n\geqslant 0$ since it is equal to the limit of $[\omega_t]^n>0$ as $t\to T$. Here, we exclude the case of $[\omega_T]^n=0$, i.e., require the flow to be (globally) volume noncollapsed. The following claim is the focus of this subsection.

Claim: for $T\in(0,\infty]$, if $[\omega_T]^n>0$, then as $t\to T$, u can't go to $-\infty$ uniformly, so $u\to u_T\in PSH_{\omega_T}(X)$ (by the essential decreasing of u and the property of plurisubharmonic function).

The proof is based on one simple observation. Let's rewrite (1.2) as follows.

$$(\omega_t+\sqrt{-1}\partial\bar{\partial}u)^n=e^{\frac{\partial u}{\partial t}+u}\Omega. \tag{2.8}$$

Assume otherwise, i.e., $u\to-\infty$ uniformly over X as $t\to T$.

Meanwhile, $[\omega_t]^n\geqslant C>0$ for $t\in[0,T)$ since $[\omega_t]^n\to[\omega_T]^n>0$ as $t\to T$. We also know $\frac{\partial u}{\partial t}\leqslant C$. In light of

$$\int_X e^u\Omega\geqslant C\int_X e^{\frac{\partial u}{\partial t}+u}\Omega=C\int_X(\omega_t+\sqrt{-1}\partial\bar{\partial}u)^n=C[\omega_t]^n\geqslant C>0,$$

we arrive at a contradiction. This argument clearly works for $T\in(0,\infty]$. Hence, we get the claim which is the convergence part of Proposition 1.2.

When $T=\infty$, then $[\omega_T]^n>0$ actually implies the uniform lower bound of u by the result in [5], [9] and [30]. Clearly, the limit $u(T)$ is bounded and certainly of minimal singularities.

When $T<\infty$, if $[\omega_T]$ is rational (which follows from $[\omega_0]$ being rational by the Rationality Theorem), it falls into the case considered in the previous subsection, so u is bounded and of minimal singularities. When $[\omega_T]$ is a real (not necessary rational) class, we leave the justification of u being of minimal singularities to the general discussion in Section 3.

Remark 2.5. *For $T = \infty$, as shown in [27], the existence of unnormalized limit for u, i.e., u not approaching $-\infty$ uniformly, is equivalent to $K_X^n > 0$ (so u and the limit have to be uniformly bounded).*

Remark 2.6. *In general, one has $[\omega_t]^n \sim C(T-t)^k$ for some $k \in \{0, \cdots, n\}$ when $T < \infty$ and $[\omega_t]^n \sim Ce^{-kt}$ for some $k \in \{0, \cdots, n\}$ when $T = \infty$ by considering the Taylor series of the explicit function $f(t) = [\omega_t]^n$ at $t = T$. In this subsection, $k = 0$. All other k values correspond to the global volume collapsed case.*

In the non-collapsed case, the following flow metric estimate, similar to that in [29], is of particular interest as illustrated in Section 5.

We begin with the following inequality from parabolic Schwarz Lemma. Let $\phi = \langle \widetilde{\omega}(t), \omega_0 \rangle > 0$. Using the computation for (1.1) in [19], one has

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log\phi \leqslant C_1\phi + 1, \tag{2.9}$$

where C_1 is a positive constant depending on the bisectional curvature of ω_0. We also have the following equation which is essentially (2.4):

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right) = -\langle \widetilde{\omega}(t), \omega_0 \rangle, \tag{2.10}$$

which gives $(e^t-1)\frac{\partial u}{\partial t} - u - nt \leqslant 0$.

Multiplying (2.10) by a constant $C_2 > C_1 + 1$ and combining with (2.9), one arrives at

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\left(\log\phi + C_2\big((e^t-1)\frac{\partial u}{\partial t} - u - nt\big)\right) &\leqslant nC_2 + 1 - (C_2 - C_1)\phi \\ &\leqslant C_3 - \phi.\end{aligned} \tag{2.11}$$

Now apply Maximum Principle for the term $\log\phi + C_2\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right)$. Considering at the (local in time) maximum value point, one has

$$\phi \leqslant C,$$

and so

$$\log\phi + C_2\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right) \leqslant C,$$

which gives

$$\widetilde{\omega}(t) \leqslant Ce^{-C_2\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right)}\omega_0 \leqslant Ce^{-C(e^t\frac{\partial u}{\partial t} - t)}\omega_0.$$

