The University of Sydney
School of Mathematics and Statistics

Algebras and Reductive Groups in Magma

MagmaMondays: 23 October 2023 — Semester 2, 2023

Web Page: `https://sites.google.com/view/magma-mondays/`
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1. Recall from the lecture that the octonions over a ring R have a basis $e_1, e_2, \ldots, e_8$ such that $e_i^2 = 1$ (for $i \geq 2$) and $e_i e_j = \varepsilon(i,j,k) e_k$ for a choice of signs $\varepsilon(i,j,k) = \pm 1$ where $\{i,j,k\}$ belongs to

```
fano := {@ <2 + n, 2 + (n+1) mod 7, 2 + (n+3) mod 7> : n in [0..6] @};
```

Let $A = \mathbb{O}(\mathbb{Q})$ denote the algebra of octonions over the rational field $\mathbb{Q}$,

(a) Let a be the matrix corresponding to the permutation $(2,3,4,5,6,7,8)$. Show that a is an automorphism of A that permutes the vectors $\pm e_i$.
Hint: PermutationMatrix(. . .)

(b) Let b_0 be the permutation $(2,7)(3,4)$. Show that b_0 is an automorphism of the 7-point plane defined by *fano*. Then find a diagonal matrix $d = \mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1)$ such that db is an automorphism of A that permutes the vectors $\pm e_i$, where b is the permutation matrix of b_0.

(c) Let G be the subgroup of $\mathrm{GL}(8,\mathbb{Q})$ generated by the matrices a and db. Show that the order of G is 1 344 and that G has a normal abelian subgroup E of order 8 such that the quotient G/E is isomorphic to $\mathrm{SL}(3,2)$. Furthermore, this extension is *non-split*; that is, there is no subgroup of G isomorphic to $\mathrm{SL}(3,2)$.

2. Let $\mathcal{M}$ be the set of elements of norm 1 in the integral octonions.

(a) Show that the elements of $\mathcal{M}$ satisfy the alternative laws: $(xy)x = x(yx)$, $x(xy) = x^2y$, $(xy)y = xy^2$ but $\mathcal{M}$ is not associative.

(b) Show that every element of $\mathcal{M}$ has an inverse.

(c) The *reflection* r_α in the hyperplane orthogonal to α is

$$vr_\alpha = v - [\![v,\alpha]\!]\alpha \text{ where } [\![v,\alpha]\!] = \frac{2(v,\alpha)}{(\alpha,\alpha)}.$$

In $\mathbb{O}(\mathbb{Q})$ we have $(u,v) = u\bar{v} + v\bar{u}$ and so for $\alpha \in \mathcal{M}$ we have $vr_\alpha = -\alpha\bar{v}\alpha$.

```
norm := func< ξ | (ξ*conj(ξ))[1] >;
ref := func< a, v | -a*conj(v)*a / norm(a) >;
refmat := func< a | MatrixRing(BaseRing(P),Dimension(P)) !
                    [ref(a,x) : x in Basis(P)] where P is Parent(a) >;
```

Use Magma to check that $\mathcal{M}$ is a root system. That is,

- $0 \notin \mathcal{M}$,
- For all $\alpha \in \mathcal{M}$ the reflection r_α leaves $\mathcal{M}$ invariant,
- For all $\alpha, \beta \in \mathcal{M}$ the *Cartan coefficient* $[\![\alpha,\beta]\!]$ is an integer.

3. If w has order 3, the map $x \mapsto \overline{w}xw$ is an automorphism of $\mathbb{O}_{\mathbb{Z}}$. The matrix of this automorphism is *autmat*(*w*), where

```
aut := func< a, v | a^3 eq 1 select a^2*v*a else 0 >;
autmat := func< a | MatrixRing(BaseRing(P), Dimension(P)) !
                      [aut(a,x) : x in Basis(P)] where P is Parent(a) >;
```

Let *gens* be the set of all automorphisms of $\mathbb{O}_{\mathbb{Z}}$ constructed from the elements of order 3 in $\mathcal{M}$ and let G be the group they generate.

(a) Show that the elements of *gens* are involutions and that G can be generated by three of them.

(b) Find the orbits of G on $\mathcal{M}$ and their lengths.

(c) Show that the set M_4 of elements of order 4 in $\mathcal{M}$ is a root system of type E_7.

(d) Let i be an element of M_4 and let G_0 be its stabiliser in G. Find the lengths of the orbits of G_0 on M_4.

4. Find all semisimple root data (up to isomorphism) of type A_3. (Hint: Let C be a Cartan matrix of type A_3 and consider factorisations $C = AB^{\top}$.)

5. The MAGMA code

```
P<x> := PolynomialRing(Rationals());
F<τ> := NumberField(x^2 - x -1);
```

creates the field F generated over the rationals by the element τ such that $\tau^2 = \tau + 1$. Then the code

```
H<i,j,k> := QuaternionAlgebra< F | -1, -1 >;
```

creates the algebra of quaternions over F with basis 1, i, j, k such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Let

```
π := (1/2)*(-1 + i + j + k);
σ := (1/2)*(τ^-1 +i +τ*j);
X := {H ! 1,π,σ};
```

and let I be the smallest multiplicatively closed subset of H containing X.

(a) Show that I is isomorphic to $\mathrm{SL}(2,5)$.

(b) Show that I is a root system (when considered as a subset of H). What is its Cartan type?

6. Let p be a prime and let S be the simply connected group of Lie type *A* and rank 1 over the finite field of p elements. For $p = 2, 3, 5$ find the dimensions of the highest weight representations of S (as computed by MAGMA)?