Yangians and their representations

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Preliminary course outline

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Lecture 1. History and background, R-matrix definition, basic structural results in type A.

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Lecture 2. Quantum determinant and quantum minor identities, center of the Yangian, Drinfeld presentation.

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Lecture 4. Yangians in arbitrary types, their representations and applications.

References

References

V. Chari and A. Pressley, A guide to quantum groups (1994),

Chapter 12.

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A. Molev, Yangians and classical Lie algebras (2007),

Chapters 1 & 3.

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History and background: Yang–Baxter equation

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Cairns, Australia, 2010

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In this case we say that R is an R-matrix.

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$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u),$$

or multiplicative,

$$R_{12}(u)\, R_{13}(uv)\, R_{23}(v) = R_{23}(v)\, R_{13}(uv)\, R_{12}(u).$$

In the theory of integrable lattice models, one considers the monodromy matrices

$$T_0(u) = R_{01}(u + c_1) \dots R_{0n}(u + c_n), \qquad c_i \in \mathbb{C},$$

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$$T_0(u) = R_{01}(u + c_1) \dots R_{0n}(u + c_n), \qquad c_i \in \mathbb{C},$$

in the vector space $V \otimes V^{\otimes n}$.

By taking trace over the 0-th copy of V, we get the transfer matrix $\operatorname{tr} T(u)$.

The Yang–Baxter equation implies

$$R_{00'}(u-v)\,T_0(u)\,T_{0'}(v) = T_{0'}(v)\,T_0(u)\,R_{00'}(u-v)$$

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By taking trace over the copies of V labelled by 0 and $0'$, we get

$$\big[\operatorname{tr} T(u), \operatorname{tr} T(v)\big] = 0.$$

The transfer matrices $\operatorname{tr} T(u)$ thus provide a commuting family of operators in $V^{\otimes n}$. One would like to find their eigenvalues and eigenvectors.

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see book by [R. Baxter 1982].

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see book by [R. Baxter 1982].

Another interpretation of the Yang–Baxter equation —
factorization property of scattering matrices;
originated in [C. N. Yang 1967].

Work of the Leningrad (St. Petersburg) school of L. Faddeev:

emergence of new algebraic structures.

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The relation

$$R_{12}(u-v)\,T_1(u)\,T_2(v)=T_2(v)\,T_1(u)\,R_{12}(u-v),$$

which is now known as the RTT-relation, defines an abstract algebra whose generators are matrix elements of $T(u)$.

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The function

$$R(u) = u\,1 + P$$

is a solution of the Yang–Baxter equation, known as the Yang R-matrix.

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$$T(u) = \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix}.$$

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$$R_{12}(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R_{12}(u-v),$$

we get the relations in terms of the matrix entries. In particular,

$$\big[A(u), A(v)\big] = 0,$$

$$(u-v)\, \big[A(u), B(v)\big] = A(v)\, B(u) - A(u)\, B(v),$$

$$(u-v)\, \big[B(u), C(v)\big] = D(v)\, A(u) - D(u)\, A(v), \quad \text{etc.}$$

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$$A(u)\,D(u+1) - C(u)\,B(u+1).$$

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This generalized the work of A. Izergin and V. Korepin 1981 for 2×2 matrices, where the quantum determinant is given by

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These elements lie in the center of the algebra.

Drinfeld's definition

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etc., so that the RTT relation defines an algebra with countably many generators.

This leads to the definition of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$ [V. Drinfeld 1985]. The name was given in honor of C. N. Yang. The Yangians $\mathrm{Y}(\mathfrak{a})$ were defined for all simple Lie algebras $\mathfrak{a}$.

Yangian for $\mathfrak{gl}_N$

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Definition.

The Yangian for $\mathfrak{gl}_N$ is the associative algebra over $\mathbb{C}$ with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where $i,j=1,\dots,N$, and the defining relations

$$\big[t_{ij}^{(r+1)},t_{kl}^{(s)}\big]-\big[t_{ij}^{(r)},t_{kl}^{(s+1)}\big]=t_{kj}^{(r)}t_{il}^{(s)}-t_{kj}^{(s)}t_{il}^{(r)},$$

where $r,s=0,1,\dots$ and $t_{ij}^{(0)}=\delta_{ij}$.

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where $r,s=0,1,\dots$ and $t_{ij}^{(0)}=\delta_{ij}$.

This algebra is denoted by $\mathrm{Y}(\mathfrak{gl}_N)$.

