BMW ALGEBRA, QUANTIZED COORDINATE ALGEBRA AND TYPE C SCHUR–WEYL DUALITY

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ABSTRACT. We prove an integral version of the Schur–Weyl duality between the specialized Birman–Murakami–Wenzl algebra $\mathfrak{B}_n(-q^{2m+1}, q)$ and the quantum algebra associated to the symplectic Lie algebra $\mathfrak{sp}_{2m}$. In particular, we deduce that this Schur–Weyl duality holds over arbitrary (commutative) ground rings, which answers a question of Lehrer and Zhang ([38]) in the symplectic case. As a byproduct, we show that, as a $\mathbb{Z}[q, q^{-1}]$-algebra, the quantized coordinate algebra defined by Kashiwara in [34] (which was denoted by $A_q^{\mathbb{Z}}(g)$ there) is isomorphic to the quantized coordinate algebra arising from a generalized Faddeev–Reshetikhin–Takhtajan construction (see [23], [29], [47]).

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1. INTRODUCTION

Let $m, n \in \mathbb{N}$. Let $\mathrm{U}(\mathfrak{gl}_m)$ be the universal enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_m(\mathbb{C})$ over $\mathbb{Q}$. Let $\mathrm{U}_{\mathbb{Z}}(\mathfrak{gl}_m)$ be the Kostant $\mathbb{Z}$-form ([37]) in $\mathrm{U}(\mathfrak{gl}_m)$. For any commutative $\mathbb{Z}$-algebra K, let $\mathrm{U}_K(\mathfrak{gl}_m) := \mathrm{U}_{\mathbb{Z}}(\mathfrak{gl}_m) \otimes_{\mathbb{Z}} K$. The natural left action of $\mathrm{U}_K(\mathfrak{gl}_m)$ on $(K^m)^{\otimes n}$ commutes with the right place permutation action of the symmetric group algebra $K\mathfrak{S}_n$. Let φ_A, ψ_A be the natural representations

$$\varphi_A : (K\mathfrak{S}_n)^{\mathrm{op}} \to \mathrm{End}_K\big((K^m)^{\otimes n}\big), \quad \psi_A : \mathrm{U}_K(\mathfrak{gl}_m) \to \mathrm{End}_K\big((K^m)^{\otimes n}\big),$$

respectively. The well-known type A Schur–Weyl duality (see [7], [9], [14], [26], [48], [49]) says that

(a) $\varphi_A\big(K\mathfrak{S}_n\big) = \mathrm{End}_{\mathrm{U}_K(\mathfrak{gl}_m)}\big((K^m)^{\otimes n}\big)$,
(b) $\psi_A\big(\mathrm{U}_K(\mathfrak{gl}_m)\big) = \mathrm{End}_{K\mathfrak{S}_n}\big((K^m)^{\otimes n}\big)$;
(c) if K is an infinite field, then

$$\mathrm{End}_{\mathrm{U}_K(\mathfrak{gl}_m)}\big((K^m)^{\otimes n}\big) = \mathrm{End}_{KGL_m(K)}\big((K^m)^{\otimes n}\big),$$

and the image of the group algebra $KGL_m(K)$ in $\mathrm{End}_K\big((K^m)^{\otimes n}\big)$ also coincides with $\mathrm{End}_{K\mathfrak{S}_n}\big((K^m)^{\otimes n}\big)$;

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(d) if K is a field of characteristic 0, then there is an irreducible $\mathrm{U}_K(\mathfrak{gl}_m)$-$K\mathfrak{S}_n$-bimodule decomposition

$$(K^m)^{\otimes n} = \bigoplus_{\substack{\lambda=(\lambda_1,\lambda_2,\cdots)\vdash n\\ \ell(\lambda)\leq m}} \Delta_\lambda \otimes S^\lambda,$$

where Δ_λ (resp., S^λ) denotes the irreducible left $\mathrm{U}_K(\mathfrak{gl}_m)$-module (resp., irreducible right $K\mathfrak{S}_n$-module) associated to λ, $\lambda \vdash n$ means λ is a partition of n, and $\ell(\lambda)$ denotes the largest integer i such that $\lambda_i \neq 0$.

There is a quantized version of the above type A Schur–Weyl duality. Let q be an indeterminate over $\mathbb{Z}$. Let $\mathscr{A} = \mathbb{Z}[q, q^{-1}]$ be the Laurent polynomial ring in q. Let $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{gl}_m)$ be the quantized enveloping algebra of $\mathfrak{gl}_m$ over $\mathbb{Q}(q)$ ([19], [32], [33]), where q is the quantum parameter. Let $\mathbb{U}_{\mathscr{A}}(\mathfrak{gl}_m)$ be the Lusztig's $\mathscr{A}$-form ([40]) in $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{gl}_m)$. Let $\mathscr{H}_{\mathscr{A}}(\mathfrak{S}_n)$ be the Iwahori–Hecke algebra associated to the symmetric group $\mathfrak{S}_n$, defined over $\mathscr{A}$ and with parameter q. By definition, $\mathscr{H}_{\mathscr{A}}(\mathfrak{S}_n)$ is generated by $\widehat{T}_1, \cdots, \widehat{T}_{n-1}$ which satisfy the well-known braid relations as well as the relation $(\widehat{T}_i - q)(\widehat{T}_i + q^{-1}) = 0$, for $i = 1, 2, \cdots, n-1$. For any commutative $\mathscr{A}$-algebra K, we use ζ to denote the natural image of q in K, and we define $\mathbb{U}_K(\mathfrak{gl}_m) := \mathbb{U}_{\mathscr{A}}(\mathfrak{gl}_m) \otimes_{\mathscr{A}} K$, $\mathscr{H}_K(\mathfrak{S}_n) := \mathscr{H}_{\mathscr{A}}(\mathfrak{S}_n) \otimes_{\mathscr{A}} K$. Then, there is a left action of $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{gl}_m)$ on $\mathbb{Q}(q)^m$ which quantizes the natural representation of $\mathfrak{gl}_m(\mathbb{C})$. Via the coproduct, we get an action of $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{gl}_m)$ on $(\mathbb{Q}(q)^m)^{\otimes n}$. Furthermore, this action actually gives rise to an action of $\mathbb{U}_{\mathscr{A}}(\mathfrak{gl}_m)$ on $(\mathscr{A}^m)^{\otimes n}$ ([20]). By base change, we get an action of $\mathbb{U}_K(\mathfrak{gl}_m)$ on $(K^m)^{\otimes n}$ for any commutative $\mathscr{A}$-algebra K. There is also a right action of $\mathscr{H}_K(\mathfrak{S}_n)$ on $(K^m)^{\otimes n}$. Let φ_A, ψ_A be the natural representations

$$\varphi_A : (\mathscr{H}_K(\mathfrak{S}_n))^{\mathrm{op}} \to \mathrm{End}_K\big((K^m)^{\otimes n}\big), \quad \psi_A : \mathbb{U}_K(\mathfrak{gl}_m) \to \mathrm{End}_K\big((K^m)^{\otimes n}\big),$$

respectively. Then by [2], [20], [21] and [33],

(a') $\varphi_A\big(\mathscr{H}_K(\mathfrak{S}_n)\big) = \mathrm{End}_{\mathbb{U}_K(\mathfrak{gl}_m)}\big((K^m)^{\otimes n}\big)$;
(b') $\psi_A\big(\mathbb{U}_K(\mathfrak{gl}_m)\big) = \mathrm{End}_{\mathscr{H}_K(\mathfrak{S}_n)}\big((K^m)^{\otimes n}\big)$;
(c') if K is a field of characteristic 0 and ζ is not a root of unity in K, then there is an irreducible $\mathbb{U}_K(\mathfrak{gl}_m)$-$\mathscr{H}_K(\mathfrak{S}_n)$-bimodules decomposition

$$(K^m)^{\otimes n} = \bigoplus_{\substack{\lambda=(\lambda_1,\lambda_2,\cdots)\vdash n\\ \ell(\lambda)\leq m}} \Delta_\lambda \otimes S^\lambda,$$

where Δ_λ (resp., S^λ) denotes the irreducible left $\mathbb{U}_K(\mathfrak{gl}_m)$-module (resp., irreducible right $\mathscr{H}_K(\mathfrak{S}_n)$-module) associated to λ.

The algebra $\mathrm{End}_{\mathscr{H}_K(\mathfrak{S}_n)}\big((K^m)^{\otimes n}\big)$ is called "q-Schur algebra", which forms an important class of quasi-hereditary algebra and has been extensively studied by Dipper–James and many other people. It plays an important role in the modular representation theory of finite groups of Lie type (cf. [11], [12], [24]). The significance of the above results lies in that it provide a bridge between the representation theory of type A quantum groups and of type A Hecke algebras at an integral level. Note that in the semisimple case, the above Schur–Weyl duality follows easily from the complete reducibility. The difficult part lies in the non-semisimple case, where the surjectivity of φ_A was established in [21] by making use of Kazhdan–Lusztig bases of type A Hecke algebra, while the proof of the surjectivity of ψ_A relies heavily on the amazing work of [2], where the quantized enveloping algebra of $\mathfrak{gl}_m$ is realized as certain "limit" of q-Schur algebras. To the best of our knowledge, there is no alternative approach for this part.

A natural question arises: how about the Schur–Weyl dualities in other types? The answer is: there do exist Schur–Weyl dualities in types B, C, D in semisimple case (for both classical and quantized versions). However, it is an open question (see [29, Page80, Line1], [38, Abstract]) whether or not these Schur–Weyl dualities hold in an integral or characteristic free setting (like the type A situation).