Since $\widetilde{\omega}(t)^n = e^{\frac{\partial u}{\partial t}+u}\Omega$, one further concludes that

$$Ce^{C(e^t\frac{\partial u}{\partial t}-t)}\omega_0 \leqslant \widetilde{\omega}(t) \leqslant Ce^{-C(e^t\frac{\partial u}{\partial t}-t)}\omega_0. \tag{2.12}$$

Now we restrict to the case of $T < \infty$. Combining with the upper bound of $\frac{\partial u}{\partial t}$, we have for $t \in [0, T)$,

$$Ce^{C\frac{\partial u}{\partial t}}\omega_0 \leqslant \widetilde{\omega}(t) \leqslant Ce^{-C\frac{\partial u}{\partial t}}\omega_0. \tag{2.13}$$

So the control of flow metric can be reduced to the lower control of $\frac{\partial u}{\partial t}$. Although we know from [29] that it's impossible to have a uniform lower bound for $\frac{\partial u}{\partial t}$, this control of metric is pointwise and helpful in light of the higher order estimates in [15]. This is useful for the discussion in Section 5.

Remark 2.7. *In the global collapsed case, this metric estimate blows up from both directions as $\frac{\partial u}{\partial t} \to -\infty$ uniformly as $t \to T < \infty$, as discussed at the end of Subsection 3.2.*

2.3 Cases with curvature assumption

Now we inspect several cases with assumptions on Riemannian curvature tensor, all for the case of $T < \infty$. They help to motivate our consideration from Riemannian geometry point of view.

- **Case 1:** Ricci lower bound, i.e., $\mathrm{Ric}\big(\widetilde{\omega}(t)\big) \geqslant -C\widetilde{\omega}(t)$ for all $t \in [0, T)$.

  Clearly, $\widetilde{\omega}(t) \leqslant C\omega_0$ by (1.1) and the assumption. So we have

  $$-e^{-t}(\omega_0-\omega_\infty)+\sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t} = \frac{\partial\widetilde{\omega}(t)}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}(t))-\widetilde{\omega}(t) \leqslant C\widetilde{\omega}(t) \leqslant C\omega_0,$$

  which gives

  $$C\omega_0 + \sqrt{-1}\partial\bar{\partial}\left(-\frac{\partial u}{\partial t}\right) \geqslant 0.$$

  Applying the Hörmander-Tian Inequality in [22], there exist uniform positive constants α and C such that

  $$\int_X e^{\alpha\left(\max_X(-\frac{\partial u}{\partial t})-(-\frac{\partial u}{\partial t})\right)}\Omega \leqslant C, \text{ uniformly for any } t \in [0, T),$$

  where we can certainly choose $\alpha < 1$. It is

  $$\int_X e^{\alpha(-\min_X \frac{\partial u}{\partial t}+\frac{\partial u}{\partial t})}\Omega \leqslant C.$$

  So we have

  $$\int_X e^{\alpha\frac{\partial u}{\partial t}}\Omega \leqslant Ce^{\alpha\min_X\left(\frac{\partial u}{\partial t}\right)} \leqslant Ce^{\alpha\frac{\partial u}{\partial t}(x_{\min}(t),t)} \tag{2.14}$$

where $x_{\min}(t)$ is a point where $u(\cdot, t)$ takes the spatial minimum. Define the Lipschitz function $U(t) := \min_{X\times\{t\}} u$, and we have $\frac{dU}{dt} = \frac{\partial u}{\partial t}(x_{\min}(t), t)$ in a proper sense.

Meanwhile, by Remark 2.6, we know

$$\int_X e^{\frac{\partial u}{\partial t}+u}\Omega = [\omega_t]^n \geqslant C(T-t)^k$$

for some $k \in \{0, \cdots, n\}$. Together with $\alpha < 1$ and the uniform upper bounds for u and $\frac{\partial u}{\partial t}$, we arrive at

$$\int_X e^{\alpha\frac{\partial u}{\partial t}}\Omega \geqslant C\int_X e^{\frac{\partial u}{\partial t}+u}\Omega \geqslant C(T-t)^k. \tag{2.15}$$

Now combining (2.14) with (2.15), we have

$$\frac{dU}{dt} \geqslant C\log(T-t) - C$$

which gives $U \geqslant -C$. So there is a uniform L^∞-bound for u. Hence, we conclude the following proposition.

