

ON THE WEAK KÄHLER-RICCI FLOW

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ABSTRACT. In this paper, we define and study Kähler-Ricci flow with initial data not being smooth with some natural applications.

1. INTRODUCTION

Let M be an n -dimensional compact (without boundary) Kähler manifold. A Kähler metric g can be represented by its Kähler form ω_g on M . Consider the Kähler-Ricci flow on $M \times [0, T)$

$$(1.1) \quad \frac{\partial \omega_{g(t)}}{\partial t} = -\text{Ric}(\omega_{g(t)}),$$

where $g(t)$ is a family of Kähler metrics and $\text{Ric}(\omega_g)$ denotes the Ricci curvature of g . It is known that for any smooth Kähler metric g_0 , there is a unique solution $g(t)$ of (1.1) for some maximal time $T > 0$ with $g(0) = g_0$. In general, T will depend on the initial metric g_0 . However, in Kähler manifold case, this only depends on the Kähler class and the first Chern class. This observation plays an important role in our introduction of weak Ricci flow over Kähler manifold.

The Ricci flow was introduced by R. Hamilton in [20]. Extensive research has been done in the case of the smooth flow (c.f. [22] [6] [14] [21] [25] [12] [13] [30] or [15] for complete updated references). In order to prove the uniqueness of extremal Kähler metrics in full generality, in [9], the first two named authors were led to the study of Kähler-Ricci flow in the weak sense. More precisely, we proved in [9] that for any initial L^∞ -bounded Kähler metric, there is a uniformly L^∞ -bounded solution of (1.1) with the given initial "metric" in a suitable sense. Moreover, the volume form of the solution converges strongly to the volume form of the initial Kähler "metric" in L^2 -topology as $t \rightarrow 0$.

As one may expect, such a weak solution of (1.1) should become smooth immediately after $t > 0$. Indeed, we confirm this in this note in a more general setting.

Definition 1.1. For any Kähler current g_0 with $C^{1,1}$ bounded potential on M , there is a unique smooth solution $g(t)$ ($t \in (0, T)$) of (1.1) such that $\lim_{t \rightarrow 0^+} g(t) = g_0$ in $C^{1,\alpha}$ ($\forall \alpha \in (0, 1)$) norm at the potential level.

A special case of Riemann surfaces was also considered in [11] with a completely different proof.

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It was known that any Kähler metric with constant scalar curvature is the absolute minimizer of the K-energy on the space of all Kähler metrics with a fixed Kähler class ([8], [18], [24] and [9])¹. From analytic point of view, it is an extremely difficult problem to prove the existence of Kähler metric with constant scalar curvature in general cases. One approach is to construct weak minimizers of the K-energy by applying certain variational or continuous methods. This seems to be a plausible direct approach but very hard. However, even if we obtained a $C^{1,1}$ minimizer of the K-energy functional, we would still face the regularity problem. In [8], the first named author made the following conjecture: *Any $C^{1,1}$ minimizers of the K-energy in a given Kähler class must be smooth.* As a consequence of the above theorem, we can solve this conjecture in canonically polarized cases.

Example 1.2. In a Kähler class which is proportional to the first Chern class, any L^∞ Kähler metric which minimizes the K-energy functional must be smooth.

There is another motivation for this short paper. On a general Kähler manifold, the Kähler-Ricci flow (1.1) may develop singularity at finite time (see [31]). In an on-going project with his collaborators, the second named author proposed some problems of studying how the Kähler-Ricci flow extends across the finite time singularity (see [28] for more discussions). One of them involves constructing solutions of the Kähler-Ricci flow with much weaker initial metrics, possibly on spaces with mild singularity. Our second result gives a partial solution to this problem.

First we recall some standard facts: by Hodge Theorem, the space of all Kähler metrics in a fixed Kähler class given by a smooth Kähler metric ω is

$$P(M, \omega) = \{\varphi \in C^\infty(M) \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } M\}.$$

We denote by $Cl_{L^\infty}P(M, \omega)$ all bounded functions plurisubharmonic with respect to ω . Then for any $\varphi \in Cl_{L^\infty}P(M, \omega)$, there is a well-defined (Borel) measure $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n \geq 0$ using the understanding from pluripotential theory about Monge-Ampère operator introduced in [2]. One can also think of this as a volume form in a weak sense. Now we can state our second result.

Definition 1.3. For any Kähler potential $\varphi_0 \in Cl_{L^\infty}P(M, \omega)$ whose volume form is $L^p(M, \omega)(p \geq 3)$, there is a unique smooth solution $g(t)$ of (1.1) for $t \in (0, T)$ such that $\frac{\omega_{g(t)}^n}{\omega^n}$ converges to $\frac{\omega_{\varphi_0}^n}{\omega^n}$ strongly in the L^2 -topology.

The organization of this note is as follows: in Section 2, we show the existence of the weak solution for the induced potential flow which is equivalent to the Kähler-Ricci flow. C^0 -estimates will be derived for the solution and its time derivative. We also examine convergence problem for these weak solutions as t tends to 0. In Section 3, we derive the 2nd and 3rd order estimates for the weak solutions. Then Theorem 1.1 and 1.3 follow. In last section, we prove the corollary.

2. WEAK SOLUTIONS OF KÄHLER-RICCI FLOW

First we reduce the Kähler-Ricci flow (1.1) to a scalar flow. To start with, we note that $[\omega_{g_0}] - tc_1(M)$ represents a Kähler class whenever $0 \leq t \leq T$ for a sufficiently small $T > 0$ depending only on $[\omega_{g_0}]$ and $c_1(M)$. Choose a smooth family of Kähler forms ω_t for $t \in [0, T]$ such that $\omega_0 = \omega$ and $[\omega_t] = [\omega_{g_0}] - tc_1(M)$.

¹For Kähler-Einstein metrics, cf. [1], [17] and [29].

