The MacMahon Master Theorem and higher Sugawara operators

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joint work with

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Manin matrices

A matrix $Z = [z_{ij}]$ with entries in an algebra is a Manin matrix if

$$[z_{ij}, z_{kl}] = [z_{kj}, z_{il}] \qquad \text{for all} \quad i, j, k, l \in \{1, \dots, m\}.$$

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In the super case, z_{ij} are elements of a superalgebra and

$$[z_{ij}, z_{kl}] = [z_{kj}, z_{il}](-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}} \qquad \text{for all} \quad i, j, k, l \in \{1, \ldots, m+n\},$$

where $\bar{\imath} = 0$ for $i \leqslant m$ and $\bar{\imath} = 1$ for $i > m$, $\quad \deg z_{ij} = \bar{\imath} + \bar{\jmath}$.

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- $Z = e^{-\partial_u}(u + E), \qquad z_{ij} = e^{-\partial_u}\left(u\, \delta_{ij} + e_{ij}\right).$

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- $Z = e^{-\partial_u}(u + E), \qquad z_{ij} = e^{-\partial_u}\left(u\, \delta_{ij} + e_{ij}\right).$

- $Z = e^{-\partial_u} T(u), \qquad T(u)$ is the generator matrix

  for the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$.

MacMahon's Master Theorem

The right quantum superalgebra $\mathcal{M}_{m|n}$ is generated by

elements z_{ij}, subject to the Manin matrix relations:

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Regard the matrix $Z = [\, z_{ij} \,]$ as the element

$$Z = \sum_{i,j=1}^{m+n} e_{ij} \otimes z_{ij} \, (-1)^{\bar{\imath}\,\bar{\jmath}+\bar{\jmath}} \in \mathrm{End}\, \mathbb{C}^{m|n} \otimes \mathcal{M}_{m|n}.$$

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Denote by Z_a the matrix Z in the a-th copy of $\mathrm{End}\, \mathbb{C}^{m|n}$ in

$$\underbrace{\mathrm{End}\, \mathbb{C}^{m|n} \otimes \ldots \otimes \mathrm{End}\, \mathbb{C}^{m|n}}_{k} \otimes \mathcal{M}_{m|n}.$$

Introduce the normalized symmetrizer and antisymmetrizer

$$\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\sigma\in\mathbb{C}[\mathfrak{S}_k], \qquad \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}\sigma\cdot\sigma\in\mathbb{C}[\mathfrak{S}_k].$$

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Set

$$\mathrm{Bos}=1+\sum_{k=1}^{\infty}\operatorname{str}H_kZ_1\ldots Z_k,$$

$$\mathrm{Ferm}=1+\sum_{k=1}^{\infty}(-1)^k\operatorname{str}A_kZ_1\ldots Z_k,$$

taking supertrace with respect to all k copies of $\operatorname{End}\mathbb{C}^{m|n}$.

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- Garoufalidis, Lê and Zeilberger, 2006.
- Other proofs: Konvalinka and Pak, 2007, Foata and Han, 2007, Hai and Lorenz, 2007.
- The proof in the super case relies on the matrix form

$$(1 - P_{12})\,[Z_1, Z_2] = 0$$

of the defining relations of $\mathcal{M}_{m|n}$.

Noncommutative Berezinian

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$$\mathrm{Ber}\, Z = \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}\, \sigma \cdot z_{\sigma(1)1} \dots z_{\sigma(m)m}$$

$$\times \sum_{\tau \in \mathfrak{S}_n} \mathrm{sgn}\, \tau \cdot z'_{m+1,m+\tau(1)} \dots z'_{m+n,m+\tau(n)}.$$

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In the supercommutative specialization,

$$\mathrm{Ber} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \cdot \det(D - CA^{-1}B)^{-1}.$$

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Moreover, we have the noncommutative Newton identities:

$$\big[\mathrm{Ber}\,(1 + u\,Z)\big]^{-1}\, \partial_u \mathrm{Ber}\,(1 + u\,Z) = \sum_{k=0}^{\infty} (-u)^k\, \mathrm{str}\, Z^{k+1}.$$

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$$\begin{aligned}\big[e_{ij}[r], e_{kl}[s]\big] &= \delta_{kj}\, e_{i\,l}[r+s] - \delta_{i\,l}\, e_{kj}[r+s](-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}\\ &\quad + K\Big(\delta_{kj}\delta_{i\,l}(-1)^{\bar{\imath}} - \frac{\delta_{ij}\delta_{kl}}{m-n}(-1)^{\bar{\imath}+\bar{k}}\Big)\, r\,\delta_{r,-s},\end{aligned}$$

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the element K is even and central, and $e_{ij}[r] := e_{ij}t^r$.