The purpose of this paper is to give an affirmative answer to the above open question in the quantized type C case. That is, we shall prove an integral version of quantized type C Schur–Weyl duality. Note that there is no counterpart in type C of the work [2] in the literature. It turns out that our approach provides a new and general framework to prove integral Schur–Weyl dualities for all classical types. Before stating the main results in this paper, we first recall the known results for the classical type C Schur–Weyl duality. Let K be an infinite field. Let V be a $2m$-dimensional K-linear space equipped with a skew bilinear form $(,)$. Let $GSp(V)$ (resp., $Sp(V)$) be the symplectic similitude group (resp., the symplectic group) on V ([15], [28]). For any integer i with $1 \leq i \leq 2m$, set $i' := 2m+1-i$. We fix an ordered basis $\{v_1, v_2, \cdots, v_{2m}\}$ of V such that

$$(v_i, v_j) = 0 = (v_{i'}, v_{j'}),\ \ (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall\ 1 \leq i, j \leq m.$$

Let $\mathfrak{B}_n(-2m)$ be the specialized Brauer algebra over K. This algebra contains the group algebra $K\mathfrak{S}_n$ as a subalgebra. There is a right action of $\mathfrak{B}_n(-2m)$ on $V^{\otimes n}$ which extends the sign permutation action of $\mathfrak{S}_n$. We refer the reader to [10] for definitions of $\mathfrak{B}_n(-2m)$ and its action. Let φ_C, ψ_C be the natural representations

$$\varphi_C : (\mathfrak{B}_n(-2m))^{\rm op} \to \mathrm{End}_K\big(V^{\otimes n}\big), \quad \psi_C : KGSp(V) \to \mathrm{End}_K\big(V^{\otimes n}\big),$$

respectively.

Theorem 1.1. ([3], [4], [5])

(1) *The natural left action of $GSp(V)$ on $V^{\otimes n}$ commutes with the right action of $\mathfrak{B}_n(-2m)$. Moreover, if $K = \mathbb{C}$, then*

$$\varphi_C\big(\mathfrak{B}_n(-2m)\big) = \mathrm{End}_{\mathbb{C}GSp(V)}\big(V^{\otimes n}\big) = \mathrm{End}_{\mathbb{C}Sp(V)}\big(V^{\otimes n}\big),$$
$$\psi_C\big(\mathbb{C}GSp(V)\big) = \psi_C\big(\mathbb{C}Sp(V)\big) = \mathrm{End}_{\mathfrak{B}_n(-2m)}\big(V^{\otimes n}\big),$$

(2) *if $K = \mathbb{C}$, then there is an irreducible $\mathbb{C}GSp(V)$–$\mathfrak{B}_n(-2m)$–bimodule decomposition*

$$V^{\otimes n} = \bigoplus_{f=0}^{[n/2]} \bigoplus_{\substack{\lambda \vdash n-2f \\ \ell(\lambda) \leq m}} \Delta(\lambda) \otimes D(\lambda^t),$$

where $\Delta(\lambda)$ (resp., $D(\lambda^t)$) denotes the irreducible left $\mathbb{C}GSp(V)$-module (resp., the irreducible right $\mathfrak{B}_n(-2m)$-module) corresponding to λ (resp., corresponding to λ^t), λ^t denotes the transpose of λ.

By the work of [9], [10] and [46], the complex field $\mathbb{C}$ used in part (1) of the above theorem can be replaced by arbitrary infinite field. That is, we have a characteristic free version of type C Schur–Weyl duality in group case.

Theorem 1.2. ([9], [10], [46]) *Let K be an arbitrary infinite field. Then*

(1) $\psi_C\big(KGSp(V)\big) = \mathrm{End}_{\mathfrak{B}_n(-2m)}\big(V^{\otimes n}\big)$*;*
(2) $\varphi_C\big(\mathfrak{B}_n(-2m)\big) = \mathrm{End}_{KGSp(V)}\big(V^{\otimes n}\big) = \mathrm{End}_{KSp(V)}\big(V^{\otimes n}\big)$.

For the quantized type C Schur–Weyl duality, we require V to be a $2m$ dimensional vector space over $\mathbb{Q}(q)$ equipped with a skew bilinear form $(,)$. We fix an ordered basis $\{v_i\}_{i=1}^{2m}$ as before. Let $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})$ be the quantized enveloping algebra of $\mathfrak{sp}_{2m}(\mathbb{C})$ over $\mathbb{Q}(q)$, where q is the quantum parameter. Let $\mathfrak{B}_n(-q^{2m+1}, q)$

be the specialized Birman–Murakami–Wenzl algebra (specialized BMW algebra for short) over $\mathbb{Q}(q)$. There is a right action of $\mathfrak{B}_n(-q^{2m+1},q)$ on $V^{\otimes n}$ which quantizes the right action of $\mathfrak{B}_n(-2m)$. We refer the reader to Section 3 for precise definitions of $\mathfrak{B}_n(-q^{2m+1},q)$ and its action. Let φ_C,ψ_C be the natural representations

$$\begin{aligned}\varphi_C:\ &(\mathfrak{B}_n(-q^{2m+1},q))^{\rm op}\to \operatorname{End}_{\mathbb{Q}(q)}\big(V^{\otimes n}\big),\\ \psi_C:\ &\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})\to \operatorname{End}_{\mathbb{Q}(q)}\big(V^{\otimes n}\big),\end{aligned}$$

respectively.

Theorem 1.3. ([8, 10.2], [38])

(1) *The natural left action of $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})$ on $V^{\otimes n}$ commutes with the right action of $\mathfrak{B}_n(-q^{2m+1},q)$. Moreover,*

$$\begin{aligned}\varphi_C\big(\mathfrak{B}_n(-q^{2m+1},q)\big)&=\operatorname{End}_{\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})}\big(V^{\otimes n}\big),\\ \psi_C\big(\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})\big)&=\operatorname{End}_{\mathfrak{B}_n(-q^{2m+1},q)}\big(V^{\otimes n}\big);\end{aligned}$$

(2) *there is an irreducible $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})$-$\mathfrak{B}_n(-q^{2m+1},q)$-bimodule decomposition*

$$V^{\otimes n}=\bigoplus_{f=0}^{[n/2]}\bigoplus_{\substack{\lambda\vdash n-2f\\ \ell(\lambda)\leq m}}\Delta(\lambda)\otimes D(\lambda^t),$$

where $\Delta(\lambda)$ (respectively, $D(\lambda^t)$) denotes the irreducible left $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})$-module (respectively, the irreducible right $\mathfrak{B}_n(-q^{2m+1},q)$-module) corresponding to λ (resp., corresponding to λ^t).

Let $\mathbb{U}_{\mathscr{A}}(\mathfrak{sp}_{2m})$ be the Lusztig's $\mathscr{A}$-form in $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})$. Let $V_{\mathscr{A}}$ be the free $\mathscr{A}$-module spanned by $\{v_i\}_{i=1}^{2m}$. Note that $\mathfrak{B}_n(-q^{2m+1},q)$ has a natural $\mathscr{A}$-form $\mathfrak{B}_n(-q^{2m+1},q)_{\mathscr{A}}$. For any commutative $\mathscr{A}$-algebra K, let ζ be the natural image of q in K, and we define $\mathbb{U}_K(\mathfrak{sp}_{2m}):=\mathbb{U}_{\mathscr{A}}(\mathfrak{sp}_{2m})\otimes_{\mathscr{A}}K$, $\mathfrak{B}_n(-\zeta^{2m+1},\zeta):=\mathfrak{B}_n(-q^{2m+1},q)_{\mathscr{A}}\otimes_{\mathscr{A}}K$. The representation ψ_C naturally gives rise to an action of $\mathbb{U}_{\mathscr{A}}(\mathfrak{sp}_{2m})$ on $V_{\mathscr{A}}^{\otimes n}$ which commutes with the right action of $\mathfrak{B}_n(-q^{2m+1},q)_{\mathscr{A}}$. By base change, for any commutative $\mathscr{A}$-algebra K, we get an action of $\mathbb{U}_K(\mathfrak{sp}_{2m})$ on $V_K^{\otimes n}$ which commutes with the right action of $\mathfrak{B}_n(-\zeta^{2m+1},\zeta)$;

The main results in this paper are the following two theorems.

Theorem 1.4. *For any commutative $\mathscr{A}$-algebra K,*

$$\psi_C\Big(\mathbb{U}_K(\mathfrak{sp}_{2m})\Big)=\operatorname{End}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V_K^{\otimes n}\big).$$

Theorem 1.5. *For any commutative $\mathscr{A}$-algebra K,*

$$\varphi_C\Big(\mathfrak{B}_n(-\zeta^{2m+1},\zeta)\Big)=\operatorname{End}_{\mathbb{U}_K(\mathfrak{sp}_{2m})}\big(V_K^{\otimes n}\big).$$

Note that if we specialize the parameter q to $1_K\in K$, then the BMW algebra $\mathfrak{B}_n(-q^{2m+1},q)$ becomes the specialized Brauer algebra $\mathfrak{B}_n(-2m)$, and the action of $\mathfrak{B}_n(-q^{2m+1},q)$ on n-tensor space becomes the action of $\mathfrak{B}_n(-2m)$ used in [10]. Applying the above two theorem, we get the following corollary.

Corollary 1.6. *For any commutative $\mathbb{Z}$-algebra K,*

(1) $\psi_C\big(\mathrm{U}_K(\mathfrak{sp}_{2m})\big)=\operatorname{End}_{\mathfrak{B}_n(-2m)_K}\big(V_K^{\otimes n}\big)$;
(2) $\varphi_C\Big(\mathfrak{B}_n(-2m)_K\Big)=\operatorname{End}_{\mathrm{U}_K(\mathfrak{sp}_{2m})}\big(V_K^{\otimes n}\big)$.

Note that this corollary can also be deduced from the main result in [10] by using the equivalence between the category of rational $Sp_{2m}(K)$-modules and the category of locally finite $U_K(\mathfrak{sp}_{2m})$-modules.

The algebra $S_K^{sy}(2m,n) := \mathrm{End}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V_K^{\otimes n}\big)$ is called "symplectic ζ-Schur algebra" by Oehms ([47]). It is a cellular (in the sense of [25]) and quasi-hereditary K-algebra. The strategy that we use to prove Theorem 1.4 is to inspect the induced natural homomorphism $\widetilde{\psi}_C$ from Lusztig's modified quantum algebra (see [41]) $\dot{\mathbb{U}}_K(\mathfrak{sp}_{2m})$ to the symplectic q-Schur algebra $S_K^{sy}(2m,n)$, and (roughly speaking) to interpret $\widetilde{\psi}_C$ as the dual of the natural map from the nth homogeneous component of the quantized coordinate algebra of $\mathrm{SpM}_{2m}(K)$ (symplectic monoid) to the quantized coordinate algebra of $\mathrm{Sp}_{2m}(K)$ (symplectic group). It turns out that the kernel of $\widetilde{\psi}_C$ is spanned by the canonical basis elements it contains. As a consequence, we deduce the following result, which is announced in [16] without proof.