Write $\omega_{g(t)} = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi(t)$. Then, one can show by standard argument that $g(t)$ solves (1.1) if and only if $\varphi(t)$ solves the following scalar flow:

$$(2.1) \quad \frac{\partial\varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_t^n} - h_{\omega_t},$$

where h_{ω_t} is defined by

$$\text{Ric}(\omega_t) + \frac{\partial\omega_t}{\partial t} = \sqrt{-1}\partial\bar{\partial}h_{\omega_t}, \text{ and } \int_X (e^{h_{\omega_t}} - 1)\omega_t^n = 0.$$

Clearly such a $h(\omega_t)$ does exist and is unique. As usual, we will regard either (1.1) or (2.1) as the Kähler-Ricci flow on M . For simplicity, we denote by ω_φ the Kähler form $\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$.

To construct a weak solution to the Kähler-Ricci flow equation with non-smooth initial Kähler potentials, as usual, we use smooth approximations of the initial data. Let φ_0 be a bounded Kähler potential such that its volume form is in L^p for some $p > 1$. This actually implies Hölder continuity of φ_0 from the result in [27]. Let $\varphi_0(s)$ ($0 \leq s \leq 1$) be a 1-parameter Kähler potentials such that the following properties hold:

- (1) $\varphi_0(0) = \varphi_0$ and $\varphi_0(s) \in P(M, \omega)$ for $0 < s \leq 1$.
- (2) If φ_0 has $C^{1,1}$ bound, then $\varphi_0(s)$ has uniform $C^{1,1}$ upper bound and $\varphi_0(s)(s > 0) \rightarrow \varphi_0$ strongly in $W^{2,q}(M, \omega)$ for q sufficiently large.
- (3) $\varphi_0(s)$ converges to φ_0 uniformly in L^∞ -topology and the volume form ratio converges to $\frac{\omega_{\varphi_0}^n}{\omega^n}$ strongly in $L^p(p > 1)$.

Later in Section 3, less regularity will be assumed, and so the approximation would also be less restrictive.

It is known (cf. [31]) that for any φ in $P(M, \omega)$, there is a unique smooth solution of (2.1) on $[0, T]$ with φ as the initial value. Therefore, we have $\varphi(s, t) \in P(M, \omega)$ ($0 < s \leq 1, t \in [0, T]$) satisfying:

$$(2.2) \quad \frac{\partial\varphi(s, t)}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_t^n} - h_{\omega_t},$$

$$(2.3) \quad \varphi(s, 0) = \varphi_0(s).$$

Clearly, for each $s > 0$ fixed, there exists a uniform $C^{2,\alpha}$ bound for $\varphi(s, t)$ ($0 \leq t \leq T$). However, the upper bound may well depend on s and so may blow up when t, s are both small. If there is a limit

$$\varphi(t) = \lim_{s \rightarrow 0} \varphi(s, t), \quad \forall t \in [0, T],$$

then we can regard $\varphi(t)$ as a weak solution of (2.1) with initial value φ_0 .

Lemma 2.1. *The solutions $\varphi(s, t)$ converges uniformly to a family of functions $\varphi(t)$ ($t \in [0, T]$) as s tends to 0.*

Proof. For any positive s and s' , put $\psi(t) = \varphi(s', t) - \varphi(s, t)$, then we have

$$\frac{\partial\psi}{\partial t} = \log \frac{(\omega_{\varphi(s,t)} + \sqrt{-1}\partial\bar{\partial}\psi)^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi(s, t))^n}.$$

Then the Maximum Principle implies that $\sup_M |\psi(t)| \leq \sup_M |\psi(0)|$, that is,

$$\sup_M |\varphi(s', t) - \varphi(s, t)| \leq \sup_M |\varphi_0(s') - \varphi_0(s)|.$$

The lemma then follows easily from our choice of the initial data $\varphi_0(s)$. \square

We shall show that $\varphi(t)$ solves the Kähler-Ricci flow and consequently is a weak solution in a suitable sense. For this purpose, one needs to justify that $\varphi(t)$ is a smooth family for $t > 0$ which solves (2.1) for $t > 0$ such that $\lim_{t \rightarrow 0^+} \varphi(t) = \varphi_0$. Of course, this is the core part of this paper. First we derive a few estimates on $\varphi(s, t)$ uniform for $s \in (0, 1]$ (Sometimes we abbreviate $\varphi(s, t)$ as φ for simplicity). These estimates would eventually be passed on to $\varphi(t)$ by taking limit as s tends to 0 in some proper sense.

Lemma 2.2. *If φ_0 is bounded, then there exists a uniform constant C such that for any $s \in (0, 1]$ and $t \in [0, T]$,*

$$|\varphi(s, t)| \leq C.$$

Proof. Choose c such that $|h_{\omega_t}|_{C^0} \leq c$ for all $t \in [0, T]$. Applying the Maximum Principle to $\varphi(s, t) \pm ct$, we can obtain

$$\min \varphi_0(s) - ct \leq \varphi(s, t) \leq \max \varphi_0(s) + ct, \quad \forall s > 0.$$

Then the bound on φ follows. \square

The next lemma gives an estimate on the volume form for the Kähler-Ricci flow when $t > 0$. In this note, Δ_φ always stands for Laplacian with respect to the flow metric $\omega_\varphi = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$. Traditionally, C might stand for different positive constants at different places.

Lemma 2.3. *For $t > 0$, the volume form has uniform upper and positive lower bounds which may depend on t .*

Proof. Differentiating the Kähler-Ricci flow equation

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_t^n} - h_{\omega_t},$$

we get

$$\begin{aligned} \varphi'' &= \Delta_\varphi \varphi' - h'_{\omega_t} - \text{tr}_{\omega_t} \left(\frac{\partial}{\partial t} \omega_t \right) + \text{tr}_\varphi \left(\frac{\partial}{\partial t} \omega_t \right) \\ &\leq \Delta_\varphi \varphi' + C(1 + \text{tr}_\varphi \omega). \end{aligned}$$

Similarly, we have

$$\varphi'' \geq \Delta_\varphi \varphi' - C(1 + \text{tr}_\varphi \omega),$$

which will be used for lower bound later in this proof.