Introduce the formal generating series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]].$$

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The defining relations take the form

$$(u - v) \left[t_{ij}(u), t_{kl}(v) \right] = t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u) :$$

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Proposition. The defining relations can be written equivalently as

$$\big[t_{ij}^{(r)}, t_{kl}^{(s)}\big] = \sum_{a=1}^{\min\{r,s\}} \Big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)} \Big).$$

Proof. Write the relations in the form

$$\left[t_{ij}(u), t_{kl}(v)\right] = \frac{1}{u-v}\left(t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u)\right).$$

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Expand

$$\frac{1}{u-v} = \sum_{p=1}^{\infty} u^{-p}\,v^{p-1}.$$

Taking the coefficients of $u^{-r}v^{-s}$ on both sides gives

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{r} \Big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\Big).$$

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This agrees with the formula in the case $r \leqslant s$. If $r > s$ then

$$\sum_{a=s+1}^{r} \Big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\Big) = 0.$$

□

Introduce the permutation operator

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N,$$

where $e_{ij} \in \mathrm{End}\,\mathbb{C}^N$ are the standard matrix units.

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The rational function

$$R(u) = 1 - P\, u^{-1}$$

with values in $\operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N$ is called the Yang R-matrix.

Proposition.

The Yang R-matrix is a solution of the Yang–Baxter equation

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Proof. Multiplying both sides by $uv(u+v)$ we come to verify the identity

$$(u - P_{12})(u + v - P_{13})(v - P_{23}) = (v - P_{23})(u + v - P_{13})(u - P_{12}).$$

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It is immediate from the relations in the group algebra $\mathbb{C}[\mathfrak{S}_3]$.

For instance, for the constant term we have

$$P_{12}\, P_{13}\, P_{23} = P_{23}\, P_{13}\, P_{12}. \qquad \square$$

Matrix form of the defining relations

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Introduce the $N \times N$ matrix $T(u)$ whose ij-th entry is the series $t_{ij}(u)$:

$$T(u) = \begin{bmatrix} t_{11}(u) & t_{12}(u) & \dots & t_{1N}(u) \\ t_{21}(u) & t_{22}(u) & \dots & t_{2N}(u) \\ \dots & \dots & \dots & \dots \\ t_{N1}(u) & t_{N2}(u) & \dots & t_{NN}(u) \end{bmatrix}.$$

Note that for any algebra $\mathcal{A}$ we have the isomorphism

$$\mathrm{Mat}_N(\mathcal{A}) \cong \mathrm{End}\,\mathbb{C}^N \otimes \mathcal{A}.$$

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and introduce its elements by

$$T_1(u) = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes t_{ij}(u) \quad \text{and} \quad T_2(u) = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes t_{ij}(u).$$

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Proposition.

The defining relations of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ can be written in the form of RTT-relation

$$R(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R(u-v).$$

Proof. Recalling the definition of the Yang R-matrix, we can write the relation in the form

$$\big[T_1(u), T_2(v)\big] = \frac{1}{u-v}\,\big(P\,T_1(u)\,T_2(v) - T_2(v)\,T_1(u)\,P\big).$$

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to get

$$\big[t_{ij}(u), t_{kl}(v)\big] = \frac{1}{u-v}\,\big(t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u)\big). \qquad \square$$

Connection with $\mathrm{U}(\mathfrak{gl}_N)$

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Proposition. The assignment

$$\mathrm{ev} : t_{ij}^{(1)} \mapsto E_{ij} \quad \text{and} \quad t_{ij}^{(r)} \to 0 \quad \text{for} \quad r \geqslant 2,$$

defines an epimorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$.

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defines an epimorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$. Equivalently,

$$\mathrm{ev} : t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1}.$$

Proof. Introduce the matrix

$$E = \begin{bmatrix} E_{11} & E_{12} & \dots & E_{1N} \\ E_{21} & E_{22} & \dots & E_{2N} \\ \dots & \dots & \dots & \dots \\ E_{N1} & E_{N2} & \dots & E_{NN} \end{bmatrix}$$

with entries in $\mathrm{U}(\mathfrak{gl}_N)$.

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by setting

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij}.$$

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where

$$E_1 = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes E_{ij} \quad \text{and} \quad E_2 = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes E_{ij}.$$

Hence the map ev is written in the matrix form as

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We need to verify that

$$\begin{aligned}\big[1 + E_1\,u^{-1}, 1 + E_2\,v^{-1}\big] \\ &= \frac{1}{u-v}\Big(P\,(1 + E_1\,u^{-1})\,(1 + E_2\,v^{-1}) \\ &\qquad - (1 + E_2\,v^{-1})\,(1 + E_1\,u^{-1})\,P\Big).\end{aligned}$$

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Note the relations

$$PE_1 = E_2P \qquad \text{and} \qquad PE_2 = E_1P.$$

We come to checking that

$$\left[E_1\,,E_2\right] = \frac{1}{u-v}\Big((u+E_2)\,(v+E_1)\,P - (v+E_2)\,(u+E_1)\,P\Big).$$

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The map $\mathrm{ev} : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$ is known as

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Proposition. The assignment

$$\imath : E_{ij} \mapsto t_{ij}^{(1)}$$

defines an embedding $\mathrm{U}(\mathfrak{gl}_N) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_N)$.

