Feigin–Frenkel center for classical types

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where $h^\vee$ is the dual Coxeter number.

For the classical types,

$$h^\vee = \begin{cases} n & \text{for} \quad \mathfrak{g} = \mathfrak{sl}_n, \\ N-2 & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N, \\ n+1 & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

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with the commutation relations

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\, K,$$

where $X[r] = X\,t^r$ for any $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

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where $X[r] = X\,t^r$ for any $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

The vacuum module at the critical level $V(\mathfrak{g})$ over $\widehat{\mathfrak{g}}$ is the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K + h^{\vee}$.

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The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is commutative.

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The subspace $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $V(\mathfrak{g})$ is T-invariant.

Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a Segal–Sugawara vector.

Theorem (Feigin–Frenkel, 1992).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}(t^{-1}\mathfrak{g}[t^{-1}])$

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$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[T^k S_l \mid l = 1, \dots, n, \ \ k \geqslant 0],$$

where $n = \operatorname{rank} \mathfrak{g}$ and the symbols $\overline{S}_1, \dots, \overline{S}_n$ coincide with the images of certain algebraically independent generators of the algebra of invariants $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ under the embedding $\mathrm{S}(\mathfrak{g}) \hookrightarrow \mathrm{S}(t^{-1}\mathfrak{g}[t^{-1}])$ defined by $X \mapsto X[-1]$.

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We call $S_1,\dots,S_n$ a complete set of Segal–Sugawara vectors.

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Let $P = P(Y_1, \ldots, Y_l)$ be a $\mathfrak{g}$-invariant in $\mathrm{S}(\mathfrak{g})$.

Set $Y_i(z) = \sum_{r<0} Y_i[r]\, z^{-r-1}$ and write

$$P\big(Y_1(z), \ldots, Y_l(z)\big) = \sum_{r<0} P_{(r)}\, z^{-r-1}.$$

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Then each $P_{(r)}$ is a $\mathfrak{g}[t]$-invariant in $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

Moreover, $k!\, P_{(-k-1)} = T^k P\big(Y_1[-1], \ldots, Y_l[-1]\big)$ for $k \geqslant 0$.

Theorem (Beilinson–Drinfeld, 1997). If $P_1, \dots, P_n$ are algebraically independent generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then the elements $P_{1,(r)}, \dots, P_{n,(r)}$ with $r < 0$ are algebraically independent generators of $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

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Earlier work: R. Goodman and N. Wallach, 1989, type A; T. Hayashi, 1988, types A, B, C; V. Kac and D. Kazhdan, 1979.

Explicit formulas for Segal–Sugawara vectors

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We will need the extended Lie algebra $\widehat{\mathfrak{g}} \oplus \mathbb{C}\tau$, where for the element $\tau = -\partial_t$ we have the relations

$$\big[\tau, X[r]\big] = -r\, X[r-1], \qquad \big[\tau, K\big] = 0.$$

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Consider the $n \times n$ matrix $\tau + E[-1]$ given by

$$\tau + E[-1] = \begin{bmatrix} \tau + E_{11}[-1] & E_{12}[-1] & \dots & E_{1n}[-1] \\ E_{21}[-1] & \tau + E_{22}[-1] & \dots & E_{2n}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1}[-1] & E_{n2}[-1] & \dots & \tau + E_{nn}[-1] \end{bmatrix}.$$

Theorem (Chervov–Talalaev, 2006; also Chervov–M., 2009).

The coefficients $S_1, \dots, S_n$ of the polynomial

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^n + S_1\,\tau^{n-1} + \cdots + S_{n-1}\,\tau + S_n$$

form a complete set of Segal–Sugawara vectors in $V(\mathfrak{gl}_n)$.