For the upper bound, let's consider

$$F_+ = -\varphi + t\varphi'.$$

The t -derivative gives

$$\begin{aligned} \frac{\partial}{\partial t} F_+ &= -\varphi' + \varphi' + t\varphi'' \\ &\leq t\Delta_\varphi \varphi' + Ct(1 + \text{tr}_\varphi \omega) \\ &= \Delta_\varphi (-\varphi + t\varphi') + \Delta_\varphi \varphi + Ct(1 + \text{tr}_\varphi \omega) \\ &= \Delta_\varphi F_+ + n - \text{tr}_\varphi \omega_t + Ct(1 + \text{tr}_\varphi \omega) \\ &\leq \Delta_\varphi F_+ + C - (C - Ct)(1 + \text{tr}_\varphi \omega). \end{aligned}$$

For t small enough, we have

$$\frac{\partial}{\partial t} F_+ \leq \Delta_\varphi F_+ + C.$$

Since $F_+(0)$ is uniformly bounded from above, it follows from the Maximum Principle that F_+ has a uniform upper bound.

For the lower bound, set

$$F_- = \varphi + t\varphi' = \varphi - th_\omega + t \log \frac{\omega_\varphi^n}{\omega_t^n}.$$

Then, using the same C to denote different constants at places,

$$\begin{aligned} \frac{\partial}{\partial t} F_- &= \varphi' + \varphi' + t\varphi'' \\ &\geq 2\varphi' + t(\Delta_\varphi \varphi' - C(1 + \text{tr}_\varphi \omega)) \\ &= \Delta_\varphi(\varphi + t\varphi') - \Delta_\varphi \varphi + 2\varphi' - Ct(1 + \text{tr}_\varphi \omega) \\ &= \Delta_\varphi F_- - n + \text{tr}_\varphi \omega_t + 2 \log \frac{\omega_\varphi^n}{\omega_t^n} - 2h_{\omega_t} - Ct(1 + \text{tr}_\varphi \omega) \\ &\geq \Delta_\varphi F_- - C + (C - Ct)(1 + \text{tr}_\varphi \omega) + 2 \log \frac{\omega_\varphi^n}{\omega_t^n} \\ &\geq \Delta_\varphi F_- - C + (C - Ct) \left(\frac{\omega_\varphi^n}{\omega_t^n} \right)^{-\frac{1}{n}} + 2 \log \frac{\omega_\varphi^n}{\omega_t^n}. \end{aligned}$$

It's easy to see for $x > 0$, with the understanding that these two C 's are not the same,

$$x^{-\frac{1}{n}} + C \log x \geq -C.$$

It follows that for t small, we have

$$\frac{\partial}{\partial t} F_- \geq \Delta_\varphi F_- - C.$$

Then the Maximum Principle implies that $F_- > -C$ for some uniform constant C , that is, the volume form $\frac{\omega_\varphi^n}{\omega_t^n}$ has a uniform positive lower bound when considered uniformly away from initial time. \square

Next we exam whether $\lim_{t \rightarrow 0^+} \varphi(t) = \varphi_0$. To make sure of this, we derive some integral estimates.

Lemma 2.4. *If φ_0 is bounded and $\frac{\omega_\varphi^n}{\omega^n}$ is in $L^p(p \geq 1)$, then for any $s \in (0, 1]$, there exists a positive function $C(t)$ of t such that*

$$\left\| \frac{\omega_\varphi^n}{\omega^n} \right\|_{L^p} \leq C(t).$$

Moreover, $C(t)$ is independent of s and is uniformly bounded for $t \in [0, T]$ with $T < \infty$.

Proof. For $\lambda > 0$, set the modified volume ratio as

$$f_\varphi = \frac{\omega_\varphi^n}{\omega_t^n} e^{-\lambda \varphi}.$$

We are considering classic smooth solution. Let C be a constant satisfying:

$$\left| \frac{\partial}{\partial t} \omega_t \right|_{\omega_t} + |\text{Ric}(\omega_t)|_{\omega_t} + |\text{Ric}(\omega)|_{\omega_t} + |\partial \bar{\partial} h_{\omega_t}|_{\omega_t} \leq C.$$

Use ∇ to denote the gradient with respect to ω_φ . Then for λ sufficiently large,

$$\begin{aligned} \frac{\partial \log f_\varphi}{\partial t} &= \operatorname{tr}_\varphi \left(\frac{\partial}{\partial t} \omega_t \right) + \Delta_\varphi \varphi' - \operatorname{tr}_{\omega_t} \left(\frac{\partial}{\partial t} \omega_t \right) - \lambda \varphi' \\ &= \Delta_\varphi \log f_\varphi + \operatorname{tr}_\varphi \left(\frac{\partial}{\partial t} \omega_t + \lambda \sqrt{-1} \partial \bar{\partial} \varphi - \sqrt{-1} \partial \bar{\partial} h_{\omega_t} \right) + R_{\omega_t} - \operatorname{tr}_{\omega_t} h_{\omega_t} - \lambda \varphi' \\ &\leq \Delta_\varphi \log f_\varphi + \lambda - (\lambda - 2C) \cdot \operatorname{tr}_\varphi \omega_t - \lambda \log f_\varphi + \lambda h_{\omega_t} + 2C \\ &\leq \frac{\Delta_\varphi f_\varphi}{f_\varphi} - \frac{|\nabla f_\varphi|_\varphi^2}{f_\varphi^2} + 2C - \lambda \log f_\varphi. \end{aligned}$$

Thus, we have

$$\frac{\partial f_\varphi}{\partial t} \leq \Delta_\varphi f_\varphi - \frac{|\nabla f_\varphi|_\varphi^2}{f_\varphi} + 2C f_\varphi - \lambda f_\varphi \log f_\varphi.$$