Proposition 2.8. *Consider the Kähler-Ricci flow (1.1). If it develops finite time singularities with Ricci curvature uniformly bounded from below, then the metric potential in (1.2) has a uniform L^∞-bound. When approaching the time of singularity, the flow metric weakly converges to a positive* $(1,1)$*-current with bounded (local) potential.*

Remark 2.9. *By the same argument, one can replace the assumption of a uniform Ricci lower bound by* $\mathrm{Ric}\big(\widetilde{\omega}(t)\big) \geqslant \alpha$ *for a smooth* $(1,1)$*-form* α*. It is a priori a weaker assumption but less geometric. Meanwhile, by the result in [28], we know that the assumption in the above proposition actually forces the global volume to collapse, i.e.,* $[\omega_T]^n = 0$*, which is indeed the difficult case in light of the discussion in Subsection 2.2.*

- **Case 2:** Ricci form upper bound, i.e., $\mathrm{Ric}\big(\widetilde{\omega}(t)\big) \leqslant \alpha$ for a smooth $(1,1)$-form α.

  This assumption is less geometric but still natural, as appearing in [10]. Notice that the upper bound $\mathrm{Ric}(\widetilde{\omega}(t)) \leqslant C\widetilde{\omega}(t)$ indicates a positive lower bound for the flow metric for any finite time, which rules out the finite time singularities as discussed in [29].

  By the flow equation (1.1),

$$\alpha \geqslant \mathrm{Ric}\left(\widetilde{\omega}(t)\right) = -\frac{\partial\widetilde{\omega}(t)}{\partial t} - \widetilde{\omega}(t) = -\omega_\infty - \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t} + u\right).$$

  So we have for some $C > 0$,

$$C\omega_0 + \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t} + u\right) \geqslant 0.$$

The standard argument using Green's function then gives

$$\int_X \left(\frac{\partial u}{\partial t} + u\right) \omega_0^n \geqslant C \max\left(\frac{\partial u}{\partial t} + u\right) - C \geqslant C \log(T-t) - C$$

where the last step makes use of

$$\int_X e^{\frac{\partial u}{\partial t}+u}\Omega = [\widetilde{\omega}(t)]^n] = [\omega_t]^n \sim (T-t)^k$$

for some $k \in \{0, \cdots, n\}$. So we conclude

$$\int_X u\omega_0^n \geqslant -C,$$

which prevents u from going to $-\infty$ uniformly as $t \to T$. We leave the discussion for minimal singularity to the general argument in Section 3.

- **Case 3:** Type I singularity on scalar curvature, i.e., $|R(\widetilde{\omega}(t))| \leqslant \frac{C}{T-t}$.

  This is weaker than the usual Type I singularity on Riemannian curvature. In light of the volume evolution equation

  $$\frac{\partial \widetilde{\omega}_t^n}{\partial t} = (-R-n)\widetilde{\omega}_t^n,$$

  we have

  $$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) \geqslant -\frac{C}{T-t}.$$

  This implies $\frac{\partial u}{\partial t} + u \geqslant C\log(T-t) - C$, and so

  $$u \geqslant -C,$$

  sufficient for the conclusion of Theorem 1.1.

3 The General Case

Recall that in order to obtain the uniqueness of sequential limit, we consider instead the existence problem of flow weak limit. For this purpose, we are left to rule out the possibility of $u \to -\infty$ uniformly over X as $t \to T < \infty$. So far, we have justified this under various assumptions motivated by algebraic geometry and Riemannian geometry interests. Here, we provide a general argument for the finite time singularity case, and then illustrate the difference between the finite and infinite time singularity cases.

3.1 The proof of Theorem 1.1

Consider $T < \infty$, i.e., the finite time singularity case. The following argument works in general. Of course, in some of the special cases discussed earlier, we have better control on the metric potential for the conclusion.

To begin with, as $[\omega_T]$ is on the boundary of the Kähler cone for X, there exists $\varphi \in PSH_{\omega_T}(X)$. For this fact, one can for example use the sequential limit construction in [23].

The fundamental regularization result by Demailly (Theorem 1.6 in [4]) provides us with a decreasing approximation sequence for φ, $\{\varphi_m\}_{m=1}^{\infty}$, satisfying:

- $\varphi_m \in PSH_{\omega_T + \frac{1}{m}\omega_0}(X)$;
- $\varphi_m \in C^{\infty}(X \setminus Z_m)$ with $Z_m \subset Z_{m+1}$ being analytic subvarieties of X. Furthermore, φ_m has logarithmic poles along Z_m, i.e. locally being the logarithm of the sum of squares of finitely many holomorphic functions, all vanishing along Z_m.

We start with the following combination of (2.4) and (2.7),

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{m}[(1-e^t)\frac{\partial u}{\partial t}+u]+[(1-e^{t-T})\frac{\partial u}{\partial t}+u]\right)\\ &=\Delta\left(\frac{1}{m}[(1-e^t)\frac{\partial u}{\partial t}+u]+[(1-e^{t-T})\frac{\partial u}{\partial t}+u]\right)\\ &\quad-\frac{n(1+m)}{m}+\langle\widetilde{\omega}(t),\frac{1}{m}\omega_0+\omega_T\rangle.\end{aligned}$$

Then let's modify it using φ_m as follows.