Corollary 1.7. *For any commutative $\mathscr{A}$-algebra K, $S_K^{sy}(2m,n)$ is isomorphic to the generalized q-Schur algebra ${}_K\mathbf{S}(\pi)$ defined in* [16]*, where π is the set of dominant weights occurring in $V^{\otimes n}$. In particular, if specializing q to* 1*, then we recover the symplectic Schur algebra studied in* [13] *and* [15].

The strategy that we use to prove Theorem 1.5 is similar to that used in [10]. We first prove the equality under the assumption that $m \geq n$. Then we reduce the case $m < n$ to the case $m = n$ via a commutative diagram. Finally, we convert the task of proving the equality concerning φ_C to a purely type C quantum algebra representation theorietic problem which involves no BMW algebras. However, the direct generalization from [10] does not work here. In our quantized case the proof is much more difficult. We expect that our approach for both equalities can be applied to prove integral versions of various other Schur–Howe–Weyl dualities in Lie theory.

The paper is organized as follows. In Section 2, we collect some basic knowledge about the usual and the modified form of the quantized enveloping algebra of $\mathfrak{sp}_{2m}(\mathbb{C})$ as well as their actions on the n-tensor space $V^{\otimes n}$. The new result is Lemma 2.3, which enables us to reduce the proof of the equality concerning ψ_C to the proof of an equality concerning $\widetilde{\psi}_C$. In Section 3, we show that each finite truncation $A_{\mathscr{A}}^{sy}(2m, \leq n)$ of the quantized coordinate algebra $A_{\mathscr{A}}^{sy}(2m)$ of $\mathrm{SpM}_{2m}(K)$ is a cellular coalgebra. The two-sided simple comodule decomposition of the quantized coordinate algebra $\widetilde{A}_{\mathbb{Q}(q)}^{sy}(2m)$ of $\mathrm{Sp}_{2m}(K)$ is obtained, which actually coincides with Peter–Weyl's decomposition proved by Kashiwara ([34]). In Section 4, after proving that the tensor product $\nabla_K^r(\lambda) \otimes \nabla_K(\lambda)$ of a cell left comodule and a cell right comodule of $A_K^{sy}(2m, \leq n)$ is actually a co-Weyl module of the quantum algebra $\mathbb{U}_K(\mathfrak{g} \oplus \mathfrak{g})$ (Lemma 4.13), we are able to identify the type C quantized coordinate algebra $A_q^{\mathbb{Z}}(\mathfrak{g})$ defined by Kashiwara in [34] with the quantized coordinate algebra $\widetilde{A}_{\mathscr{A}}^{sy}(2m)$ arising from generalized FRT construction. The proof relies on some nice properties of the upper global crystal basis (i.e., the dual canonical basis) of the quantized coordinate algebra $A_q^{\mathbb{Z}}(\mathfrak{g})$ introduced by Kashiwara. Then we give a proof of our first main result—Theorem 1.4. In Section 5, we give a proof of our second main result—Theorem 1.5 in the case where $m \geq n$, following a similar idea (but different and more difficult arguments than) in [10, Section 3]. The case where $m < n$ is dealt with in Section 6. We reduce the proof of Theorem 1.5 to the proof of the surjectivity of a map between coinvariants of two tensor spaces and the commutativity of a certain diagram of maps (Lemma 6.1). For the former, we use Lusztig's theory on based modules ([41]) as in [10, Section 4]. The proof of the latter turns out to be quite delicate and more interesting than in the classical case.

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2. Tensor space and modified quantized enveloping algebra of type C

In this section we shall recall some basic facts about the usual and the modified form of the quantized enveloping algebra of type C as well as their actions on the n-tensor space $V^{\otimes n}$. We shall show that, for any commutative $\mathscr{A}$-algebra K, the image of the quantum algebra $\mathbb{U}_K(\mathfrak{sp}_{2m})$ in the endomorphism algebra of the n-tensor space $V_K^{\otimes n}$ coincides with the image of the corresponding modified quantum algebra $\dot{\mathbb{U}}_K(\mathfrak{sp}_{2m})$.

Recall the Dynkin diagram of $\mathfrak{sp}_{2m}$

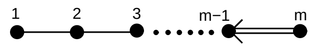

Figure 1.1

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where each vertex labeled by i represents a simple root α_i. For each integer i with $1 \leq i \leq m$, let $\alpha_i^{\vee}$ be the corresponding simple co-root. Let X be the weight lattice of $\mathfrak{sp}_{2m}$. We realize X as a free $\mathbb{Z}$-module with basis $\varepsilon_1, \cdots, \varepsilon_m$. Then $\alpha_m = 2\varepsilon_m,\ \ \alpha_i = \varepsilon_i - \varepsilon_{i+1},\ i = 1, 2, \cdots, m-1$.

Throughout this paper, we shall identify the weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_m\varepsilon_m \in X$ with the sequence $(\lambda_1, \cdots, \lambda_m)$ of integers. We shall also write $\lambda = (\lambda_1, \cdots, \lambda_m)$ and $|\lambda| := \lambda_1 + \cdots + \lambda_m$. Let

$$X^+ := \{\lambda \in X | \langle \lambda, \alpha_i \rangle \geq 0, \forall\, 1 \leq i \leq m\},$$

i.e., the set of dominant weights. Then $\lambda = (\lambda_1, \cdots, \lambda_m) \in X^+$ if and only if λ is a partition. Let $(,)$ be the symmetric bilinear form X defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$ for any i, j. The Cartan matrix $A = (a_{i,j})$ (where $a_{i,j} := 2(\alpha_j, \alpha_i)/(\alpha_i, \alpha_i)$) of $\mathfrak{sp}_{2m}$ is:

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & . & . & & \\ & & . & . & . & \\ & & & . & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}$$

with rows and columns indexed by $\{1, ..., m\}$. Given a fixed indeterminate q set

$$q_i = \begin{cases} q, & \text{if } i \neq m \\ q^2, & \text{if } i = m \end{cases},$$

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}} \text{ and } [k]_i^! = [k]_i[k-1]_i \cdots [1]_i.$$

The quantized enveloping algebra $\mathbb{U}_{\mathbb{Q}(q)} := \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})$ is the associative unital algebra over $\mathbb{Q}(q)$ generated by e_i, f_i, k_i, k_i^{-1} $i = 1, ..., m$, subject to the relations:

$$k_ik_i^{-1} = k_i^{-1}k_i = 1, \qquad k_i^{\pm 1}k_j^{\pm 1} = k_j^{\pm 1}k_i^{\pm 1},$$
$$k_ie_jk_i^{-1} = q^{a_{i,j}}e_j, \qquad k_if_jk_i^{-1} = q^{-a_{i,j}}f_j,$$
$$[e_i, f_i] = \frac{\widetilde{k}_i - \widetilde{k}_i^{-1}}{q_i - q_i^{-1}}\delta_{i,j} \text{ where } \widetilde{k}_i = \begin{cases} k_i, & \text{if } i \neq m \\ k_i^2, & \text{if } i = m \end{cases},$$
$$\text{if } i \neq j\ , \quad \sum_{k=0}^{1-a_{i,j}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{i,j}-k)} = 0,$$
$$\text{if } i \neq j\ , \quad \sum_{k=0}^{1-a_{i,j}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{i,j}-k)} = 0,$$

where $e_i^{(k)} = e_i^k/[k]_i^!$ and $f_i^{(k)} = f_i^k/[k]_i^!$.

$\mathbb{U}_{\mathbb{Q}(q)}$ is a Hopf algebra with coproduct Δ, counit ε and antipode S defined on generators by

$$\Delta(e_i) = e_i \otimes 1 + \widetilde{k}_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \widetilde{k}_i^{-1}, \quad \Delta(k_i) = k_i \otimes k_i,$$
$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon(k_i) = 1,$$
$$S(e_i) = -\widetilde{k}_i^{-1}e_i, \quad S(f_i) = -f_i\widetilde{k}_i, \quad S(k_i) = k_i^{-1}.$$

Recall our definition of $V_{\mathscr{A}}$ in the first paragraph below Theorem 1.3. For each $1 \leq i \leq 2m$, let $i' := 2m + 1 - i$. The action of the generators of $\mathbb{U}_{\mathbb{Q}(q)}$ on $V_{\mathbb{Q}(q)} := V_{\mathscr{A}} \otimes_{\mathscr{A}} \mathbb{Q}(q)$ is as follows (cf. [29, (4.16)])[1].

$$e_iv_j := \begin{cases} v_i, & \text{if } j = i+1, \\ -v_{(i+1)'}, & \text{if } j = i', \\ 0, & \text{otherwise;} \end{cases} \qquad e_mv_j := \begin{cases} v_m, & \text{if } j = m', \\ 0, & \text{otherwise,} \end{cases}$$
$$f_iv_j := \begin{cases} v_{i+1}, & \text{if } j = i, \\ -v_{i'}, & \text{if } j = (i+1)', \\ 0, & \text{otherwise;} \end{cases} \quad f_mv_j := \begin{cases} v_{m'}, & \text{if } j = m, \\ 0, & \text{otherwise,} \end{cases}$$
$$k_iv_j := \begin{cases} qv_j, & \text{if } j = i \text{ or } j = (i+1)', \\ q^{-1}v_j, & \text{if } j = i+1 \text{ or } j = i', \\ v_j, & \text{otherwise,} \end{cases}$$
$$k_mv_j := \begin{cases} qv_j, & \text{if } j = m, \\ q^{-1}v_j, & \text{if } j = m', \\ v_j, & \text{otherwise,} \end{cases}$$

where $1 \leq i < m$, $j \in \{1, \cdots, m\} \cup \{m', \cdots, 1'\}$. Via the coproduct, we get an action of $\mathbb{U}_{\mathbb{Q}(q)}$ on $V_{\mathbb{Q}(q)}^{\otimes n}$. Let $\mathbb{U}_{\mathscr{A}} := \mathbb{U}_{\mathscr{A}}(\mathfrak{sp}_{2m})$ be the Lusztig's $\mathscr{A}$-form in $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{sp}_{2m})$. As an $\mathscr{A}$-algebra, $\mathbb{U}_{\mathscr{A}}$ is generated by

$$e_i^{(a)},\ f_i^{(a)},\ k_i,\ k_i^{-1},\ a = 0, 1, 2, \cdots, 1 \leq i \leq m.$$

Lemma 2.1. *The above action of $\mathbb{U}_{\mathbb{Q}(q)}$ on $V_{\mathbb{Q}(q)}^{\otimes n}$ naturally gives rise to an action of $\mathbb{U}_{\mathscr{A}}$ on $V_{\mathscr{A}}^{\otimes n}$.*

Proof. It is well-known that $\mathbb{U}_{\mathscr{A}}$ is an $\mathscr{A}$-Hopf algebra. Hence it suffices to show that $\mathbb{U}_{\mathscr{A}}V_{\mathscr{A}} \subseteq V_{\mathscr{A}}$. However, this follows from direct verification. □

[1]Note that our $\widetilde{k}_m = k_m^2$ in this paper corresponds to k_m in the notation of [29].