For any $p \geq 1$, we have

$$\begin{aligned} &\frac{d}{dt} \int_M f_\varphi^p e^{\lambda \varphi} \omega_t^n \\ &= p \int_M f_\varphi^{p-1} \frac{\partial f_\varphi}{\partial t} e^{\lambda \varphi} \omega_t^n + \int_M f_\varphi^p e^{\lambda \varphi} \frac{\partial \omega_t^n}{\partial t} + \lambda \int_M f_\varphi^p e^{\lambda \varphi} \frac{\partial \varphi}{\partial t} \omega_t^n \\ &= p \int_M f_\varphi^{p-1} \frac{\partial f_\varphi}{\partial t} e^{\lambda \varphi} \omega_t^n + \int_M f_\varphi^p (-R_{\omega_t} + \operatorname{tr}_{\omega_t} h_{\omega_t}) e^{\lambda \varphi} \omega_t^n \\ &\quad + \lambda \int_M f_\varphi^p e^{\lambda \varphi} \left(\log \frac{\omega_\varphi^n}{\omega_t^n} - h_{\omega_t} \right) \omega_t^n \\ &\leq p \int_M f_\varphi^{p-1} \left(\Delta_\varphi f_\varphi - \frac{|\nabla f_\varphi|_\varphi^2}{f_\varphi} + f_\varphi (2C - \lambda \log f_\varphi) \right) e^{\lambda \varphi} \omega_t^n + C \int_M f_\varphi^p e^{\lambda \varphi} \omega_t^n \\ &\quad + \lambda \int_M f_\varphi^p e^{\lambda \varphi} \left(\log \frac{\omega_\varphi^n}{\omega_t^n} - h_{\omega_t} \right) \omega_t^n \\ &\leq \int_M f_\varphi^{p-2} \left(p \left(\Delta_\varphi f_\varphi - \frac{|\nabla f_\varphi|_\varphi^2}{f_\varphi} \right) + f_\varphi (2pC - \lambda(p-1) \log f_\varphi) \right) \omega_\varphi^n + C_\lambda \int_M f_\varphi^p e^{\lambda \varphi} \omega_t^n \\ &\leq -p(p-1) \int_M f_\varphi^{p-2} |\nabla f_\varphi|_\varphi^2 e^{\lambda \varphi} \omega_t^n + C_\lambda \int_M (f_\varphi^p + f_\varphi^{p-1}) e^{\lambda \varphi} \omega_t^n. \end{aligned}$$

Here we have used the fact that the function $x \log x$ has a lower bound for $x > 0$.

Since φ is uniformly bounded, it follows that $\frac{\omega_\varphi^n}{\omega_t^n}$ is uniformly bounded in L^p for all $t \in [0, T]$. \square

It follows from this lemma that the L^p -norm of $\varphi(t)$ is uniformly bounded. By the work of Kolodziej [27], we know that $\|\varphi(t)\|_{C^\alpha}$ is uniformly bounded, where $\alpha = \alpha(p) > 0$ may depend on $p > 1$. Then for any sequence $t_i \rightarrow 0$, there is a subsequence, still denoted by t_i for simplicity, such that $\varphi(t_i)$ converges to a C^α -function $\tilde{\varphi}_0$ in the C^β -topology for some $\beta \in (0, \alpha)$. A priori, this limit potential might well depend on the sequence we choose.

Lemma 2.5. *If $p \geq 3$, we have $\tilde{\varphi}_0 = \varphi_0$. Moreover, $\frac{\omega_\varphi^n(t)}{\omega_t^n}$ converges to $\frac{\omega_{\varphi_0}^n}{\omega_t^n}$ strongly in the L^2 -topology, where a smooth volume form is used to define the L^2 -topology and the choice clearly doesn't matter.*

Proof. First we assume that φ_0 is smooth, i. e., one of $\varphi_0(s)$ ($s > 0$). In the proof of the preceding lemma, set $p = 3$ and we have

$$\frac{d}{dt} \int_M f_\varphi^3 e^{\lambda \varphi} \omega_t^n \leq -6 \int_M f_\varphi |\nabla f_\varphi|_\varphi^2 e^{\lambda \varphi} \omega_t^n + C \int_M f_\varphi^3 e^{\lambda \varphi} \omega_t^n + C,$$

which implies

$$(2.4) \quad \int_0^t \int_M f_\varphi |\nabla f_\varphi|_\varphi^2 \omega_t^n du \leq C.$$

Now set

$$f = \frac{\omega_\varphi^n}{\omega_t^n}.$$

Claim 1: We have for t in some finite interval,

$$(2.5) \quad \int_0^t \int_M |\nabla f|_\varphi^2 \omega_\varphi^n du = \int_0^t \int_M f |\nabla f|_\varphi^2 \omega_t^n du \leq C$$

$$(2.6) \quad \int_0^t \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n du \leq C.$$

Proof of Claim 1: Since φ is uniformly bounded, we can choose a uniform constant c such that $\varphi - c \leq 0$. Then we have

$$\begin{aligned} & \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n \\ &= -2 \int_M f_\varphi (\nabla f_\varphi \cdot \nabla \varphi)_\varphi (\varphi - c) \omega_\varphi^n - \int_M f_\varphi^2 (\varphi - c) \Delta_\varphi \varphi \omega_\varphi^n \\ &\leq \frac{1}{2} \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n + C \int_M |\nabla f_\varphi|_\varphi^2 \omega_\varphi^n - n \int_M f_\varphi^2 (\varphi - c) \omega_\varphi^n + \int_M f_\varphi^2 (\varphi - c) \operatorname{tr}_\varphi \omega_t \omega_\varphi^n \\ &\leq \frac{1}{2} \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n + C \int_M |\nabla f_\varphi|_\varphi^2 \omega_\varphi^n + C \int_M f_\varphi^2 \omega_\varphi^n \\ &= \frac{1}{2} \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n + C \int_M f_\varphi |\nabla f_\varphi|_\varphi^2 \omega_t^n + C \int_M f_\varphi^3 e^{\lambda \varphi} \omega_t^n. \end{aligned}$$

It follows by integrating over t and (2.4) that

$$\int_0^t \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n du \leq C.$$

To prove the first inequality, we observe

$$f = e^{\lambda \varphi} f_\varphi.$$

Then,

$$\begin{aligned} \int_M f |\nabla f|_\varphi^2 \omega_t^n &= \int_M |\nabla(f_\varphi e^{\lambda \varphi})|_\varphi^2 \omega_\varphi^n \\ &\leq 2 \int_M |\nabla f_\varphi|^2 e^{2\lambda \varphi} \omega_\varphi^n + 2\lambda^2 \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 e^{2\lambda \varphi} \omega_\varphi^n \\ &\leq C \left(\int_M |\nabla f_\varphi|_\varphi^2 \omega_\varphi^n + \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n \right). \end{aligned}$$

Thus the first inequality follows from the second one and (2.4). **Claim 1** is proved.