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{m}[(1-e^t)\frac{\partial u}{\partial t}+u]+[(1-e^{t-T})\frac{\partial u}{\partial t}+u]-\varphi_m\right)\\ &=\Delta\left(\frac{1}{m}[(1-e^t)\frac{\partial u}{\partial t}+u]+[(1-e^{t-T})\frac{\partial u}{\partial t}+u]-\varphi_m\right)\\ &\quad-\frac{n(1+m)}{m}+\langle\widetilde{\omega}(t),\frac{1}{m}\omega_0+\omega_T+\sqrt{-1}\partial\bar{\partial}\varphi_m\rangle.\end{aligned}\tag{3.1}$$

where $\frac{1}{m}\omega_0+\omega_T+\sqrt{-1}\partial\bar{\partial}\varphi_m$ is smooth and positive over $X \setminus Z_m$. Since $\varphi_m \in C^{\infty}(X \setminus Z_m)$ and has $-\infty$ poles along Z_m, the spatial minimum of

$$\frac{1}{m}[(1-e^t)\frac{\partial u}{\partial t}+u]+[(1-e^{t-T})\frac{\partial u}{\partial t}+u]-\varphi_m$$

is always achieved in $X \setminus Z_m$, where everything is smooth. So we can apply the standard Maximum Principle argument to conclude

$$\frac{1}{m}[(1-e^t)\frac{\partial u}{\partial t}+u]+[(1-e^{t-T})\frac{\partial u}{\partial t}+u]-\varphi_m \geqslant -C,$$

which is uniform for all m's over $X \times [0,T)$. Here, we make use of the uniform upper bound of φ_m's. This inequality can be reformulated as follows:

$$\left(1+\frac{1}{m}\right)u+\left(\frac{1}{m}(1-e^t)+(1-e^{t-T})\right)\frac{\partial u}{\partial t}\geqslant -C+\varphi_m.$$

For any $t\in[0,T)$, we could choose $m(t)$ large enough with

$$0<\frac{2}{m(t)}(1-e^t)+(1-e^{t-T})<1.$$

Combining with $u\leqslant C$, $\frac{\partial u}{\partial t}\leqslant C$ and $\varphi_{m(t)}\geqslant\varphi$, we conclude

$$u\geqslant -C+\varphi,$$

which is enough to exclude the possibility of $u\to-\infty$ uniformly as $t\to T$.

This argument works for any $\varphi\in PSH_{\omega_T}(X)$, and so the flow limit $u(T)\in PSH_{\omega_T}(X)$ is of minimal singularities among all elements in $PSH_{\omega_T}(X)$. Hence we have completed the proof of Theorem 1.1.

Remark 3.1. *By Theorem 1.1, the flow provides a smooth descreasing approximation for the flow limit which is of minimal singularities. This interpretation will be more interesting after the discussion in Section 4. Also, there is similar discussion in the final version of [3], where the focus is certainly different.*

3.2 The difference between $T<\infty$ and $T=\infty$ cases

We start by an example showing that for the infinite time collapsed case, i.e. $T=\infty$ and $[\omega_T]^n=0$, $u\to-\infty$ uniformly as $t\to\infty$.

Example 3.2. *Suppose that $K_X=[\omega_\infty]$ gives a fibration structure of X with the generic fibre of dimension $0<k\leqslant n$, i.e., $P:X\to\mathbb{CP}^N$ with $mK_X=P^*[\omega_{FS}]$ and $P(X)$ of complex dimension $n-k$. Then the flow (1.1) exists forever and $u\sim -kt$ as $t\to\infty$.*

This is justified as follows. Begin with the following scalar potential flow

$$\frac{\partial v}{\partial t}=\log\frac{(\omega_t+\sqrt{-1}\partial\bar\partial v)^n}{\Omega}-v+kt,\qquad v(\cdot,0)=0.$$

Clearly, it still corresponds to the same metric flow (1.1) and the relation between u in (1.2) and v is

$$u=v+f(t)\quad\text{with}\quad\frac{df}{dt}+f=-kt,\quad f(0)=0.$$

We easily see $f(t)\sim -kt$ and $\frac{df(t)}{dt}\sim -k$ as $t\to\infty$. Rewrite the equation of v as follows,

$$(\omega_t+\sqrt{-1}\partial\bar\partial v)^n=e^{-kt}e^{\frac{\partial v}{\partial t}+v}\Omega.$$

With the upper bound of $\frac{\partial u}{\partial t}$, as in (2.5), and the above description of $\frac{df}{dt}$, we have the uniform upper bound of $\frac{\partial v}{\partial t}$. Then applying the L^∞ estimate in [8] and [5], we have $|v| \leqslant C$ for all time. Hence $u \sim -kt$, tending $-\infty$ as $t \to \infty$.