For any commutative $\mathscr{A}$-algebra K, we define $\mathbb{U}_K = \mathbb{U}_K(\mathfrak{sp}_{2m}) := \mathbb{U}_{\mathscr{A}} \otimes_{\mathscr{A}} K$. By base change, we see that there is a representation

$$\psi_C : \mathbb{U}_K \to \mathrm{End}_K\big(V_K^{\otimes n}\big).$$

In Lusztig's book [41, Part IV], a "modified form" $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ of $\mathbb{U}_{\mathbb{Q}(q)}$ was introduced. The algebra $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ in general does not have a unit element. But it does have a family $\{1_\lambda\}_{\lambda\in X}$ of orthogonal idempotents such that $\dot{\mathbb{U}}_{\mathbb{Q}(q)} = \oplus_{\lambda,\mu\in X} 1_\lambda \dot{\mathbb{U}}_{\mathbb{Q}(q)} 1_\mu$. In a sense, the family $\{1_\lambda\}$ serves as a replacement for the identity. Let $\pi_{\lambda,\mu}$ be the canonical projection from $\mathbb{U}_{\mathbb{Q}(q)}$ onto $1_\lambda \dot{\mathbb{U}}_{\mathbb{Q}(q)} 1_\mu$ (see [41, (23.1.1)]). As a $\mathbb{Q}(q)$-algebra, $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ is generated by the elements $e_i 1_\lambda$, $f_i 1_\lambda$ and 1_λ with $i \in \{1, 2, \cdots, m\}$ and $\lambda \in X$, where the following relations are satisfied.

$$1_\lambda 1_\mu = 1_\mu 1_\lambda = \delta_{\lambda,\mu} 1_\lambda, \quad e_i 1_\lambda = 1_{\lambda+\alpha_i}(e_i 1_\lambda), \quad f_i 1_\lambda = 1_{\lambda-\alpha_i}(f_i 1_\lambda),$$

$$(e_i 1_{\lambda-\alpha_j})(f_j 1_\lambda) - (f_j 1_{\lambda+\alpha_i})(e_i 1_\lambda) = \delta_{i,j}[\langle\lambda, \alpha_i^\vee\rangle]_i 1_\lambda,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0 \quad \text{if} \quad i \neq j,$$

where $\langle\lambda, \alpha_i^\vee\rangle := 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$, and the last identity is understood as its canonical image under $\pi_{\lambda,\mu}$ for any $\lambda, \mu \in X$.

Let $\dot{\mathbb{U}}_{\mathscr{A}}$ be the $\mathscr{A}$-subalgebra of $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ generated by $e_i^{(k)} 1_\lambda$, $f_i^{(k)} 1_\lambda$ for $i = 1, 2, \cdots, m$, $k = 0, 1, 2, \cdots$, $\lambda \in X$. Then by [41, (23.2)], $\dot{\mathbb{U}}_{\mathscr{A}}$ is a free $\mathscr{A}$-module, and in fact $\dot{\mathbb{U}}_{\mathscr{A}}$ is an $\mathscr{A}$-form of $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$.

Since $V_{\mathbb{Q}(q)}^{\otimes n}$ is a finite dimensional integrable module over $\mathbb{U}_{\mathbb{Q}(q)}$, it follows that $V_{\mathbb{Q}(q)}^{\otimes n}$ naturally becomes a unital $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$-module in the sense of [41, (23.1.4)]. For each $\lambda \in X$, we define p_λ to be the projection operator from $V^{\otimes n}$ onto its λ-weight space (with respect to the subalgebra generated by $k_1^{\pm 1}, \cdots, k_{m-1}^{\pm 1}, k_m^{\pm 1}$).

Lemma 2.2. *Let $\widetilde{\psi}_C$ be the map*

$$1_\lambda \mapsto p_\lambda, \quad e_i 1_\lambda \mapsto \psi_C(e_i) p_\lambda, \quad f_i 1_\lambda \mapsto \psi_C(f_i) p_\lambda, \ i = 1, 2, \cdots, m, \ \lambda \in X.$$

Then $\widetilde{\psi}_C$ can be naturally extended to a representation of $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ on $V^{\otimes n}$ such that $\widetilde{\psi}_C(P 1_\lambda) = \psi_C(P) p_\lambda$ for any $P \in \mathbb{U}_{\mathbb{Q}(q)}$ and $\lambda \in X$.

Proof. This follows directly from the definition of $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ and the fact that $V_{\mathbb{Q}(q)}^{\otimes n}$ is a direct sum of its weight spaces. □

By restriction and applying Lemma 2.1, we see that $V_{\mathscr{A}}^{\otimes n}$ naturally becomes an $\dot{\mathbb{U}}_{\mathscr{A}}$-module. For any commutative $\mathscr{A}$-algebra K, we define $\dot{\mathbb{U}}_K = \dot{\mathbb{U}}_K(\mathfrak{sp}_{2m}) := \dot{\mathbb{U}}_{\mathscr{A}} \otimes_{\mathscr{A}} K$. By base change, we get a representation

$$\widetilde{\psi}_C : \dot{\mathbb{U}}_K \to \mathrm{End}_K\big(V_K^{\otimes n}\big),$$

Lemma 2.3. *With the above notation, for any commutative $\mathscr{A}$-algebra K,*

$$\psi_C\Big(\mathbb{U}_K\Big) = \widetilde{\psi}_C\Big(\dot{\mathbb{U}}_K\Big).$$

Proof. It suffices to show that $\psi_C\Big(\mathbb{U}_{\mathscr{A}}\Big) = \widetilde{\psi}_C\Big(\dot{\mathbb{U}}_{\mathscr{A}}\Big)$.

Let X_n be the set of weights (with respect to the Cartan part of $\mathfrak{sp}_{2m}$) in $V^{\otimes n}$. Obviously, X_n is a finite set. As linear operators on $V^{\otimes n}$, it is easy to check that for $i = 1, 2, \cdots, m$ and $a = 0, 1, 2, \cdots$,

$$\psi_C\big(e_i^{(a)}\big) = \widetilde{\psi}_C\Big(\sum_{\lambda\in X_n} e_i^{(a)}1_\lambda\Big), \quad \psi_C\big(f_i^{(a)}\big) = \widetilde{\psi}_C\Big(\sum_{\lambda\in X_n} f_i^{(a)}1_\lambda\Big),$$
$$\psi_C(k_i) = \widetilde{\psi}_C\Big(\sum_{\lambda\in X_n} q^{\langle\lambda,\alpha_i^\vee\rangle}1_\lambda\Big).$$

As a result, we deduce that $\psi_C\Big(\mathbb{U}_{\mathscr{A}}\Big) \subseteq \widetilde{\psi}_C\Big(\dot{\mathbb{U}}_{\mathscr{A}}\Big)$. It remains to show that $\widetilde{\psi}_C\Big(\dot{\mathbb{U}}_{\mathscr{A}}\Big) \subseteq \psi_C\Big(\mathbb{U}_{\mathscr{A}}\Big)$.

It suffices to show that $p_\lambda = \widetilde{\psi}_C(1_\lambda) \in \psi_C\Big(\mathbb{U}_{\mathscr{A}}\Big)$ for each $\lambda \in X_n$. For each integer i with $1 \le i \le m$, we set (following Lusztig)

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{\widetilde{k}_i q_i^{c-s+1} - \widetilde{k}_i^{-1} q_i^{-c+s-1}}{q_i^s - q_i^{-s}}, \qquad \begin{bmatrix} K_i; c \\ 0 \end{bmatrix} = 1,$$

for $t \ge 1$, $c \in \mathbb{Z}$. By [39, Lemma 4.4], we know that

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} \in \mathbb{U}_{\mathscr{A}}.$$

Let $\lambda \in X_n$. We write $\hat{\lambda}_i := \langle\lambda, \alpha_i^\vee\rangle$ for each i. Note that for each i,

$$\hat{\lambda}_i \in \begin{cases} \big\{-n, -n+1, \cdots, -1, 0, 1, \cdots, n-1, n\big\}, & \text{if } i \ne m, \\ \big\{-2n, -2n+2, \cdots, -2, 0, 2, \cdots, 2n-2, 2n\big\}, & \text{if } i = m. \end{cases}$$

We define

$$p'_\lambda := \Big(\prod_{i=1}^{m-1} \begin{bmatrix} K_i; -\hat{\lambda}_i - 1 \\ 2n \end{bmatrix} \begin{bmatrix} K_i; -\hat{\lambda}_i + 2n \\ 2n \end{bmatrix}\Big) \times \Big(\begin{bmatrix} K_m; -\hat{\lambda}_m - 1 \\ 4n \end{bmatrix} \begin{bmatrix} K_m; -\hat{\lambda}_m + 4n \\ 4n \end{bmatrix}\Big).$$

Clearly, $p'_\lambda \in \mathbb{U}_{\mathscr{A}}$. For each $\mu \in X_n$, we use $V_{\mathbb{Q}(q)}^{\otimes n}[\mu]$ to denote the μ-weight space of $V_{\mathbb{Q}(q)}^{\otimes n}$. One can verify directly that

$$\psi_C(p'_\lambda)(x) = \begin{cases} x, & \text{if } x \in V_{\mathbb{Q}(q)}^{\otimes n}[\lambda], \\ 0, & \text{if } x \in V_{\mathbb{Q}(q)}^{\otimes n}[\mu] \text{ with } \mu \ne \lambda. \end{cases}$$

As a result, we deduce that $p_\lambda = \widetilde{\psi}_C(1_\lambda) = \psi_C(p'_\lambda) \in \psi_C\Big(\mathbb{U}_{\mathscr{A}}\Big)$, as required. This completes the proof of the lemma. □

Remark 2.4. Lemma 2.3 enables use to reduce the proof of the equality concerning ψ_C to the proof of the equality concerning $\widetilde{\psi}_C$. Note that the arguments used in the proof of Lemma 2.3 actually work in all types.