Claim 2: For any smooth non-negative cut-off function χ (fixed), we have,

$$(2.7) \quad \int_M f |\nabla \chi|_\varphi^2 \omega_t^n = \int_M |\nabla \chi|_\varphi^2 \omega_\varphi^n \leq C(|\nabla \chi|_{\omega, L^\infty}).$$

Proof of Claim 2:

$$\begin{aligned} \int_M f |\nabla \chi|_\varphi^2 \omega_t^n &= \int_M |\nabla \chi|_\varphi^2 \omega_\varphi^n \\ &\leq \int_M |\nabla \chi|_{\omega_t}^2 \operatorname{tr}_\varphi \omega_t \omega_\varphi^n \\ &\leq C \int_M \operatorname{tr}_\varphi (\omega_\varphi - \sqrt{-1} \partial \bar{\partial} \varphi) \omega_\varphi^n \\ &= C \int_M (n - \Delta_\varphi \varphi) \omega_\varphi^n \leq C, \end{aligned}$$

where the first inequality follows from an elementary inequality and the second inequality makes use of the positivity of ω_t .

In other words, any smooth cut-off function automatically has uniform $W^{1,2}$ norm with respect to any Kähler metric in any given Kähler class.

Claim 3: We have

$$(2.8) \quad \int_M f |\nabla\varphi|_\varphi^2 \omega_t^n = \int_M |\nabla\varphi|_\varphi^2 \omega_\varphi^n \leq C(|\varphi|_{L^\infty}).$$

Proof of Claim 3: Let c be the same as in the proof of **Claim 1**.

$$\begin{aligned} \int_M |\nabla\varphi|_\varphi^2 \omega_\varphi^n &= -\int_M (\varphi - c)\Delta_\varphi\varphi \omega_\varphi^n \\ &= \int_M (\varphi - c)\text{tr}_\varphi\omega_t \cdot \omega_\varphi^n - n \int_M (\varphi - c) \omega_\varphi^n \\ &\leq C. \end{aligned}$$

In the last inequality, we have used the fact that φ is uniformly bounded.

Claim 4: For any positive χ , the following inequality holds

$$(2.9) \quad \int_0^t \int_M \chi f^2 \text{tr}_\varphi\omega_t \omega_t^n \, d\mathbf{u} \leq C_1 t + C_2 \sqrt{t},$$

where both constants depend only on $|\chi|_{L^\infty}$ and $|\nabla\chi|_{\omega, L^\infty}$.

Proof of Claim 4:

$$\begin{aligned} &\int_M \chi f^2 \text{tr}_\varphi \omega_t \omega_t^n \\ &= \int_M \chi f \omega_t \wedge \omega_\varphi^{n-1} \\ &= \int_M \chi f \omega_\varphi^n - \int_M \chi f \sqrt{-1}\partial\bar{\partial}\varphi \wedge \omega_\varphi^{n-1} \\ &\leq C + \int_M \sqrt{-1}\chi \partial f \wedge \bar{\partial}\varphi \wedge \omega_\varphi^{n-1} + \int_M \sqrt{-1}f \partial\chi \wedge \bar{\partial}\varphi \wedge \omega_\varphi^{n-1} \\ &\leq C \left(1 + \left(\int_M |\nabla f|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} \left(\int_M |\nabla\varphi|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} + \left(\int_M |\nabla\chi|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} \cdot \left(\int_M f^2 |\nabla\varphi|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} \right) \\ &\leq C \left(1 + \left(\int_M |\nabla f|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} + \left(\int_M f^2 |\nabla\varphi|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} \right). \end{aligned}$$

We have used **Claims 2** and **Claim 3** in deriving the last inequality.

Then **Claim 4** follows from integrating the above inequality from 0 to t and using **Claim 1** and the Schwartz inequality.

A straightforward computation shows

$$\begin{aligned} \frac{\partial \log f}{\partial t} &= \text{tr}_\varphi \left(\frac{\partial}{\partial t} \omega_t \right) + \Delta_\varphi \varphi' - \text{tr}_{\omega_t} \frac{\partial}{\partial t} \omega_t \\ &= \Delta_\varphi \log f + \text{tr}_\varphi \left(-\partial\bar{\partial}h_{\omega_t} + \frac{\partial}{\partial t} \omega_t \right) - \text{tr}_{\omega_t} \frac{\partial}{\partial t} \omega_t \\ &\leq \frac{\Delta_\varphi f}{f} - \frac{|\nabla f|_\varphi^2}{f^2} + C \text{tr}_\varphi \omega_t + C. \end{aligned}$$

Using this, we deduce

$$\begin{aligned} \frac{d}{dt} \int_M \chi f^2 \omega_t^n &= \int_M \chi \left(2f \frac{\partial f}{\partial t} + f^2 \text{tr}_{\omega_t} \frac{\partial}{\partial t} \omega_t \right) \omega_t^n \\ &\leq 2 \int_M \chi f \left(\Delta_\varphi f - \frac{|\nabla f|_\varphi^2}{f} + C f \text{tr}_\varphi \omega_t + C f \right) \omega_t^n \\ &\leq -2 \int_M (\nabla\chi, \nabla f)_\varphi \omega_\varphi^n + C \int_M \chi f^2 \omega_t^n + C \int_M \chi f^2 \text{tr}_\varphi \omega_t \omega_t^n \\ &\leq 2 \sqrt{\int_M |\nabla\chi|_\varphi^2 \omega_\varphi^n} \sqrt{\int_M |\nabla f|_\varphi^2 \omega_\varphi^n} + C \int_M \chi f^2 \omega_t^n + C \int_M \chi f^2 \text{tr}_\varphi \omega_t \omega_t^n \\ &\leq C \sqrt{\int_M |\nabla f|_\varphi^2 \omega_\varphi^n} + C \int_M \chi f^2 \omega_t^n + C \int_M \chi f^2 \text{tr}_\varphi \omega_t \omega_t^n. \end{aligned}$$

Note that

$$\int_M \chi \cdot f^2 \omega_t^n \leq C, \quad \forall t \in [0, T].$$

Integrating the above inequality from $t = 0$ to $t = t_i$ and using **Claim 4**, we have

$$\int_M \chi f^2 \omega_t^n|_{t_i} \leq \int_M \chi f^2 \omega_t^n|_0 + C(t_i + \sqrt{t_i}).$$

In deriving the above inequality, we used the fact that φ_0 is smooth. For general φ_0 as given, applying the above to $\varphi(s, t_i)$ for any $s > 0$ and then taking the limit as s tends to 0, we get for any $\varphi(t_i)$,

$$\int_M \chi \left(\frac{\omega_{\varphi(t_i)}^n}{\omega_{t_i}^n} \right)^2 \omega_{t_i}^n \leq \int_M \chi \left(\frac{\omega_{\varphi_0}^n}{\omega^n} \right)^2 \omega^n + C(t_i + \sqrt{t_i}).$$

Here one can use Fatou's Lemma and the strong convergence for the measure at initial time.