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Example. For $n=2$

$$\begin{aligned}\mathrm{cdet}\big(\tau+E[-1]\big)&=\big(\tau+E_{11}[-1]\big)\big(\tau+E_{22}[-1]\big)-E_{21}[-1]E_{12}[-1]\\&=\tau^2+S_1\,\tau+S_2\end{aligned}$$

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$$= \tau^2 + S_1\,\tau + S_2$$

with

$$S_1 = E_{11}[-1] + E_{22}[-1],$$

$$S_2 = E_{11}[-1]\,E_{22}[-1] - E_{21}[-1]\,E_{12}[-1] + E_{22}[-2].$$

Corollary. For any $k \geqslant 0$ all coefficients $P_{k\,l}$ in the expansion

$$\operatorname{tr}(\tau + E[-1])^{k} = P_{k\,0}\,\tau^{k} + P_{k\,1}\,\tau^{k-1} + \cdots + P_{k\,k}$$

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Remark. These results generalize to the Lie superalgebra $\mathfrak{gl}_{m|n}$. The column-determinant is replaced by a noncommutative Berezinian (M.–Ragoucy, 2009).

Types B, C and D

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Denote by F the $N \times N$ matrix whose (i, j) entry is F_{ij}. Regard F as the element

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Introduce elements of $\operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N \cong \operatorname{End} (\mathbb{C}^N \otimes \mathbb{C}^N)$ by

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}, \qquad Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij}.$$

The defining relations of the algebra $\mathrm{U}(\mathfrak{o}_N)$ have the form

$$F_1\,F_2 - F_2\,F_1 = (P - Q)\,F_2 - F_2\,(P - Q)$$

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together with the relation $F + F^{\,t} = 0$, where both sides are regarded as elements of the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{o}_N)$ and

$$F_1 = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes F_{ij}, \qquad F_2 = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes F_{ij}.$$

In the affine Kac–Moody algebra $\widehat{\mathfrak{o}}_N = \mathfrak{o}_N[t, t^{-1}] \oplus \mathbb{C}K$ set

$F_{ij}[r] = F_{ij}\, t^r$ for any $r \in \mathbb{Z}$.

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The defining relations of the algebra $\mathrm{U}(\widehat{\mathfrak{o}}_N)$ can be written as

$$F[r]_1\, F[s]_2 - F[s]_2\, F[r]_1 = (P - Q)\, F[r+s]_2 - F[r+s]_2\, (P - Q)$$

$$+ \, r\, \delta_{r,-s}\, (P - Q)\, K.$$

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Note that

$$F_{ij}[-1] + F_{ji}[-1] = 0.$$

For each $a \in \{1, \ldots, m\}$ define the element Φ_a of the algebra

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by

$$\Phi_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes \Phi_{ij},$$

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where $\Phi_{ij} = \delta_{ij}\tau + F_{ij}[-1]$.

The trace map $\mathrm{tr} : \mathrm{End}\,\mathbb{C}^N \to \mathbb{C}$ is defined by $\mathrm{tr} : e_{ij} \mapsto \delta_{ij}$.

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by

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1} \Big),$$

the product is taken in the lexicographic order on the pairs (a, b), and P_{ab} and Q_{ab} act as the respective operators P and Q in the a-th and b-th copies of $\mathbb{C}^N$ and as the identity operators in all the remaining copies.

Properties: for $1 \leqslant a < b \leqslant m$ we have

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Equivalent formula:

$$S^{(m)} = \frac{1}{m!} \prod_{1\leqslant a<b\leqslant m} \Big(1 - \frac{Q_{ab}}{N+a+b-3}\Big) \prod_{1\leqslant a<b\leqslant m} \Big(1 + \frac{P_{ab}}{b-a}\Big).$$

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Remark. $S^{(m)}$ is the idempotent associated with the trivial representation of the Brauer algebra $\mathcal{B}_m(N)$. In particular, $(S^{(m)})^2 = S^{(m)}$.

In a reduced form,

$$S^{(m)} = H^{(m)} \sum_{r=0}^{\lfloor m/2 \rfloor} \frac{(-1)^r}{2^r\, r!} \binom{N/2+m-2}{r}^{-1} \sum_{a_i<b_i} Q_{a_1 b_1} \dots Q_{a_r b_r},$$

where $H^{(m)}$ is the symmetrizer in the group algebra $\mathbb{C}[\mathfrak{S}_m]$.