Let's also point out that by the result in [17], $\frac{\partial v}{\partial t}$ and also $\frac{\partial u}{\partial t}$ are uniformly bounded from below for all time.

As mentioned before, by the result in [27], if $T = \infty$ and $[\omega_T]^n = 0$, the unnormalized metric potential u always tends to $-\infty$ uniformly as $t \to \infty$. Furthermore, if Abundance Conjecture is true as most people believe, the assumption in the above example holds for K_X nef. and $K_X^n = 0$, and so the more clearly described behavior of u and $\frac{\partial u}{\partial t}$ is universal for infinite time collapsed singularity. By all means, in the collapsed setting, the $T = \infty$ case needs to be treated differently from the $T < \infty$ case. The difference can be illustrated in a very intuitive way as follows.

For the finite time collapsed case, one has the following flow for v,

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}v)^n = (T-t)^k e^{\frac{\partial v}{\partial t} + v}\Omega,$$

which corresponds to the same metric flow (1.1) and

$$u = v + h(t) \quad \text{with} \quad \frac{dh}{dt} + h = k\log(T-t), \quad h(0) = 0.$$

So we have $|h(t)| \leqslant C$ and $\frac{dh}{dt} \sim k\log(T-t)$. In principle, we expect that v stays bounded or at least doesn't tend to $-\infty$ uniformly (comparable with the situation in the example), and so that's also expected for u, which is true by the main result in Theorem 1.1.

Moreover, the difference also exists in the behaviour of $\frac{\partial u}{\partial t}$. In fact, for the finite time collapsed case, one can justify $\frac{\partial u}{\partial t} \to -\infty$ uniformly as $t \to T$, different from the $T = \infty$ as in the example. This is seen by the following simple argument. The essentially decreasing limit of $\frac{\partial u}{\partial t} + u$ as $t \to T$ is $-\infty$ almost everywhere by simply considering the volume. This limit is essentially (i.e., ignoring measure 0 sets) upper semi-continuous as the essentially decreasing limit of the smooth function $\frac{\partial u}{\partial t} + u$. So the limit of $\frac{\partial u}{\partial t} + u$ is indeed $-\infty$ over X, and the convergence is then uniform by the classic considersation as for Dini's theorem. Finally, by (2.5), we conclude $\frac{\partial u}{\partial t} \to -\infty$ uniformly as $t \to T < \infty$.

4 Applications to Any Nef. Class

In this section, we generalize the discussion in Section 3 to the study of a general nef. (i.e., numerically effective) class α, which is a real $(1,1)$-class on the boundary of the Kähler cone of X.

For any Kähler metric ω_0, we can choose a class $\beta \in H^{1,1}(X;\mathbb{R})$ in the complement of the closure of the Kähler cone of X such that the interval joining $[\omega_0]$ and β intersects the boundary of the Kähler cone right at α. In other words,

$$\alpha = (1-a)[\omega_0] + a\beta \text{ for } a = \sup\{s \mid (1-s)[\omega_0] + s\beta \text{ Kähler}\} \in (0,1).$$

Of course, the choice of β is not unique even after fixing $[\omega_0]$. However, this is not an issue for our purpose.

We then pick a smooth real $(1,1)$-form L representing β, and consider the following evolution equation of Kähler-Ricci flow type:

$$\frac{\partial \widetilde{\omega}(t)}{\partial t} = -\mathrm{Ric}\left(\widetilde{\omega}(t)\right) - \widetilde{\omega}(t) + \mathrm{Ric}(\Omega) + L, \quad \widetilde{\omega}(0) = \omega_0, \tag{4.1}$$

where Ω is a smooth volume form over X. This equation was introduced in [26] and further studied in [24] and [31]. It shares a lot of common features with (1.1), especially when considering the parabolic complex Monge-Ampère equation for the metric potential.

By the ODE consideration in $H^{1,1}(X,\mathbb{R})$, we know $[\widetilde{\omega}(t)] = \beta + e^{-t}([\omega_0]-\beta)$, and the corresponding optimal existence result in [24] tells that the flow metric solution exists as long as the class $\beta + e^{-t}([\omega_0]-\beta)$ remains Kähler. So by our construction, the flow must develop finite time singularities.