On the other hand, using the assumption that $\varphi(t_i)$ converges to $\tilde{\varphi}_0$, we can show that $\frac{\omega_{\varphi(t_i)}^n}{\omega_{t_i}^n}$ converges weakly to $\frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n}$ in $L^2(M, \omega)$. Then by taking t_i to 0, we can justify the convergence of integration in a similar way and have

$$(2.10) \quad \int_M \chi \left(\frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n} \right)^2 \omega^n \leq \int_M \chi \left(\frac{\omega_{\varphi_0}^n}{\omega^n} \right)^2 \omega^n.$$

Since this holds for any non-negative smooth cut-off function χ , we have

$$0 \leq \frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n} \leq \frac{\omega_{\varphi_0}^n}{\omega^n}$$

a. e. in M . However,

$$\int_M \frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n} \omega^n = \int_M \frac{\omega_{\varphi_0}^n}{\omega^n} \omega^n = \text{vol}(M)!$$

It follows

$$(2.11) \quad \omega_{\tilde{\varphi}_0}^n \equiv \omega_{\varphi_0}^n$$

in $L^2(M, \omega)$. The uniqueness of Monge-Ampère equation for C^0 solution by Kolodziej as in [26] implies that $\tilde{\varphi}_0 = \varphi_0$. Since $\{t_i\}$ is any sequence going to 0, we have proved that $\varphi(t)$ converges to φ_0 as t tends to 0.

Furthermore, we have

$$\int_M \chi \left(\frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n} \right)^2 \omega^n = \lim_{i \rightarrow \infty} \int_M \chi \left(\frac{\omega_{\varphi_i}^n}{\omega^n} \right)^2 \omega^n.$$

Together with weak convergence, we conclude $\frac{\omega_{\varphi_i}^n}{\omega^n}$ converges strongly to $\frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n}$ in $L^2(M, \omega)$. Hence Lemma 2.5 is proved. □

So far, we have shown that with assumption on the volume form, the solution of the weak Kähler-Ricci flow really goes (back) to the initial data as $t \rightarrow 0$.

3. HIGHER ORDER ESTIMATES

In this section, we prove the regularity of the weak Ricci flow under appropriate assumptions on the initial Kähler potential φ_0 .

3.1. The $C^{1,1}$ case. In this subsection, we want to prove the following Laplacian estimates.

Proposition 3.1. [9] If $\varphi_0 \in Cl_{L^\infty}P(M, \omega)$ is $C^{1,1}$ -bounded, then

$$0 \leq \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi(t) \leq C, \quad \forall t \in [0, T].$$

In other words, $\varphi(t)$ is uniformly $C^{1,1}$ -bounded. Moreover, $\frac{\omega_{\varphi^n(t)}}{\omega^n}$ converges to $\frac{\omega_{\varphi_0^n}}{\omega^n}$ strongly in L^2 space as $t \rightarrow 0$.

Proof. The $C^{1,1}$ assumption on φ_0 gives a L^∞ (upper) bound for the volume form. In sight of Lemma 2.5, it suffices to prove the Laplacian estimate.

Let $\varphi_0(s)$ be the smooth approximations of φ_0 and $\varphi(s, t)$ be the associated solutions as before. Since φ_0 is in $C^{1,1}$, we can arrange $\varphi_0(s)$ with uniformly bounded $C^{1,1}$ -norms. Thus, we only need to bound $\varphi(s, t)$ uniformly in $C^{1,1}$. For simplicity, φ stands for $\varphi(s, \cdot)$ below. By Lemma 2.2, it's uniformly bounded.

We first derive a pointwise uniform upper bound on the volume form for $t \in [0, T]$ as follows.

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial t} \right) &\leq \Delta \left(\frac{\partial \varphi}{\partial t} \right) + C(1 + \text{tr}_\varphi \omega_t) \\ &\leq \Delta \left(\frac{\partial \varphi}{\partial t} - C\varphi \right) + C + C(\text{tr}_\varphi(\sqrt{-1}\partial\bar{\partial}\varphi) + \text{tr}_\varphi \omega_t) \\ &\leq \Delta \left(\frac{\partial \varphi}{\partial t} - C\varphi \right) + C + n \\ &\leq \Delta \left(\frac{\partial \varphi}{\partial t} - C\varphi \right) + C + C\varphi. \end{aligned}$$

It can be reformulated as

$$\frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial t} - C\varphi \right) \leq \Delta \left(\frac{\partial \varphi}{\partial t} - C\varphi \right) + C - C \left(\frac{\partial \varphi}{\partial t} - C\varphi \right).$$

Applying the Maximum Principle, one can easily derive a uniform upper bound for $\frac{\partial \varphi}{\partial t} - C\varphi$. Since φ is uniformly bounded, we get a uniform upper bound on $\frac{\partial \varphi}{\partial t}$, so the volume form is uniformly bounded from above.