In the case $m = n$ the singular term is omitted.

For any $\kappa \in \mathbb{C}$ the vacuum module $V_\kappa(\mathfrak{gl}_{m|n})$ of level κ is the quotient of $\mathrm{U}(\widehat{\mathfrak{gl}}_{m|n})$ by the left ideal generated by $\mathfrak{gl}_{m|n}[t]$ and the element $K - \kappa$.

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Any element b of the $\widehat{\mathfrak{gl}}_{m|n}$-module $V_\kappa(\mathfrak{gl}_{m|n})$ satisfying $\mathfrak{gl}_{m|n}[t]\, b = 0$ is called a Segal–Sugawara vector.

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Hence, the subspace $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ spanned by the Segal–Sugawara vectors is a commutative associative algebra which can be identified with a commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$.

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Lemma. The matrix

$$\tau + \widehat{E}\,[-1] = \big[\delta_{ij}\tau + e_{ij}[-1](-1)^{\bar{\imath}}\big]$$

with the entries in $\mathrm{U}(\widehat{\mathfrak{gl}}_{m|n} \oplus \mathbb{C}\tau)$ is a Manin matrix.

Theorem. For any $k \geqslant 0$ all coefficients s_{kl} in the expansion

$$\mathrm{str}(\tau + \widehat{E}\,[-1])^k = s_{k\,0}\,\tau^k + s_{k1}\,\tau^{k-1} + \cdots + s_{kk}$$

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Examples.

$$\mathrm{str}(\tau + \widehat{E}\,[-1]) = (m-n)\,\tau + \sum_{i=1}^{m+n} e_{ii}[-1],$$

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$$\begin{aligned}\mathrm{str}(\tau + \widehat{E}\,[-1])^2 = (m-n)\,\tau^2 + 2\sum_{i=1}^{m+n} e_{ii}[-1]\,\tau \\ + \sum_{i,k=1}^{m+n} e_{ik}[-1]\,e_{ki}[-1]\,(-1)^{\bar{k}} + \sum_{i=1}^{m+n} e_{ii}[-2].\end{aligned}$$

Corollary. All the coefficients $b_{k\,l}, \sigma_{k\,l}, h_{k\,l} \in \mathrm{U}(t^{-1}\mathfrak{gl}_{m|n}[t^{-1}])$ in the expansions

$$\mathrm{Ber}\left(1 + u(\tau + \widehat{E}\,[-1])\right) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} b_{k\,l}\, u^k \tau^{k-l},$$

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Moreover, $b_{k\,l} = \sigma_{k\,l}$ for all k and l.

Sugawara operators

The application of the state-field correspondence map

$$Y : V_{n-m}(\mathfrak{gl}_{m|n}) \to \mathrm{End}\, V_{n-m}(\mathfrak{gl}_{m|n})[[z, z^{-1}]],$$

to the Segal–Sugawara vectors produces elements of the center of the local completion $\mathrm{U}_{n-m}(\widehat{\mathfrak{gl}}_{m|n})_{\mathrm{loc}}$ at the critical level.

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Under the state-field correspondence, $\tau \mapsto \partial_z$,

$$Y : e_{ij}[-1] \mapsto e_{ij}(z) := \sum_{r \in \mathbb{Z}} e_{ij}[r]\, z^{-r-1}.$$

Set $\widehat{E}(z) = [e_{ij}(z)(-1)^{\bar{\imath}}]$ and $T(z) = \partial_z + \widehat{E}(z)$.

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Corollary. All Fourier coefficients of the fields

$s_{kl}(z)$, $b_{kl}(z)$, $\sigma_{kl}(z)$ and $h_{kl}(z)$ defined by

$$: \mathrm{str}\, T(z)^k : \; = s_{k\,0}(z)\,\partial_z^k + s_{k1}(z)\,\partial_z^{k-1} + \dots + s_{kk}(z),$$

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:\operatorname{str} A_k\,T_1(z)\dots T_k(z): \; &= \sigma_{k\,0}(z)\,\partial_z^k + \sigma_{k1}(z)\,\partial_z^{k-1} + \dots + \sigma_{kk}(z),\\
:\operatorname{str} H_k\,T_1(z)\dots T_k(z): \; &= h_{k\,0}(z)\,\partial_z^k + h_{k1}(z)\,\partial_z^{k-1} + \dots + h_{kk}(z)
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$$:\mathrm{str}\, A_k\,T_1(z)\dots T_k(z): \; = \sigma_{k\,0}(z)\,\partial_z^k + \sigma_{k1}(z)\,\partial_z^{k-1} + \dots + \sigma_{kk}(z),$$

$$:\mathrm{str}\, H_k\,T_1(z)\dots T_k(z): \; = h_{k\,0}(z)\,\partial_z^k + h_{k1}(z)\,\partial_z^{k-1} + \dots + h_{kk}(z)$$

are Sugawara operators for $\widehat{\mathfrak{gl}}_{m|n}$. Moreover, $\sigma_{kl}(z) = b_{kl}(z)$.