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In terms of the Jucys–Murphy elements:

$$S^{(m)} = \prod_{b=2}^{m} \frac{1}{b(N+2b-4)} \Big(1 + \sum_{a=1}^{b-1} (P_{ab} - Q_{ab})\Big) \times \Big(N + b - 3 + \sum_{a=1}^{b-1} (P_{ab} - Q_{ab})\Big).$$

Theorem. The elements $\phi_{m\,a} \in \mathrm{U}(t^{-1}\mathfrak{o}_N[t^{-1}])$ defined by

$$\operatorname{tr} S^{(m)} \Phi_1 \dots \Phi_m = \phi_{m\,0}\, \tau^m + \phi_{m\,1}\, \tau^{m-1} + \dots + \phi_{m\,m}$$

are Segal–Sugawara vectors for $\mathfrak{o}_N$.

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Moreover, $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n\,2n}$ is a complete set of

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Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$,

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Segal–Sugawara vectors for $\mathfrak{o}_{2n}$, where $\phi'_n = \operatorname{Pf} F[-1]$ is the

Pfaffian of the skew-symmetric matrix $F[-1]$.

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In the case of $\mathfrak{o}_{2n}$ the Pfaffian $\operatorname{Pf} F[-1]$ is

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Example. For $\mathfrak{o}_4$ we have

$$\mathrm{Pf}\, F[-1] = F_{1\,2}[-1]\, F_{3\,4}[-1] - F_{1\,3}[-1]\, F_{2\,4}[-1] + F_{1\,4}[-1]\, F_{2\,3}[-1].$$

For the proof of the theorem we show that

$$F[0]_0 \operatorname{tr} S^{(m)} \Phi_1 \ldots \Phi_m = 0 \qquad \text{and} \qquad F[1]_0 \operatorname{tr} S^{(m)} \Phi_1 \ldots \Phi_m = 0$$

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in the module

$$\underbrace{\operatorname{End} \mathbb{C}^N \otimes \dots \otimes \operatorname{End} \mathbb{C}^N}_{m+1} \otimes V(\mathfrak{o}_N)[\tau]$$

with the copies of $\operatorname{End} \mathbb{C}^N$ labelled by $0, 1, \dots, m$.

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Then

$$\operatorname{tr} S^{(2k)} X_1 \dots X_{2k} = \frac{N + 4k - 2}{N + 2k - 2}\, h_k(x_1^2, \dots, x_n^2),$$

h_k is the complete symmetric polynomial.

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It is determined by

$$Y(X[-1], z) = \sum_{r \in \mathbb{Z}} X[r] z^{-r-1} =: X(z).$$

For any $r_i \geqslant 0$ we have

$$Y(X_1[-r_1 - 1] \dots X_m[-r_m - 1], z) = \frac{1}{r_1! \dots r_m!} : \partial_z^{r_1} X_1(z) \dots \partial_z^{r_m} X_m(z) :,$$

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Suppose that $S_1, \dots, S_n$ is a complete set of Segal–Sugawara vectors in $\mathfrak{z}(\widehat{\mathfrak{g}})$. Apply the state-field correspondence map:

$$Y(S_l, z) = \sum_{r \in \mathbb{Z}} S_{l,r}\, z^{-r-1}.$$

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Applications: Singular vectors in Verma modules and Weyl modules over $\widehat{\mathfrak{g}}$ (E. Frenkel and D. Gaitsgory, 2006, 2007).

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The S_p are the Sugawara operators

$$
S_p = \sum_{i,j=1}^{N} \Big(\sum_{r<0} F_{ij}[r]\, F_{ji}[p-r] + \sum_{r\geqslant 0} F_{ji}[p-r]\, F_{ij}[r] \Big)
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commuting with $\widehat{\mathfrak{o}}_N$.

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$$\partial_z + E(z) = \begin{bmatrix} \partial_z + E_{11}(z) & E_{12}(z) & \dots & E_{1n}(z) \\ E_{21}(z) & \partial_z + E_{22}(z) & \dots & E_{2n}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1}(z) & E_{n2}(z) & \dots & \partial_z + E_{nn}(z) \end{bmatrix}.$$

Expand the normally ordered column-determinant

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The coefficients $S_{l,r}$ of the $S_l(z)$ are Sugawara operators for $\widehat{\mathfrak{gl}}_n$.