By standard computation as in [32], we have

$$e^{C\varphi} \left(\Delta_{\omega_\varphi} - \frac{\partial}{\partial t} \right) (e^{-C\varphi} \text{tr}_{\omega_t}(\omega_\varphi)) \geq -C + \left(C \frac{\partial \varphi}{\partial t} - C \right) \text{tr}_{\omega_t}(\omega_\varphi) + C e^{-\frac{1}{n-1} \frac{\partial \varphi}{\partial t}} \text{tr}_{\omega_t}(\omega_\varphi)^{\frac{n}{n-1}}.$$

Suppose that the maximum of $e^{-C\varphi} \text{tr}_{\omega_t}(\omega_\varphi)$ is attained at some (p, t) in $M \times [0, T]$. At that point, assuming $t > 0$ without loss of generality, we have

$$0 \geq -C + \left(C \frac{\partial \varphi}{\partial t} - C \right) \text{tr}_{\omega_t}(\omega_\varphi) + C e^{-\frac{1}{n-1} \frac{\partial \varphi}{\partial t}} \text{tr}_{\omega_t}(\omega_\varphi)^{\frac{n}{n-1}}.$$

Using the upper bound for $\frac{\partial \varphi}{\partial t}$, we have

$$\begin{aligned} \text{tr}_{\omega_t}(\omega_\varphi)^{\frac{n}{n-1}} &\leq C e^{\frac{1}{n-1} \frac{\partial \varphi}{\partial t}} + C e^{\frac{1}{n-1} \frac{\partial \varphi}{\partial t}} \text{tr}_{\omega_t}(\omega_\varphi) - C \frac{\partial \varphi}{\partial t} e^{\frac{1}{n-1} \frac{\partial \varphi}{\partial t}} \text{tr}_{\omega_t}(\omega_\varphi) \\ &\leq C + C \text{tr}_{\omega_t}(\omega_\varphi). \end{aligned}$$

This implies a uniform upper bound of $\text{tr}_{\omega_t}(\omega_\varphi)$ at (p, t) . Since φ is uniformly bounded, it follows that this trace is uniformly bounded. The proposition is thus proved. \square

3.2. The L^∞ case. We are going to finish proving Theorems 1.1 and 1.3 in this subsection. The estimates will be local for $t > 0$ under the weaker assumption that the initial Kähler potential is merely bounded.

Let's point out that in this case, one can still define weak flow using smooth decreasing approximation of the initial bounded Kähler potential². These approximation flows have potentials decreasing to the weak flow pointwisely. Moreover, for two choices of such approximations, it's not hard to see the limits would be the same because one-sided relation at the initial time will be preserved along the flow. All these can be justified by applying Maximum Principle as in Lemma 2.1. Thus we have the uniqueness of the weak flow. This also makes sure what we are discussed below is the same weak flow as before when there is more regularity assumption.

Recall that ω_t is a smooth family of Kähler metrics with $[\omega_t] = [\omega] - tc_1(M)$ and $\omega_\varphi = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$. Moreover, the Kähler-Ricci flow is reduced to the following equation for φ :

$$\frac{\partial\varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_t^n} - h_{\omega_t}, \quad \varphi(0, \cdot) = \varphi_0.$$

$\varphi(t)$ is obtained by taking the limit of $\varphi(s, t)$ as s tends to 0. In order to prove the theorem, we only need to get uniform estimates for higher order derivatives of $\varphi(s, t)$ for all $s > 0$ small. As before, for simplicity, we can simply assume that φ_0 is smooth (as one of the $\varphi(s, 0)$).

Choose any interval $[t_1, t_0]$ with $T > t_1 > t_0 > 0$. By Lemmas 2.2 and 2.3, we have the uniform bound for φ and $\frac{\partial\varphi}{\partial t}$ for $t \in [t_0, t_1]$.

Now we derive the Laplacian and higher order derivative estimates by the standard methods (see [32], also see [6] or [34]). The estimates may depend on t_0 .

As in the last subsection, we have

$$e^{C\varphi}(\Delta_{\omega_\varphi} - \frac{\partial}{\partial t})(e^{-C\varphi}\text{tr}_{\omega_t}(\omega_\varphi)) \geq -C - C\text{tr}_{\omega_t}(\omega_\varphi) + C\text{tr}_{\omega_t}(\omega_\varphi)^{\frac{n}{n-1}},$$

where the bound on $\frac{\partial\varphi}{\partial t}$ for $t \in [t_0, t_1]$ has been used. The bound for $e^{-C\varphi}\text{tr}_{\omega_t}(\omega_\varphi)$ at $t - t_0 = 0$ is not available. So we consider the following quantity instead

$$(t - t_0)^{n-1}e^{-C\varphi}\text{tr}_{\omega_t}(\omega_\varphi).$$

Straightforward computations then gives

$$\begin{aligned} & e^{C\varphi}(\Delta_{\omega_\varphi} - \frac{\partial}{\partial t})((t - t_0)^{n-1}e^{-C\varphi}\text{tr}_{\omega_t}(\omega_\varphi)) \\ & \geq (t - t_0)^{n-1}(-C - C\text{tr}_{\omega_t}(\omega_\varphi) + C\text{tr}_{\omega_t}(\omega_\varphi)^{\frac{n}{n-1}}) - (n-1)(t - t_0)^{n-2}\text{tr}_{\omega_t}(\omega_\varphi). \end{aligned}$$

At the maximum value point (p, \tilde{t}) of $(t - t_0)^{n-1}e^{-C\varphi}\text{tr}_{\omega_t}(\omega_\varphi)$ in $M \times [t_0, t_1]$, where clearly $\tilde{t} > t_0$,

$$\begin{aligned} 0 & \geq (\tilde{t} - t_0)^{n-1}(-C - C\text{tr}_{\omega_{\tilde{t}}}(\omega_\varphi) + C\text{tr}_{\omega_{\tilde{t}}}(\omega_\varphi)^{\frac{n}{n-1}}) - (n-1)(\tilde{t} - t_0)^{n-2}\text{tr}_{\omega_{\tilde{t}}}(\omega_\varphi) \\ & = -C(\tilde{t} - t_0)^{n-1} - (C\tilde{t} + C)(\tilde{t} - t_0)^{n-2}\text{tr}_{\omega_{\tilde{t}}}(\omega_\varphi) + C(\tilde{t} - t_0)^{n-1}\text{tr}_{\omega_{\tilde{t}}}(\omega_\varphi)^{\frac{n}{n-1}}. \end{aligned}$$

Multiply $\tilde{t} - t_0$ on both sides and reformulate the above to be

$$0 \geq -C(\tilde{t} - t_0)^n - (C\tilde{t} + C)(\tilde{t} - t_0)^{n-1}\text{tr}_{\omega_{\tilde{t}}}(\omega_\varphi) + C((\tilde{t} - t_0)^{n-1}\text{tr}_{\omega_{\tilde{t}}}(\omega_\varphi))^{\frac{n}{n-1}}.$$

²This is provided by the result in [3] as a simple version of the more general result in [16] for this case.