3. BMW algebras and a generalized FRT construction

In this section we shall first recall the definitions of specialized BMW algebras and Oehms's results on a generalized Faddeev–Reshetikhin–Takhtajan (FRT for short) construction. Then we shall analyze the structure of each finite truncation $A_{\mathscr{A}}^{sy}(2m, \le n)$ of the quantized coordinate algebra $A_{\mathscr{A}}^{sy}(2m)$ of SpM_{2m}. Using Oehms's symplectic bideterminant basis for the quantized coordinate algebra of symplectic monoid scheme SpM_{2m}, we shall conclude that $A^{sy}_{\mathscr{A}}(2m, \leq n)$ is a cellular coalgebra. The two-sided simple comodule decomposition of the quantized coordinate algebra $\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)$ is obtained.

Let x, r, z be three indeterminates over $\mathbb{Z}$. Let R be the ring

$$R := \mathbb{Z}[r, r,^{-1}, z, x]/\big((1-x)z + (r - r^{-1})\big).$$

For simplicity, we shall use the same letters r, r^{-1}, z, x to denote their images in R respectively.

Definition 3.1. ([6], [43], [44]) *The Birman–Murakami–Wenzl algebra (or BMW algebra for short) $\mathfrak{B}_n(r, x, z)$ is a unital associative R-algebra with generators T_i, E_i, $1 \leq i \leq n-1$ and relations*

(1) $T_i - T_i^{-1} = z(1 - E_i)$, *for* $1 \leq i \leq n-1$,
(2) $E_i^2 = xE_i$, *for* $1 \leq i \leq n-1$,
(3) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, *for* $1 \leq i \leq n-2$,
(4) $T_iT_j = T_jT_i$, *for* $|i-j| > 1$,
(5) $E_iE_{i+1}E_i = E_i,\ E_{i+1}E_iE_{i+1} = E_{i+1}$, *for* $1 \leq i \leq n-2$,
(6) $T_iT_{i+1}E_i = E_{i+1}E_i,\ T_{i+1}T_iE_{i+1} = E_iE_{i+1}$, *for* $1 \leq i \leq n-2$,
(7) $E_iT_i = T_iE_i = r^{-1}E_i$, *for* $1 \leq i \leq n-1$.
(8) $E_iT_{i+1}E_i = rE_i,\ E_{i+1}T_iE_{i+1} = rE_{i+1}$, *for* $1 \leq i \leq n-2$.

In [43], Morton and Wassermann proved that $\mathfrak{B}_n(r, x, z)$ is isomorphic to the Kauffman's tangle algebra [36] whose R-basis is indexed by Brauer diagrams. As a consequence, they show that $\mathfrak{B}_n(r, x, z)$ is a free R-module with rank $(2n-1)!!$. In fact, the same is still true if one replaces the ring R by any commutative R-algebra K. Note that if we specialize r to 1 and z to 0, then $\mathfrak{B}_n(r, x, z)$ will become the usual Brauer algebra with parameter x.

We regard $\mathscr{A}$ as an R-algebra by sending r to $-q^{2m+1}$, z to $q - q^{-1}$ and x to $1 - \sum_{i=-m}^{m} q^{2i}$. The resulting $\mathscr{A}$-algebra will be denoted by $\mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}}$. We set $\mathfrak{B}_n(-q^{2m+1}, q) = \mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}} \otimes_{\mathscr{A}} \mathbb{Q}(q)$, and we call it *the specialized Birman–Murakami–Wenzl algebra* (or *specialized BMW algebra* for short). Note that if we specialize further q to 1, then $\mathfrak{B}_n(-q^{2m+1}, q)$ will become the specialized Brauer algebra $\mathfrak{B}_n(-2m)$ used in [10] and [31].

There is an action of the algebra $\mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}}$ on the n-tensor space $V_{\mathscr{A}}^{\otimes n}$ which we now recall. We set

$$(\rho_1, \cdots, \rho_{2m}) := (m, m-1, \cdots, 1, -1, \cdots, -m+1, -m),$$

and $\epsilon_i := \mathrm{sign}(\rho_i)$. For any $i, j \in \{1, 2, \cdots, 2m\}$, we use $E_{i,j}$ to denote the corresponding basis of matrix units for $\mathrm{End}_{\mathbb{Q}(q)}(V_{\mathbb{Q}(q)})$. Let " $-$ " be the ring involution defined on $\mathscr{A}$ by $\overline{q^{\pm 1}} = q^{\mp 1}$, $\overline{k} = k$ for any $k \in \mathbb{Z}$. The involution " $-$ " can be uniquely extended to an involution of $\mathrm{End}_{\mathscr{A}}\big(V_{\mathscr{A}}^{\otimes 2}\big)$ such that

$$\overline{E_{i,j} \otimes E_{k,l}} = E_{i,j} \otimes E_{k,l},\ \overline{rx} = \overline{r}\,\overline{x},$$

for any integers $1 \leq i, j, k, l \leq 2m$, any $r \in \mathscr{A}$ and any $x \in \mathrm{End}_{\mathscr{A}}\big(V_{\mathscr{A}}^{\otimes 2}\big)$. Following [47, Section 2], we set

$$\begin{aligned}\beta := &\sum_{1 \leq i \leq 2m} \Big(q^2 E_{i,i} \otimes E_{i,i} + E_{i,i'} \otimes E_{i',i}\Big) + q \sum_{\substack{1 \leq i,j \leq 2m \\ i \neq j,j'}} E_{i,j} \otimes E_{j,i} + \\ &(q^2 - 1) \sum_{1 \leq j < i \leq 2m} \Big(E_{i,i} \otimes E_{j,j} - q^{\rho_i - \rho_j} \epsilon_i \epsilon_j E_{i,j'} \otimes E_{i',j}\Big), \\ \gamma := &\sum_{1 \leq i,j \leq 2m} q^{\rho_i - \rho_j} \epsilon_i \epsilon_j E_{i,j'} \otimes E_{i',j}.\end{aligned}$$

We define $\beta' := \overline{q\beta^{-1}}$, $\gamma' := \overline{\gamma}$. By direct verification, we have that

$$\beta' := \sum_{1\leq i\leq 2m}\Big(qE_{i,i}\otimes E_{i,i}+q^{-1}E_{i,i'}\otimes E_{i',i}\Big)+\sum_{\substack{1\leq i,j\leq 2m\\ i\neq j,j'}}E_{i,j}\otimes E_{j,i}+$$
$$(q-q^{-1})\sum_{1\leq i<j\leq 2m}\Big(E_{i,i}\otimes E_{j,j}-q^{\rho_j-\rho_i}\epsilon_i\epsilon_j E_{i,j'}\otimes E_{i',j}\Big),$$
$$\gamma' := \sum_{1\leq i,j\leq 2m}q^{\rho_j-\rho_i}\epsilon_i\epsilon_j E_{i,j'}\otimes E_{i',j}.$$

Note that the operators[2] β',γ' are related to each other by the equation

$$\beta'-(\beta')^{-1}=(q-q^{-1})(\mathrm{id}_{V^{\otimes 2}}-\gamma'). \tag{3.2}$$

For $i=1,2,\cdots,n-1$, we set

$$\beta'_i := \mathrm{id}_{V^{\otimes i-1}}\otimes\beta'\otimes\mathrm{id}_{V^{\otimes n-i-1}},\quad \gamma'_i := \mathrm{id}_{V^{\otimes i-1}}\otimes\gamma'\otimes\mathrm{id}_{V^{\otimes n-i-1}}.$$

By [8, (10.2.5)] and [29, Section 4], the map φ_C which sends each T_i to β'_i and each E_i to γ'_i for $i=1,2,\cdots,n-1$ can be naturally extended to a right action of $\mathfrak{B}_n(-q^{2m+1},q)_{\mathscr{A}}$ on $V_{\mathscr{A}}^{\otimes n}$ such that all the statements in Theorem 1.3 hold. Note that if we specialize q to 1, then T_i degenerates to $-s_i\in\mathfrak{B}_n(-2m)$ for each i and this action of $\mathfrak{B}_n(-q^{2m+1},q)$ becomes the action studied in [10] of the specialized Brauer algebra $\mathfrak{B}_n(-2m)$ on $V_{\mathbb{Z}}^{\otimes n}$.