Proof. Recall the defining relations,

$$\big[t_{ij}^{(r)}, t_{kl}^{(s)}\big] = \sum_{a=1}^{\min\{r,s\}} \Big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\Big),$$

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Note that the composition $\mathrm{ev} \circ \imath$ of the map $\imath : \mathrm{U}(\mathfrak{gl}_N) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_N)$ and the evaluation homomorphism $\mathrm{ev} : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$ is the identity map on $\mathrm{U}(\mathfrak{gl}_N)$.

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We thus may regard $\mathrm{U}(\mathfrak{gl}_N)$ as a subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$.

Symmetries of $\mathrm{Y}(\mathfrak{gl}_N)$

Let $f(u)$ be a formal power series in u^{-1} of the form

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]].$$

Let $c \in \mathbb{C}$ and let B be any invertible complex $N \times N$ matrix.

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Proposition. Each of the mappings

$$T(u) \mapsto f(u)\,T(u),$$

$$T(u) \mapsto T(u+c),$$

$$T(u) \mapsto B\,T(u)\,B^{-1}$$

defines an automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$.

Proof. We need to verify that each map preserves the defining relations and is invertible.

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It is clear that the maps $T(u) \mapsto f(u)\, T(u)$ and $T(u) \mapsto T(u+c)$ preserve the RTT relation

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because

$$A_1\,A_2 = A_2\,A_1 \qquad \text{and} \qquad P\,A_1\,A_2 = A_2\,A_1\,P.$$

This implies that the matrices $A\,T(u)$ and $T(u)\,A$ also satisfy the RTT relation:

$$R(u-v)\,A_1\,T_1(u)\,A_2\,T_2(v) = R(u-v)\,A_1\,A_2\,T_1(u)\,T_2(v)$$

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All three homomorphisms are obviously invertible. $\square$

Proposition. Each of the mappings

$$\sigma_N : T(u) \mapsto T(-u),$$

$$t : T(u) \mapsto T^t(u),$$

$$\mathrm{S} : T(u) \mapsto T^{-1}(u)$$

defines an anti-automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$.

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Proof. The images $t^\circ_{ij}(u)$ of the series $t_{ij}(u)$ under any anti-automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$ must satisfy the defining relations with the opposite multiplication:

$$(u-v)\,[t^\circ_{ij}(u), t^\circ_{kl}(v)] = t^\circ_{il}(u)t^\circ_{kj}(v) - t^\circ_{il}(v)t^\circ_{kj}(u).$$

These relations can be equivalently written in the matrix form:

$$R(u-v)\,T_2^{\circ}(v)\,T_1^{\circ}(u)=T_1^{\circ}(u)\,T_2^{\circ}(v)\,R(u-v),$$

where $T^{\circ}(u)$ is the $N\times N$ matrix whose ij-th entry is $t_{ij}^{\circ}(u)$.

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The relation

$$R(u-v)\,T_2(-v)\,T_1(-u) = T_1(-u)\,T_2(-v)\,R(u-v)$$

follows from the RTT relation by conjugating both sides by P and then replacing (u, v) by $(-v, -u)$.

Lemma. Let $\mathcal{A}$ be an associative algebra.

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Suppose that two elements

$$X = \sum_{i,j=1}^{N} e_{ij} \otimes X_{ij} \qquad \text{and} \qquad Y = \sum_{i,j=1}^{N} e_{ij} \otimes Y_{ij}$$

of the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathcal{A}$ satisfy the property

$$X_{ij}\, Y_{kl} = Y_{kl}\, X_{ij} \qquad \text{for all} \quad i, j, k, l.$$

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of the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathcal{A}$ satisfy the property

$$X_{ij}\, Y_{kl} = Y_{kl}\, X_{ij} \qquad \text{for all} \quad i, j, k, l.$$

Then

$$(XY)^t = Y^t X^t.$$

Proof. We have

$$Y^t X^t = \sum_{i,j,k,l=1}^{N} e_{lk} e_{ji} \otimes Y_{kl} X_{ij} = \sum_{i,j,l=1}^{N} e_{li} \otimes Y_{jl} X_{ij}.$$

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On the other hand,

$$XY = \sum_{i,j,k,l=1}^{N} e_{ij} e_{kl} \otimes X_{ij} Y_{kl} = \sum_{i,j,l=1}^{N} e_{il} \otimes Y_{jl} X_{ij},$$

so that the application of the transposition yields $Y^t X^t$. □

Apply the partial transposition operator t_1 to both sides of the RTT relation

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By the Lemma, we get

$$T_1^t(u)\,R^t(u-v)\,T_2(v) = T_2(v)\,R^t(u-v)\,T_1^t(u).$$

Since $R(u-v)$ is stable under the composition $t_2 \circ t_1$, Since $R(u-v)$ is stable under the composition $t_2 \circ t_1$, applying t_2 we get

$$T_1^t(u)\, T_2^t(v)\, R(u-v) = R(u-v)\, T_2^t(v)\, T_1^t(u),$$

showing that t is an anti-automorphism.