Using the vacuum axiom

$$: \mathrm{cdet}\big(\partial_z + E(z)\big) : 1 = \mathrm{cdet}\big(\partial_z + E(z)_+\big),$$

we get explicit generators of $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ and hence, generators of the commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{gl}_n[t^{-1}])$.

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Apply the state-field correspondence map

$$Y : \mathrm{tr}\, S^{(m)} \Phi_1 \dots \Phi_m \quad \mapsto \quad : \mathrm{tr}\, S^{(m)} \big(\partial_z + F_1(z)\big) \dots \big(\partial_z + F_m(z)\big) :$$

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$$\partial_z + F(z) = \begin{bmatrix} \partial_z & F_{12}(z) & \dots & F_{1N}(z) \\ F_{21}(z) & \partial_z & \dots & F_{2N}(z) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N1}(z) & F_{N2}(z) & \dots & \partial_z \end{bmatrix}.$$

Expand into a polynomial in ∂_z:

$$: \operatorname{tr} S^{(m)} \big(\partial_z + F_1(z)\big) \dots \big(\partial_z + F_m(z)\big) :$$

$$= f_{m\,0}(z)\,\partial_z^m + f_{m\,1}(z)\,\partial_z^{m-1} + \cdots + f_{m\,m}(z).$$

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All coefficients of the $f_{m\,a}(z)$ are Sugawara operators for $\widehat{\mathfrak{o}}_N$.

Applying them to the vacuum vector, we get explicit generators of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{o}}_N)$, and hence, generators of the commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{o}_N[t^{-1}])$.

Introduce the matrix $F(z)_- = [F_{ij}(z)_-]$ and set $L(z) = \partial_z - F(z)_-$,

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Corollary. The coefficients of all series $l_{m\,a}(z)$ with $m = 1, 2, \dots$

defined by the decompositions

$$\operatorname{tr} S^{(m)} L_1(z) \dots L_m(z) = l_{m\,0}(z)\, \partial_z^m + l_{m\,1}(z)\, \partial_z^{m-1} + \dots + l_{m\,m}(z),$$

generate a commutative subalgebra of $\mathrm{U}(\mathfrak{o}_N[t])$.

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Taking the coefficients of the powers of z we get Sugawara operators S_r, $r \in \mathbb{Z}$, of the form

$$S_r = \sum_{r_1+\dots+r_n=r} \sum_{\sigma} \mathrm{sgn}\, \sigma \cdot F_{\sigma(1)\,\sigma(2)}[r_1] \dots F_{\sigma(2n-1)\,\sigma(2n)}[r_n].$$

Harmonic polynomials

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These are polynomials annihilated by the Laplace operator $\partial_1^2 + \cdots + \partial_N^2$.

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and p satisfies $e\, p = p f = 0$.

Corollary. The Segal–Sugawara vectors $\phi_{m\,k}$ can be found from the expansion

$$\operatorname{tr} p\, \Phi^{(m)}|_{\mathcal{H}_N^m} = \phi_{m\,0}\, \tau^m + \phi_{m\,1}\, \tau^{m-1} + \dots + \phi_{m\,m}$$

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$$\Phi^{(m)} : x_{j_1} \dots x_{j_m} \mapsto \sum_{i_1 \leqslant \dots \leqslant i_m} x_{i_1} \dots x_{i_m} \otimes \Phi^{i_1,\dots,i_m}_{j_1,\dots,j_m}$$

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where

$$\Phi^{i_1,\dots,i_m}_{j_1,\dots,j_m} = \frac{1}{\alpha_1! \dots \alpha_N!\, m!} \sum_{\sigma,\pi \in \mathfrak{S}_m} \Phi_{i_{\sigma(1)} j_{\pi(1)}} \dots \Phi_{i_{\sigma(m)} j_{\pi(m)}}$$

and α_i is the multiplicity of i in the multiset $\{i_1, \dots, i_m\}$.