Set $\omega_t = L + e^{-t}(\omega_0 - L)$. It's known that $\widetilde{\omega}(t) = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ with u satisfying

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \quad u(\cdot, 0) = 0, \tag{4.2}$$

which is equivalent to (4.1).

All the computations and estimations at the beginning of Section 2 can be carried through in exactly the same way for this flow. We still denote the time of singularity by $T < \infty$ with $\alpha = [\omega_T] = \beta + e^{-T}([\omega_0]-\beta)$. The construction in [23] still works for this flow and so we know for example, $PSH_{\omega_T}(X) \neq \varnothing$. Then the argument in Section 3 justifies the existence of the flow weak limit and see it's of minimal singularities. Hence, we conclude the following result.

Theorem 4.1. *Use the above notations. For any nef. class, the evolution equation of Kähler-Ricci flow type (4.1) provides a positive* $(1,1)$*-current of minimal singularities as its representative, together with a smooth approximation.*

In light of the famous example by Serre as described in [2] about a nef. and big integral class without any positive current representative of bounded local potential, we know that the flow limit from the above construction can't always have bounded local potential.

In the following, we derive a flow metric estimate similar to (2.13) for (1.1). Let's begin with the inequality below which follows from [19],

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log\langle \omega_0, \widetilde{\omega}(t)\rangle \leqslant C\langle \widetilde{\omega}(t), \omega_0\rangle + C + \frac{\langle \omega_0, \mathrm{Ric}(\Omega) + L\rangle}{\langle \omega_0, \widetilde{\omega}(t)\rangle},$$

where the last term on the right hand side comes from the extra term in (4.1) comparing with (1.1). Then one has

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log\langle \omega_0, \widetilde{\omega}_t\rangle \leqslant C\langle \widetilde{\omega}(t), \omega_0\rangle + C + \frac{C}{\langle \omega_0, \widetilde{\omega}(t)\rangle}. \tag{4.3}$$

Let's recall (2.4) which still holds for (4.1),

$$\frac{\partial}{\partial t}\left((1-e^t)\frac{\partial u}{\partial t}+u\right)=\Delta\left((1-e^t)\frac{\partial u}{\partial t}+u\right)-n+\langle\widetilde{\omega}(t),\omega_0\rangle. \tag{4.4}$$

It implies

$$(1-e^t)\frac{\partial u}{\partial t}+u+nt\geqslant 0.$$

Combining (4.3) and (4.4), we arrive at

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\Delta\right)\left(\log\langle\omega_0,\widetilde{\omega}(t)\rangle-B\big((1-e^t)\frac{\partial u}{\partial t}+u+nt\big)\right)\\ &\leqslant(C-B)\langle\widetilde{\omega}(t),\omega_0\rangle+C+\frac{C}{\langle\omega_0,\widetilde{\omega}(t)\rangle},\end{aligned}$$

for a positive constant B fixed shortly. At the (local space-time) maximum value point of the quantity,

$$\log\langle\omega_0,\widetilde{\omega}(t)\rangle-B\big((1-e^t)\frac{\partial u}{\partial t}+u+nt\big),$$

if the point is not at the initial time (otherwise trivial), one has

$$(C-B)\langle\widetilde{\omega}(t),\omega_0\rangle+C+\frac{C}{\langle\omega_0,\widetilde{\omega}(t)\rangle}\geqslant 0.$$

Recall the elementary algebraic inequality

$$\langle\omega_0,\widetilde{\omega}(t)\rangle\cdot\langle\widetilde{\omega}(t),\omega_0\rangle\geqslant n^2,$$

and so $\frac{1}{\langle\omega_0,\widetilde{\omega}(t)\rangle}\leqslant\frac{\langle\widetilde{\omega}(t),\omega_0\rangle}{n^2}$. After choosing $B>C+\frac{1}{n^2}+1$, at the maximum value point, we have

$$\langle\widetilde{\omega}(t),\omega_0\rangle\leqslant C.$$

Now we apply another elementary inequality

$$\langle\omega_0,\widetilde{\omega}(t)\rangle\leqslant\langle\widetilde{\omega}(t),\omega_0\rangle^{n-1}\cdot\frac{\widetilde{\omega}(t)^n}{\omega_0^n}.$$

Together with $\widetilde{\omega}(t)^n=e^{\frac{\partial u}{\partial t}+u}\Omega\leqslant C\Omega$, we have

$$\langle\omega_0,\widetilde{\omega}(t)\rangle\leqslant C$$

at that point. Noticing the lower bound for $(1-e^t)\frac{\partial u}{\partial t}+u+nt$, we conclude

$$\log\langle\omega_0,\widetilde{\omega}(t)\rangle-B\big((1-e^t)\frac{\partial u}{\partial t}+u+nt\big)\leqslant C,$$

which gives

$$\widetilde{\omega}(t)\leqslant Ce^{B\left((1-e^t)\frac{\partial u}{\partial t}+u+nt\right)}\omega_0\leqslant Ce^{-C(e^t\frac{\partial u}{\partial t}-t)}\omega_0.$$