Now we recall what Oehms called "generalized Faddeev–Reshetikhin–Takhtajan construction". The basic references are [23], [29] and [47]. We concentrate on the quantized type C case. Let $u_i, 1\leq i\leq 2m$, (resp., $X_{i,j}, 1\leq i,j\leq 2m$) be a basis of V^* (resp., of $V^*\otimes V$) satisfying $u_i(v_j)=\delta_{i,j}$ (resp., $X_{i,j}=u_i\otimes v_j$) for each i,j. Set

$$I(2m,n) := \big\{(i_1,\cdots,i_n)\bigm| i_j\in\{1,2,\cdots,2m\}\text{ for each }j\big\}.$$

For each $\mathbf{i}=(i_1,\cdots,i_n)\in I(2m,n)$, we set $v_{\mathbf{i}} := v_{i_1}\otimes\cdots\otimes v_{i_n}$. An endomorphism $\mu\in\mathrm{End}(V^{\otimes n})$ is uniquely determined by its coefficient $\mu_{\mathbf{i},\mathbf{j}}$ with respect to the basis $\{v_{\mathbf{i}}\}_{\mathbf{i}\in I(2m,n)}$, that is,

$$\mu(v_{\mathbf{j}})=\sum_{\mathbf{i}\in I(2m,n)}\mu_{\mathbf{i},\mathbf{j}}v_{\mathbf{i}}.$$

For any commutative $\mathscr{A}$-algebra K, we use $F_K(2m)$ to denote the tensor algebra over $V^*\otimes V$, which can be identified with the free K-algebra generated by the $(2m)^2$ symbols $X_{i,j}$ for $i,j\in\{1,2,\cdots,2m\}$. For each $\mathbf{i},\mathbf{j}\in I(2m,n)$, we write

$$X_{\mathbf{i},\mathbf{j}} := X_{i_1,j_1}X_{i_2,j_2}\cdots X_{i_n,j_n}.$$

Following [47, Section 2], for an endomorphism $\mu\in\mathrm{End}(V^{\otimes n})$, we write

$$\mu\wr X_{\mathbf{i},\mathbf{j}}=\sum_{\mathbf{b}\in I(2m,n)}\mu_{\mathbf{i},\mathbf{b}}X_{\mathbf{b},\mathbf{j}},\quad X_{\mathbf{i},\mathbf{j}}\wr\mu=\sum_{\mathbf{b}\in I(2m,n)}X_{\mathbf{i},\mathbf{b}}\mu_{\mathbf{b},\mathbf{j}}.$$

Note that the algebra $F_K(2m)$ possesses a structure of bialgebra where comultiplication and augmentation on the generators $x_{\mathbf{i},\mathbf{j}}$ are given by

$$\Delta(X_{\mathbf{i},\mathbf{j}})=\sum_{\mathbf{b}\in I(2m,n)}X_{\mathbf{i},\mathbf{b}}\otimes X_{\mathbf{b},\mathbf{j}},\quad \varepsilon(X_{\mathbf{i},\mathbf{j}})=\delta_{\mathbf{i},\mathbf{j}}.$$

[2]The reason we use the operators β',γ' is because we want to let $\mathfrak{B}_n(-q^{2m+1},q)_{\mathscr{A}}$ act on $V_{\mathscr{A}}^{\otimes n}$ from the right hand side instead of from the left hand side.

Following Oehms [47], we can apply the generalized FRT construction with respect to the subset $\{\beta,\gamma\}\subseteq \mathrm{End}(V)\otimes\mathrm{End}(V)$ to obtain a new bialgebra $A_K^{sy}(2m)$. Precisely, we define

$$A_K^{sy}(2m):=F_K(2m)/\langle \beta\wr X_{\mathbf{i},\mathbf{j}}-X_{\mathbf{i},\mathbf{j}}\wr\beta,\ \gamma\wr X_{\mathbf{i},\mathbf{j}}-X_{\mathbf{i},\mathbf{j}}\wr\gamma,\ \mathbf{i},\mathbf{j}\in I(2m,2)\rangle,$$

which is necessarily a bialgebra as the ideal generated by $\beta\wr X_{\mathbf{i},\mathbf{j}}-X_{\mathbf{i},\mathbf{j}}\wr\beta$ and $\gamma\wr X_{\mathbf{i},\mathbf{j}}-X_{\mathbf{i},\mathbf{j}}\wr\gamma$ for $\mathbf{i},\mathbf{j}\in I(2m,2)$ is actually a bi-ideal. Clearly, $A_K^{sy}(2m)$ is a $\mathbb{Z}$-graded algebra, that is,

$$A_K^{sy}(2m)=\bigoplus_{0\leq n\in\mathbb{Z}}A_K^{sy}(2m,n).$$

For each $\mathbf{i},\mathbf{j}\in I(2m,n)$, let $x_{\mathbf{i},\mathbf{j}}$ be the canonical image of $X_{\mathbf{i},\mathbf{j}}$ in $A^{sy}(2m,n)$. By the main result in [47], the natural map $A_{\mathscr{A}}^{sy}(2m,n)\otimes_{\mathscr{A}}K\rightarrow A_K^{sy}(2m,n)$ is an isomorphism for any commutative $\mathscr{A}$-algebra K. The algebra $A_K^{sy}(2m)$ is a quantization of the coordinate ring of the symplectic monoid scheme SpM_{2m} which is defined by (for any commutative $\mathbb{Z}$-algebra K)

$$\mathrm{SpM}_{2m}(K):=\bigl\{A\in M_{2m}(K)\bigm| \exists d(A)\in K, A^tJA=AJA^t=d(A)J\bigr\},$$

where J is the Gram matrix of the given skew bilinear form with respect to the basis $\{v_i\}_{i=1}^{2m}$.

In [47], Oehms constructed a basis for $A_{\mathscr{A}}^{sy}(2m,n)$ for each n. To describe that basis, we need some more notations. For $i=1,2,\cdots,n-1$, we set

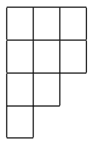
$$\beta_i:=\mathrm{id}_{V^{\otimes i-1}}\otimes\beta\otimes\mathrm{id}_{V^{\otimes n-i-1}},\quad \gamma_i:=\mathrm{id}_{V^{\otimes i-1}}\otimes\gamma\otimes\mathrm{id}_{V^{\otimes n-i-1}}.$$

Recall that for $i=1,2,\cdots,n-2$, we have the braid relation $\beta_i\beta_{i+1}\beta_i=\beta_{i+1}\beta_i\beta_{i+1}$. Recall also that (see Section 1) we have used $\mathfrak{S}_n$ to denote the symmetric group on n letters. For each $w\in\mathfrak{S}_n$ there is a well–defined element $\beta(w)\in\mathrm{End}(V^{\otimes n})$, where $\beta(w)=\beta_{i_1}\ldots\beta_{i_k}$ whenever k is minimal such that $w=(i_1,i_1+1)\ldots(i_k,i_k+1)$. For each partition λ of n, let λ^t be the transpose of λ, and let $\mathfrak{S}_{\lambda^t}$ be the corresponding Young subgroup of $\mathfrak{S}_n$ (which is the subgroup fixing the sets $\{1,2,\cdots,\lambda_1^t\},\{\lambda_1^t+1,\lambda_1^t+2,\cdots,\lambda_1^t+\lambda_2^t\},\cdots$). For each $w\in\mathfrak{S}_n$ and each pair of multi-indices $\mathbf{i},\mathbf{j}\in I(2m,n)$, we set (following Oehms)

$$T_q^\lambda(\mathbf{i}:\mathbf{j}):=\sum_{w\in\mathfrak{S}_{\lambda^t}}(-q^2)^{-\ell(w)}\beta(w)\wr x_{\mathbf{i},\mathbf{j}},$$

and call $T_q^\lambda(\mathbf{i}:\mathbf{j})$ a *quantum symplectic bideterminant.*

Recall that for each partition $\lambda=(\lambda_1,\lambda_2,\cdots,)$ one can naturally associates a Young diagram reading row lengths out of the components λ_i. For example,

$(3,3,2,1)\leftrightarrow$  .

For each positive integer k, let $\Lambda^+(m,k)$ be the set of partitions of k into not more than m parts. Let $\lambda\in\Lambda^+(m,k)$ with $0\leq k\leq n$. For each $\mathbf{i}\in I(2m,k)$, one can construct a λ-tableau $T_{\mathbf{i}}^\lambda$ by inserting the components of $\mathbf{i}$ column by column into the boxes of the Young diagram of λ. In the above example,

$T_{\mathbf{i}}^\lambda=$

i_1	i_5	i_8
i_2	i_6	i_9
i_3	i_7	
i_4		

.

Let $\mathbf{i}_\lambda$ be the unique multi-index in $I(2m,k)$ such that in the corresponding λ-tableau $T_{\mathbf{i}}^\lambda$, the jth row is filled with the number j for each integer j with $1\leq j\leq m$.

For each integer $1 \leq i \leq \lambda_1 = \ell(\lambda^t)$, we use $w_{i,0}$ to denote the unique longest element in the symmetric group $\mathfrak{S}_{\lambda_i^t}$. Let

$$w_{0,\lambda^t} := \prod_{i=1}^{\lambda_1} w_{i,0}.$$

Let $\widehat{\mathbf{i}}_\lambda \in I(2m,k)$ be such that $T^\lambda_{\widehat{\mathbf{i}}_\lambda} = T^\lambda_{\mathbf{i}} w_{0,\lambda^t}$. Following [47, Section 6], we put a new order " $\prec$ " on the set $\{1,2,\cdots,2m\}$, namely,

$$m \prec m' \prec (m-1) \prec (m-1)' \prec \cdots \prec 1 \prec 1'.$$

We define I^{mys}_λ to be the set of $\mathbf{i} \in I(2m,k)$ such that the entries in $T^\lambda_{\mathbf{i}}$ are weakly increasing along rows and strictly increasing down columns according to the order " $\prec$ ", and for each $1 \leq i \leq m$, i, i' are limited to the first $m-i+1$ rows.

In the previous example, we have

$$T^\lambda_{\mathbf{i}} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}, \qquad T^\lambda_{\widehat{\mathbf{i}}_\lambda} = \begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 3 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}.$$

Lemma 3.3. *For any $\lambda \in \Lambda^+(m,k)$, we have*

$$\widehat{\mathbf{i}}_\lambda \in I^{mys}_\lambda.$$

Proof. This follows directly from the definition of I^{mys}_λ. □

For each $\lambda \in \Lambda^+(m,k)$, let $I^<_\lambda$ be the set of multi-indices $\mathbf{i} \in I(2m,k)$ such that the entries in $T^\lambda_{\mathbf{i}}$ are strictly increasing down columns with respect to the usual order "$<$" on $\{1,2,\cdots,2m\}$. For each integer $j \geq 1$, we write

$$\Delta^{(j)} := \left(\Delta \otimes 1^{\otimes j-1}\right) \circ \cdots \circ \left(\Delta \otimes 1\right) \circ \Delta.$$

The following result was used in [47, Section 15] without proof. Since we shall use it in this paper, we include a proof here.