So we get

$$(\tilde{t} - t_0)^{n-1} \text{tr}_{\omega_t}(\omega_\varphi) \leq C.$$

Thus at (p, \tilde{t}) , using the bound for φ ,

$$(\tilde{t} - t_0)^{n-1} e^{-C\varphi} \text{tr}_{\omega_t}(\omega_\varphi) \leq C.$$

It follows that for all $t \in [t_0, t_1]$,

$$(t - t_0)^{n-1} e^{-C\varphi} \text{tr}_{\omega_t}(\omega_\varphi) \leq C, \text{ on } M$$

which implies

$$\text{tr}_{\omega_t}(\omega_\varphi) \leq \frac{C}{(t - t_0)^{n-1}}.$$

This together with the volume lower bound gives the uniform metric bound locally away from $t = 0$.

Next, we can get the third order estimate in a similar fashion. We have a similar inequality as in [6]. Of course, in order to do this, we still translate the time to guarantee the uniform metric bound. Let's say the time is translated so that the original time $t = t_0 > 0$ is now the new initial time $t = 0$. The inequality is as follows, for $S = g_\varphi^{i\bar{j}} g_\varphi^{k\bar{l}} g_\varphi^{\lambda\bar{\eta}} \varphi_{i\bar{l}\lambda} \varphi_{\bar{j}k\bar{\eta}}$,

$$(\Delta_\varphi - \frac{\partial}{\partial t})S \geq -C \cdot S - C.$$

As in [32], we also have

$$(\Delta_\varphi - \frac{\partial}{\partial t})\Delta_{\omega_t}\varphi \geq C \cdot S - C.$$

From the first one, as in the Laplacian estimate, we consider tS instead,

$$(\Delta_\varphi - \frac{\partial}{\partial t})(tS) \geq -Ct \cdot S - Ct - S.$$

Choosing $A > 0$ large enough, one has, for $t \in [0, T]$,

$$(\Delta_\varphi - \frac{\partial}{\partial t})(tS + A\Delta_{\omega_t}\varphi) \geq C \cdot S - C.$$

The function $tS + A\Delta_{\omega_t}\varphi$ has uniformly bounded at the (new) initial time, $t = 0$. By Maximum Principle, if its maximum value is achieved at some point in M and $t > 0$, then S is bounded there. Consequently, the whole function $tS + A\Delta_{\omega_t}\varphi$ is also bounded there. So we finally conclude

$$tS + A\Delta_{\omega_t}\varphi \leq C,$$

which implies

$$S \leq \frac{C}{t}.$$

This provides a local $C^{2,\alpha}$ bound for φ . After this, higher order estimates follow from the standard argument.

Therefore, by Ascoli-Arzelà Theorem, we have proved that the weak flow considered so far is actually smooth in $(0, T]$. Theorem 1.1 follows from this and the second order estimate in Subsection 3.1. Theorem 1.3 follows from this and Lemma 2.5.

4. $C^{1,1}$ K-ENERGY MINIMIZER

In an earlier paper [9] where the present work is initiated, the first two named authors proved that

Proposition 4.1. In any Kähler class, the volume form of any $C^{1,1}$ K-energy minimizer belongs to $H^{1,2}(M, \omega)$.

By Lemma 2.5 and Proposition 3.1, we can prove a stronger result.

Definition 4.2. The $C^{1,1}$ minimizer of the K-energy functional in any canonical Kähler class with positive first Chern class is necessarily smooth.

Proof. Let ϕ_0 be a $C^{1,1}$ Kähler potential which minimizes the K-energy functional in the canonical Kähler class. According to the results proved in the preceding sections, there is a unique smooth Kähler-Ricci flow $\varphi(t)$ ($t > 0$) such that

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_{\varphi}^n}{\omega^n} + \varphi - h_{\omega}.$$

Moreover, $\lim_{t \rightarrow 0} \frac{\omega_{\varphi}^n}{\omega^n} = \frac{\omega_{\varphi_0}^n}{\omega^n}$ strongly in $L^2(M, g_0)$ and the Kähler potential $\varphi(t)$ converges to φ_0 strongly in $C^{1,\alpha}(M)$ for any $\alpha \in (0, 1)$. Since the K-energy is decreasing along the Kähler-Ricci flow, we have

$$\limsup_{t \rightarrow 0^+} \mathbf{E}(\varphi(t)) \leq \mathbf{E}(\varphi_0) = \inf_{\phi \in \mathcal{H}} \mathbf{E}(\phi).$$

Since the K-energy is non-increasing for $\varphi(t)$ ($t > 0$), then for any $t > 0$, we have

$$\inf_{\phi \in \mathcal{H}} \mathbf{E}(\phi) \leq \mathbf{E}(\varphi(t)) \leq \inf_{\phi \in \mathcal{H}} \mathbf{E}(\phi).$$

In other words,

$$\mathbf{E}(\varphi(t)) = \inf_{\phi \in \mathcal{H}} \mathbf{E}(\phi), \quad t > 0.$$

Since $\omega_{\varphi(t)}$ is a smooth Kähler metric, this means that the scalar curvature of $\omega_{\varphi(t)}$ ($t > 0$) must be constant. Consequently, $\omega_{\varphi(t)}$ is a Kähler-Einstein metric for all $t > 0$. This in turns implies that $\frac{\partial \varphi}{\partial t}$ is a function of t only. Note that φ_0 is the strong $C^{1,\alpha}$ limit of $\varphi(t)$ as $t \rightarrow 0$. Therefore, $\varphi_0 - \varphi(t)$ is a constant which depends only on t . In other words, ω_{φ_0} is also a Kähler-Einstein metric and the theorem is then proved. \square

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