Since $\widetilde{\omega}(t)^n = e^{\frac{\partial u}{\partial t}+u}\Omega$, we can further conclude that

$$Ce^{C(e^t\frac{\partial u}{\partial t}-t)}\omega_0 \leqslant \widetilde{\omega}(t) \leqslant Ce^{-C(e^t\frac{\partial u}{\partial t}-t)}\omega_0. \tag{4.5}$$

which is similar to (2.12). Now we restrict to the case of $T < \infty$ and have for $t \in [0, T)$,

$$Ce^{C\frac{\partial u}{\partial t}}\omega_0 \leqslant \widetilde{\omega}(t) \leqslant Ce^{-C\frac{\partial u}{\partial t}}\omega_0. \tag{4.6}$$

which is (2.13) for this more general evolution equation. Hence just as for (1.1) in [29], there can't be any uniform lower bound for $\frac{\partial u}{\partial t}$ (or the volume form $\widetilde{\omega}^n(t) = e^{\frac{\partial u}{\partial t}+u}\Omega$) for the finite time singularity case.

Remark 4.2. *Similar to the result in [15], with the uniform control of the flow metric, higher order estimates should be available even in a local fashion.*

In all, the discussion so far for (1.1) can be naturally generalized to the more general flow (4.1).

5 Further Remarks

In this section, we discuss the implication of the lower order estimate of the metric potential in understanding the formation of singularities for the Kähler-Ricci flow. Let's begin by stating the following conjecture.

Conjecture 5.1. *For the flow (1.1) with singularities at $T < \infty$, $u \geqslant -C$ for $t \in [0, T)$.*

Notice that the conjecture is on the classic Kähler-Ricci flow, i.e., about the canonical class K_X. For the more general flow in Section 4, the situation is known to be different by Serre's example. The confidence here mostly comes from the cases from algebraic geometry consideration, in which the conjecture is known to hold as discussed in Subsection 2.1. Also, there is natural relation with the conjectures on the singularity type of the Kähler-Ricci flow in [20], as illustrated by the discussion in Subsection 2.3.

Furthermore, there is the fundamental problem regarding the singularities of the Kähler-Ricci flow: *are they always developed along analytic varieties?* A little discussion with Professor F. Campana brought this to my attention.

For the global volume collapsed case, this is obviously the case if one considers singularities in the simple-minded way, i.e. relating to uniform control on flow metric, because of the vanishing of global volume. In this case, it's more meaningful to search for global control of geometric quantities and a more precise understanding of *essential singularities* regarding the metric collapsing, i.e. after excluding the effect of *regular collapsing*, i.e. proper scaling with respect to the rate of volume collapsing.

So far, quite some progress has been made for the infinite time singularity case, for example, as in [19], [18], [11] and [25]. The remaining difficulty is to achieve global geometric control in the presence of singular fibres. While for the finite time singularity case, it's fairly open with the existing results imposing serious assumptions, for example, in [16] and [10].

The main result of this paper provides us with a flow weak limit of the metric potential which is closely related to the limiting class $[\omega_T]$. We expect the $-\infty$ locus of this limit (of minimal singularities) to characterize the essential singularities of the flow.

Meanwhile, the situation for the global volume non-collapsed case is a lot different, for which one naturally expects the singularities to develop along a subvariety of X. There are evidences for both finite and infinite time singularity cases, for example, already in [24], where the estimates are degenerate along a subvariety.

Recently, there has been substantial progress made by Collins-Tosatti in the fundamental work [3] on the finite time non-collapsed singularities of the Kähler-Ricci flow. More precisely, among other things, it's proved there that the flow stays smooth out of the subvariety $E_{nK}([\omega_T]) = \mathrm{Null}([\omega_T])$. This is done by obtaining the proper lower bound of $\frac{\partial u}{\partial t}$. At the same time, it is impossible for the flow to stay smooth around any point of that subvariety in the Riemannian sense (i.e., with the curvature staying uniformly bounded in some fixed neighbourhood, which is one way to characterize flow singularity as in [6]) by simple cohomology type consideration in light of the definition of the set $\mathrm{Null}([\omega_T])$ (i.e., the union of vanishing subvarieties with respect to $[\omega_T]$). It's quite obvious that the discussion in that work can be adapted to the more general flow (4.1) in Section 4. Of course, for the higher order estimates, one needs to accept the statement in Remark 4.1 about higher order estimates.