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$$\operatorname{str} L(z)^k = S_{k\,0}(z)\,\partial_z^k + S_{k1}(z)\,\partial_z^{k-1} + \dots + S_{kk}(z),$$

$$\operatorname{Ber}\big(1 + u\,L(z)\big) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} B_{kl}(z)\, u^k \partial_z^{k-l},$$

generate a commutative subalgebra of $\mathrm{U}(\mathfrak{gl}_{m|n}[t])$.

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$$\operatorname{cdet}\begin{bmatrix} \partial_z - e_{11}(z)_- & -e_{12}(z)_- & \dots & -e_{1m}(z)_- \\ -e_{21}(z)_- & \partial_z - e_{22}(z)_- & \dots & -e_{2m}(z)_- \\ \vdots & \vdots & \ddots & \vdots \\ -e_{m1}(z)_- & -e_{m2}(z)_- & \dots & \partial_z - e_{mm}(z)_- \end{bmatrix}.$$

Example. In the case $n = 0$ the Berezinian turns into the column determinant

$$\operatorname{cdet}\begin{bmatrix} \partial_z - e_{11}(z)_- & -e_{12}(z)_- & \dots & -e_{1m}(z)_- \\ -e_{21}(z)_- & \partial_z - e_{22}(z)_- & \dots & -e_{2m}(z)_- \\ \vdots & \vdots & \ddots & \vdots \\ -e_{m1}(z)_- & -e_{m2}(z)_- & \dots & \partial_z - e_{mm}(z)_- \end{bmatrix}.$$

This yields the commutative subalgebras of $\mathrm{U}(\mathfrak{gl}_m[t])$ first discovered by Talalaev, 2006.

Consider finite-dimensional $\mathfrak{gl}_{m|n}$-modules $M^{(1)}, \dots, M^{(k)}$ and let $a_1, \dots, a_k$ be complex parameters.

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The tensor product

$$M^{(1)} \otimes \dots \otimes M^{(k)}$$

becomes a $\mathfrak{gl}_{m|n}[t]$-module, where the images of the matrix elements of the matrix $L(z) = \partial_z - \widehat{E}(z)_-$ are found by

$$\ell_{ij}(z) = \delta_{ij}\partial_z - (-1)^{\bar{\imath}} \sum_{r=1}^{k} \frac{e_{ij}^{(r)}}{z - a_r},$$

and $e_{ij}^{(r)}$ denotes the image of e_{ij} in the $\mathfrak{gl}_{m|n}$-module $M^{(r)}$.

Set $\mathcal{L}(z) = [\ell_{ij}(z)]$.

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Corollary.

Higher Gaudin Hamiltonians associated with $\mathfrak{gl}_{m|n}$ are provided by the coefficients of the Berezinian $\mathrm{Ber}\,\big(1 + u\,\mathcal{L}(z)\big)$

Set $\mathcal{L}(z) = [\ell_{ij}(z)]$.

Corollary.

Higher Gaudin Hamiltonians associated with $\mathfrak{gl}_{m|n}$ are provided by the coefficients of the Berezinian $\mathrm{Ber}\,\big(1 + u\,\mathcal{L}(z)\big)$

and the supertrace

$$\mathrm{str}\,\mathcal{L}(z)^k = \mathcal{S}_{k\,0}(z)\,\partial_z^k + \mathcal{S}_{k1}(z)\,\partial_z^{k-1} + \cdots + \mathcal{S}_{kk}(z).$$

Example.

The quadratic Gaudin Hamiltonian $\mathcal{H}(z) = \mathcal{S}_{22}(z)$ is given by

$$\mathcal{H}(z) = 2 \sum_{r=1}^{k} \frac{\mathcal{H}^{(r)}}{z - a_r} + \sum_{r=1}^{k} \frac{\Delta^{(r)}}{(z - a_r)^2},$$

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and $\Delta^{(r)}$ denotes the eigenvalue of the Casimir element

$\sum e_{ij} e_{ji} (-1)^{\bar{\jmath}} + \sum e_{ii}$ of $\mathfrak{gl}_{m|n}$ in the representation $M^{(r)}$.