Lemma 3.4. *Let $\lambda \in \Lambda^+(m,k)$, $\mathbf{i}, \mathbf{j} \in I^{mys}_\lambda$ and $j \geq 1$ be an integer. We have that*

$$\Delta^{(j)}\Big(T^\lambda_q(\mathbf{i},\mathbf{j})\Big) = \sum_{\mathbf{h}^{(1)},\cdots,\mathbf{h}^{(j)} \in I^<_\lambda} T^\lambda_q(\mathbf{i},\mathbf{h}^{(1)}) \otimes T^\lambda_q(\mathbf{h}^{(1)},\mathbf{h}^{(2)}) \otimes \cdots \otimes T^\lambda_q(\mathbf{h}^{(j)},\mathbf{j}).$$

Proof. We only prove the special case where $j=1$. The general case follows from the same argument. For each positive integer r, we set $\omega_r := (\underbrace{1,1,\cdots,1}_{r \text{ copies}})$. Let $\lambda^t = (\mu_1,\cdots,\mu_p)$ be the transpose of λ, where $p = \lambda_1$. We split $\mathbf{j}$ into p multi-indices $\mathbf{j}^l \in I(2m,\mu_l)$, where for each $l \in \{1,2,\cdots,p\}$, the entries of $\mathbf{j}^l$ are taken from the lth column of $T^\lambda_{\mathbf{j}}$. The same thing can be done with $\mathbf{i}$. Then

$$T^\lambda_q(\mathbf{i},\mathbf{j}) = T^{\omega_{\mu_1}}_q(\mathbf{i}^1,\mathbf{j}^1) T^{\omega_{\mu_2}}_q(\mathbf{i}^2,\mathbf{j}^2) \cdots T^{\omega_{\mu_p}}_q(\mathbf{i}^p,\mathbf{j}^p),$$

hence

$$\Delta\Big(T^\lambda_q(\mathbf{i},\mathbf{j})\Big) = \Delta\Big(T^{\omega_{\mu_1}}_q(\mathbf{i}^1,\mathbf{j}^1)\Big)\Delta\Big(T^{\omega_{\mu_2}}_q(\mathbf{i}^2,\mathbf{j}^2)\Big) \cdots \Delta\Big(T^{\omega_{\mu_p}}_q(\mathbf{i}^p,\mathbf{j}^p)\Big).$$

Therefore, to prove the lemma, it suffices to consider the case where $p=1$.

Now assume that $p=1$. Then $\lambda = \omega_k$. If $\mathbf{i},\mathbf{j} \in I^<_{\omega_k}$, then the lemma follows from [47, (20)]. In general, by the definition of I^{mys}_λ ([47, Section 6]), we can find some simple reflections $s_{q_1},\cdots,s_{q_a},s_{l_1},\cdots,s_{l_b} \in \mathfrak{S}_{\lambda^t}$ such that

(1) $\widetilde{\mathbf{i}} := \mathbf{i}s_{q_1}s_{q_2}\cdots s_{q_a} \in I^<_{\omega_k}$, $\widetilde{\mathbf{j}} := \mathbf{j}s_{l_1}s_{l_2}\cdots s_{l_b} \in I^<_{\omega_k}$;

(2) For each integer $1 \leq c \leq a$, the action of s_{q_c} on $\mathbf{i}s_{q_1}s_{q_2}\cdots s_{q_{c-1}}$ does not exchange the indices i, i' for any i;

(3) For each integer $1 \leq c \leq b$, the action of s_{l_c} on $\mathbf{j}s_{l_1}s_{l_2}\cdots s_{l_{c-1}}$ does not exchange the indices i, i' for any i.

Now using [47, (13), (20), Corollary 11.9], we deduce that there exist integer $a(\mathbf{i}), b(\mathbf{j})$, such that

$$T_q^\lambda(\mathbf{i},\mathbf{j}) = (-q)^{a(\mathbf{i})+b(\mathbf{j})} T_q^\lambda(\widetilde{\mathbf{i}},\widetilde{\mathbf{j}}).$$

Therefore, applying [47, (20)], we get that

$$\begin{aligned}\Delta\big(T_q^\lambda(\mathbf{i},\mathbf{j})\big) &= (-q)^{a(\mathbf{i})+b(\mathbf{j})}\Delta\big(T_q^\lambda(\widetilde{\mathbf{i}},\widetilde{\mathbf{j}})\big)\\ &= \sum_{\mathbf{h}\in I_\lambda^<} (-q)^{a(\mathbf{i})} T_q^\lambda(\widetilde{\mathbf{i}},\mathbf{h}) \otimes (-q)^{b(\mathbf{j})} T_q^\lambda(\mathbf{h},\widetilde{\mathbf{j}})\\ &= \sum_{\mathbf{h}\in I_\lambda^<} T_q^\lambda(\mathbf{i},\mathbf{h}) \otimes T_q^\lambda(\mathbf{h},\mathbf{j}),\end{aligned}$$

as required. □

For each integer $n \geq 0$, set

$$\Lambda_n := \big\{\underline{\lambda} := (\lambda, l) \bigm| l \in \mathbb{Z}, 0 \leq l \leq n/2, \lambda \in \Lambda^+(m, n-2l)\big\}.$$

Let $d_q \in A_{\mathscr{A}}^{sy}(2m)$ be the central group-like element as defined in [47, (7)] and [29, Corollary 6.3]. By definition,

$$d_q = -q^{-\rho_k-\rho_l}\epsilon_k\epsilon_l x_{(k,k'),(l,l')} \wr \gamma \in A_{\mathscr{A}}^{sy}(2m, 2),$$

which is independent of the choices of $k, l \in \{1, 2, \cdots, 2m\}$. For each $\underline{\lambda} := (\lambda, l) \in \Lambda_n$ and each $\mathbf{i},\mathbf{j} \in I_\lambda^{mys}$, we set $D_{\mathbf{i},\mathbf{j}}^{\underline{\lambda}} := d_q^l T_q^\lambda(\mathbf{i},\mathbf{j})$. Oehms ([47, (7.1)]) proved that $A_{\mathscr{A}}^{sy}(2m, n) \otimes_{\mathscr{A}} K \cong A_K^{sy}(2m, n)$ for any commutative $\mathscr{A}$-algebra K. Indeed, $A_{\mathscr{A}}^{sy}(2m, n)$ is a free $\mathscr{A}$-module and the elements in the following set

$$\Big\{D_{\mathbf{i},\mathbf{j}}^{\underline{\lambda}} \Bigm| \underline{\lambda} = (\lambda, l) \in \Lambda_n,\, \mathbf{i},\mathbf{j} \in I_\lambda^{mys}\Big\} \tag{3.5}$$

form an $\mathscr{A}$-basis of $A_{\mathscr{A}}^{sy}(2m, n)$.

For each integer $k \geq 0$, we put an order on the set $\Lambda^+(m, k)$, write $\lambda \prec \mu$ if $\lambda_i^t = \mu_i^t$ for $i = 1, 2, \cdots, s-1$ and $\lambda_s^t < \mu_s^t$ for some s. For example, for $\lambda = (2, 2, 1), \mu = (3, 1, 1) \in \Lambda^+(3, 5)$, we have $\mu \prec \lambda$. Next, we put an order on Λ_n. For any $\underline{\lambda} = (\lambda, l), \underline{\mu} = (\mu, b) \in \Lambda_n$, write $\underline{\lambda} \prec \underline{\mu}$ if $l < b$ or $l = b$ and $\lambda \prec \mu$. For each $\underline{\lambda} = (\lambda, l) \in \Lambda_n$, we set $M(\underline{\lambda}) = I_\lambda^{mys}$. Let "$*$" be the $\mathscr{A}$-linear involution of $A_{\mathscr{A}}^{sy}(2m, n)$ which is defined on generators by $\Big(D_{\mathbf{i},\mathbf{j}}^{\underline{\lambda}}\Big)^* = D_{\mathbf{j},\mathbf{i}}^{\underline{\lambda}}$ for all $\underline{\lambda} \in \Lambda_n, \mathbf{i},\mathbf{j} \in \Lambda_n$.

Lemma 3.6. ([47, Theorem 7.1]) *With respect to the ordered set $(\Lambda_n, \prec)$, the finite set $M(\underline{\lambda})$ and the $\mathscr{A}$-linear involution "$*$", the coalgebra $A_{\mathscr{A}}^{sy}(2m, n)$ is a cellular coalgebra in the sense of* [47, Section 5, page 860] *with cellular basis given by*

$$\Big\{D_{\mathbf{i},\mathbf{j}}^{\underline{\lambda}} \Bigm| \underline{\lambda} = (\lambda, l) \in \Lambda_n,\, \mathbf{i},\mathbf{j} \in I_\lambda^{mys}\Big\}.$$

For each $\underline{\lambda} = (\lambda, l) \in \Lambda_n$, we set

$$\begin{aligned}A_{\mathscr{A}}^{sy}(2m, n)^{\succ\underline{\lambda}} &:= \mathscr{A}\text{-Span}\Big\{D_{\mathbf{i},\mathbf{j}}^{\underline{\mu}} \Bigm| \underline{\lambda} \prec \underline{\mu} = (\mu, b) \in \Lambda_n,\, \mathbf{i},\mathbf{j} \in I_\mu^{mys}\Big\},\\ A_{\mathscr{A}}^{sy}(2m, n)^{\succeq\underline{\lambda}} &:= \mathscr{A}\text{-Span}\Big\{D_{\mathbf{i},\mathbf{j}}^{\underline{\mu}} \Bigm| \underline{\lambda} \preceq \underline{\mu} = (\mu, b) \in \Lambda_n,\, \mathbf{i},\mathbf{j} \in I_\mu^{mys}\Big\}.\end{aligned}$$

By general theory of cellular coalgebra ([47, Section 5, page 860]), we know that both $A^{sy}_{\mathscr{A}}(2m,n)^{\succ\underline{\lambda}}$ and $A^{sy}_{\mathscr{A}}(2m,n)^{\succeq\underline{\lambda}}$ are two-sided coideal of $A^{sy}_{\mathscr{A}}(2m,n)$, and we have a two-sided $A^{sy}_{\mathscr{A}}(2m,n)$ comodule isomorphism

$$A^{sy}_{\mathscr{A}}(2m,n)^{\succeq\underline{\lambda}}/A^{sy}_{\mathscr{A}}(2m,n)^{\succ\underline{\lambda}}\cong\nabla^r(\underline{\lambda})\otimes\nabla(\underline{\lambda}),$$

where $\nabla(\underline{\lambda})$ (resp., $\nabla^r(\underline{\lambda})$) is the cell right (resp., left) comodule corresponding to $\underline{\lambda}$. Note that $\nabla^r(\underline{\lambda})=\nabla(\underline{\lambda})$ as $\mathscr{A}$-module, while the left $A^{sy}_{\mathscr{A}}(2m,n)$-coaction is obtained by twisting the right $A^{sy}_{\mathscr{A}}(2m,n)$-coaction using "$*$". If we extend the base ring $\mathscr{A}$ to the rational function field $\mathbb{Q}(q)$, then $\nabla(\underline{\lambda})_{\mathbb{Q}(q)}$ is an irreducible right $A^{sy}_{\mathbb{Q}(q)}(2m,n)$-comodule.