Finally, observe that the relation

$$R(u-v)\,T_2^{-1}(v)\,T_1^{-1}(u) = T_1^{-1}(u)\,T_2^{-1}(v)\,R(u-v)$$

is equivalent to the RTT relation so that S is an anti-homomorphism.

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Note now that the mappings σ_N and t are involutive and so these anti-homomorphisms are bijective.

Remark. The anti-homomorphism S is not involutive.

To complete the proof, take the composition of the anti-homomorphisms σ_N and S to get the homomorphism

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$$\omega_N\big(T(u)\big) \cdot T(-u) = 1$$

to get

$$\omega_N^2\big(T(u)\big) \cdot T^{-1}(u) = 1\,.$$

So $\omega_N^2 = 1$ and S is bijective. $\square$

Poincaré–Birkhoff–Witt theorem

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Proof. Introduce the ascending filtration on $\mathrm{Y}(\mathfrak{gl}_N)$ by $\deg t_{ij}^{(r)} = r$. By the defining relations

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the corresponding graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N)$ is commutative.

This implies that the ordered monomials in the generators span the algebra $\mathrm{Y}(\mathfrak{gl}_N)$.

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Denote by $\bar{t}_{ij}^{(r)}$ the image of $t_{ij}^{(r)}$ in the r-th component of $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N)$. It will be sufficient to show that the elements $\bar{t}_{ij}^{(r)}$ are algebraically independent.

By the defining relations, for any $M \geqslant 0$ there is a homomorphism

$$\iota_M : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{Y}(\mathfrak{gl}_{N+M}),$$

such that $t_{ij}^{(r)} \mapsto t_{ij}^{(r)}$.

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such that $t_{ij}^{(r)} \mapsto t_{ij}^{(r)}$. Take the composition

$$\zeta_M = \mathrm{ev}_{N+M} \circ \omega_{N+M} \circ \iota_M.$$

The automorphism ω_{N+M} of the algebra $\mathrm{Y}(\mathfrak{gl}_{N+M})$ is defined by

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while

$$\mathrm{ev}_{N+M} : \mathrm{Y}(\mathfrak{gl}_{N+M}) \to \mathrm{U}(\mathfrak{gl}_{N+M})$$

is the evaluation homomorphism,

$$T(u) \mapsto 1 + E\, u^{-1}.$$

Then

$$\mathrm{ev}_{N+M} \circ \omega_{N+M} : T(u) \mapsto (1 - E\, u^{-1})^{-1} = \sum_{r=0}^{\infty} E^{r}\, u^{-r}.$$

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Hence, explicitly, for the homomorphism

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we have

$$\zeta_M : t_{ij}^{(r)} \mapsto (E^r)_{ij},$$

and E is the $(N+M) \times (N+M)$ matrix whose ij-th entry is E_{ij}.

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Passing to the graded algebras, we get the homomorphism

$$\overline{\zeta}_M : \operatorname{gr}\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{S}(\mathfrak{gl}_{N+M}),$$

where $\mathrm{S}(\mathfrak{gl}_{N+M})$ is the symmetric algebra of $\mathfrak{gl}_{N+M}$.

The image of $\bar{t}_{ij}^{(r)}$ under $\overline{\zeta}_M$ is the polynomial $p_{ij}^{(r)}$ such that

$$p_{ij}^{(r)}(X) = (X^r)_{ij} \quad \text{for any} \quad X \in \mathrm{Mat}_{N+M}\,.$$

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It suffices to show that for any positive integer R there exists a large enough M such that the polynomials $p_{ij}^{(r)}$ with $1 \leqslant i, j \leqslant N$ and $1 \leqslant r \leqslant R$ are algebraically independent.

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For a large enough M, choose disjoint subsets

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and let $y_{ij}^{(r)}$ be independent parameters.

Now take

$$X = \sum_{i,j=1}^{N} \sum_{r=1}^{R} \left(e_{ia_1} + e_{a_1 a_2} + \cdots + e_{a_{r-2} a_{r-1}} + y_{ij}^{(r)} e_{a_{r-1} j} \right).$$

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For the polynomial $p_{ij}^{(r)}$ we have

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Hence, these polynomials are algebraically independent. $\square$

Hopf algebra structure

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the counit axioms

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In particular, $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$.

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where $\mu : A \otimes A \to A$ is the algebra multiplication and $\delta : \mathbb{C} \to A$ is the unit map of the algebra A; that is, $\delta(c) = c \cdot 1$ for any $c \in \mathbb{C}$.