We now provide the following point of view which is certainly related but slightly different, coming from a different characterization of flow singularities. Let's consider the more general evolution equation of Kähler-Ricci flow type (4.1) with finite time singularities, i.e., $T < \infty$.

At the first sight, the (pluripolar) set $\{u_T = -\infty\}$, looks like the natural candidate for the singular set of the flow. However, this is not true with many known cases, for example, u_T might well be bounded (and so this set is empty) in the presence of singularities. Indeed, by the discussion in Subsection 2.2, we are led to investigate the set

$$\widehat{S} := \{x \in X \mid \frac{\partial u}{\partial t} \to -\infty \text{ for some time sequence}\}.$$

By (4.6), the flow metric itself is (pointwise) bounded in the complement of $\widehat{S}$.

Since $\frac{\partial u}{\partial t} \leqslant Cu + C$ (from $\frac{\partial u}{\partial t} \leqslant \frac{u+nt}{e^t-1}$) and $u \leqslant C$, we have

$$C\frac{\partial u}{\partial t} - C \leqslant \frac{\partial u}{\partial t} + u \leqslant \frac{\partial u}{\partial t} + C. \tag{5.1}$$

By the essential decreasing of the volume form, we can define the limit of $\frac{\partial u}{\partial t} + u$ which is an (essentially) upper semi-continuous function V over X valued in

$[-\infty, C)$ for $C < \infty$, satisfying

$$\int_X e^V \Omega = [\omega_T]^n.$$

By (5.1), we have

$$\widehat{S} = \{x \in X \mid \frac{\partial u}{\partial t} \to -\infty\} = \{x \in X \mid \frac{\partial u}{\partial t} + u \to -\infty\} = \{x \in X \mid V = -\infty\}.$$

In the collapsed case, we have $V \equiv -\infty$ as discussed in Subsection 3.2, and so $\widehat{S} = X$ which coincides with the usual vision of the singular set.

Now we consider the global volume non-collapsed case. In the algebraic geometry setting, we already have in [24] that this set is contained in a subvariety of X. Indeed, by the estimate in [3], we have $\widehat{S} \subset \mathrm{Null}([\omega_T])$.

The set $\widehat{S}$ can be complicated. More precisely, we have

$$\widehat{S} = \cap_{A=1}^{\infty} \cup_{s\in[0,T)} \{x \in X \mid \frac{\partial u}{\partial t} + u + Ce^{-s} < -A \text{ at } (x, s)\}.$$

The decreasing of $\frac{\partial u}{\partial t} + u + Ce^{-t}$ tells us that the open set

$$\{x \in X \mid \frac{\partial u}{\partial t} + u + Ce^{-s} < -A \text{ at } (x, s)\}$$

is increasing as $s \to T$. Also, the result in [29] implies the open set

$$\cup_{s\in[0,T)}\{x \in X \mid \frac{\partial u}{\partial t} + u + Ce^{-s} < -A \text{ at } (x, s)\} \neq \varnothing$$

for any A. However, it's not even clear whether $\{x \in X \mid V = -\infty\} \neq \varnothing$, as the intersection of a sequence of decreasing open sets. A priori, V might not actually take the value $-\infty$, although it can't have a lower bound by the discussion in [29] for (1.1) and the natural generalization to the general flow (4.1). Nevertheless, if we consider the the lower semi-continuization V_*, the set

$$S := \{V_* = -\infty\}$$

is a closed set in X and certainly non-empty by the above discussion.

In the complement of S, V is locally bounded (by the semi-continuity of V_*), and so the flow metric is locally uniformly bounded by (4.6) and then higher order estimates are available by the result in [15], at least for the classic flow (1.1). Thus, it is reasonable to consider $\{V_* = -\infty\}$ as the singular set.

Let's point out that the function V can a priori be wild. For example, $\{V = -\infty\} = \varnothing$ and $\{V_* = -\infty\} = X$ might happen simultaneously. However, by the result in [3], $\widehat{S} \subset S \subset \mathrm{Null}([\omega_T])$. To conclude the discussion in this direction, we state the following conjecture.

Conjecture 5.2. *Use the notations above. Consider the flow (4.1) of Kähler-Ricci flow type. If there are finite time singularities, then*

$$\widehat{S} = S = \mathrm{Null}([\omega_T]).$$

This predicts a more precise and also elementary description for the blow-up along the singular set of the flow. As mentioned earlier, it's known to hold in the collapsed case.

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