For any commutative $\mathscr{A}$-algebra K, we set

$$S^{sy}_K(2m,n):=\mathrm{End}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\Big(V_K^{\otimes n}\Big),$$

where ζ is the image of q in K. The algebra $S^{sy}_K(m,n)$ is called "symplectic ζ-Schur algebra" by Oehms ([47]). By [29, Proposition 2.1], there is a non-degenerate pairing between $F_{\mathbb{Q}(q)}(2m)$ and $\mathrm{End}_{\mathbb{Q}(q)}(V^{\otimes n}_{\mathbb{Q}(q)})$ such that for any $X_{\mathbf{i},\mathbf{j}}\in F_{\mathbb{Q}(q)}(2m,n)$, $f\in\mathrm{End}_{\mathbb{Q}(q)}(V^{\otimes\widehat{n}}_{\mathbb{Q}(q)})$, where $\mathbf{i},\mathbf{j}\in I(2m,n),n,\widehat{n}\in\mathbb{Z}^{\geq 0}$,

$$\langle X_{\mathbf{i},\mathbf{j}},f\rangle_0:=\begin{cases}u_{\mathbf{i}}\Big(f\big(v_{\mathbf{j}}\big)\Big), & \text{if } n=\widehat{n};\\ 0, & \text{otherwise.}\end{cases}$$

where $u_{\mathbf{i}}:=u_{i_1}\otimes\cdots\otimes u_{i_n}$, $\{u_i\}$ is the basis of V^* dual to $\{v_i\}$. By [29, Proposition 2.1], it induces a non-degenerate pairing $\langle,\rangle_0$ between $A^{sy}_{\mathbb{Q}(q)}(2m)$ and

$$\bigoplus_{0\leq n\in\mathbb{Z}}S^{sy}_{\mathbb{Q}(q)}(2m,n).$$

Furthermore, it induces a pairing $\langle,\rangle_0$ between $A^{sy}_{\mathbb{Q}(q)}(2m)$ and $\mathbb{U}_{\mathbb{Q}(q)}$. Precisely, for any $x_{\mathbf{i},\mathbf{j}}\in A^{sy}_{\mathbb{Q}(q)}(2m,n)$, $u\in\mathbb{U}_{\mathbb{Q}(q)}$, where $\mathbf{i},\mathbf{j}\in I(2m,n),n\in\mathbb{Z}^{\geq 0}$,

$$\langle x_{\mathbf{i},\mathbf{j}},u\rangle_0:=\langle x_{\mathbf{i},\mathbf{j}},\psi_C(u)\rangle_0,$$

where ψ_C is the canonical homomorphism

$$\psi_C:\mathbb{U}_{\mathbb{Q}(q)}\to S^{sy}_{\mathbb{Q}(q)}(2m,n):=\mathrm{End}_{\mathfrak{B}_n(-q^{2m+1},q)}\Big(V^{\otimes n}_{\mathbb{Q}(q)}\Big).$$

In [46] and [47], Oehms proved the pairing $\langle,\rangle_0$ actually induces an $\mathscr{A}$-algebra isomorphism $S^{sy}_{\mathscr{A}}(2m,n)\cong\big(A^{sy}_{\mathscr{A}}(2m,n)\big)^*$ for each $n\in\mathbb{Z}^{\geq 0}$. Note that our $F(2m)$ is just $T(E)$ in the notation of [29, Section 2], while the ideal generated by $\beta\wr X_{\mathbf{i},\mathbf{j}}-X_{\mathbf{i},\mathbf{j}}\wr\beta$ for $\mathbf{i},\mathbf{j}\in I(2m,2)$ in our paper is the same as the ideal generated by $\mathrm{Im}\big(\mathrm{id}_{E\otimes E}-\beta_E\big)$ in [29, Section 2]. Note also that over $\mathscr{A}$, the algebra $A^{sy}_{\mathscr{A}}(2m)$ is only a homomorphic image of $S(E)$ in [29, Proposition 2.1]. However, if we work over $\mathbb{Q}(q)$, then $A^{sy}_{\mathbb{Q}(q)}(2m)$ coincides with $S(E)$ in the notation of [29, Proposition 2.1] because of the relation (3.2). Therefore, for each $\underline{\lambda}=(\lambda,l)\in\Lambda_n$, $\nabla(\underline{\lambda})_{\mathbb{Q}(q)}$ is also an irreducible left $S^{sy}_{\mathbb{Q}(q)}(2m,n)$-module, and hence an irreducible left $\mathbb{U}_{\mathbb{Q}(q)}$-module.

For any commutative $\mathscr{A}$-algebra K, let ζ be the natural image of q in K, we define $\widetilde{A}^{sy}_K(2m):=A^{sy}_K(2m)/\langle d_\zeta-1\rangle$, where d_ζ is the natural image of d_q in $A^{sy}_K(2m)$. Note that $A^{sy}_K(2m)$ is a quantized version of the coordinate algebra of the symplectic monoid $\mathrm{SpM}_{2m}(K)$, while $\widetilde{A}^{sy}_K(2m)$ is a quantized version of the coordinate algebra of the symplectic group $\mathrm{Sp}_{2m}(K)$. The algebra $\widetilde{A}^{sy}_K(2m)$ will play a key role in the proof of Theorem 1.4 in the next section. We use π_C to denote the natural projection

$$\pi_C:\ A^{sy}_K(2m)\twoheadrightarrow\widetilde{A}^{sy}_K(2m).$$

For each integer $n \geq 0$ and each commutative $\mathscr{A}$-algebra K, we define

$$\Lambda^+(m, \leq n) := \bigsqcup_{0\leq k\leq n} \Lambda^+(m,k), \quad A_K^{sy}(2m, \leq n) := \bigoplus_{0\leq k\leq n} A_K^{sy}(2m,k).$$

We shall call $A_{\mathscr{A}}^{sy}(2m, \leq n)$ *a finite truncation of the quantized coordinate algebra* $A_{\mathscr{A}}^{sy}(2m)$ *of* SpM_{2m}. Clearly, $A_{\mathscr{A}}^{sy}(2m, \leq n)$ is a sub-coalgebra of $A_{\mathscr{A}}^{sy}(2m)$ as well as a free $\mathscr{A}$-submodule with basis

$$\Big\{ T_q^{\lambda}(\mathbf{i},\mathbf{j}) \Bigm| \lambda \in \Lambda^+(m, \leq n),\ \ \mathbf{i},\mathbf{j} \in I_{\lambda}^{mys} \Big\}.$$

The author is grateful to Professor S. Doty and the referee for the first part of the following corollary.

Corollary 3.7. *Let $n \geq 0$ be an integer and K be a commutative $\mathscr{A}$-algebra.*

1) The maps

$$\pi_C{\downarrow}_{A_K^{sy}(2m,n)}\colon\ A_K^{sy}(2m,n) \to \pi_C\big(A_K^{sy}(2m,n)\big)$$
$$\pi_C{\downarrow}_{A_K^{sy}(2m,\leq n)}\colon\ A_K^{sy}(2m,\leq n) \to \pi_C\big(A_K^{sy}(2m,\leq n)\big)$$

are both isomorphisms.

2) $\widetilde{A}_{\mathscr{A}}^{sy}(2m)$ is a free $\mathscr{A}$-module and the elements in the following set

$$\left\{ \pi_C\Big(T_q^{\lambda}(\mathbf{i},\mathbf{j})\Big) \;\middle|\; \begin{array}{c} \lambda \in \Lambda^+(m, n-2l) \text{ for some integer } n, l \\ \text{with } n \geq 0,\ 0 \leq l \leq n/2,\ \mathbf{i},\mathbf{j} \in I_{\lambda}^{mys} \end{array} \right\}$$

form an $\mathscr{A}$-basis of $\widetilde{A}_{\mathscr{A}}^{sy}(2m)$. Moreover, the canonical map

$$\widetilde{A}_{\mathscr{A}}^{sy}(2m) \otimes_{\mathscr{A}} K \to \widetilde{A}_K^{sy}(2m)$$

is an isomorphism.

Proof. Suppose that $0 \neq x \in \mathrm{Ker}\big(\pi_C{\downarrow}_{A_K^{sy}(2m,n)}\big)$. Since d_q is central, it follows that $x = (d_q - 1)y$ for some $y \in A_K^{sy}(2m)$. Note that $d_q \neq 0$ is homogeneous of degree 2, while x is homogeneous of degree n. By Lemma 3.6, the elements in the following set

$$\Big\{ d_q^l T_q^{\lambda}(\mathbf{i},\mathbf{j}) \Bigm| \underline{\lambda} = (\lambda, l) \in \Lambda_k \text{ for some } k \geq 0 \text{ and } 0 \leq l \leq k/2,\ \mathbf{i},\mathbf{j} \in I_{\lambda}^{mys} \Big\}$$

form a homogeneous basis of $A_K^{sy}(2m)$. Expressing y into a linear combination of this basis and comparing the degree of each homogeneous component, we get a contradiction. This proves that $\pi_C{\downarrow}_{A_K^{sy}(2m,n)}$ is injective and hence an isomorphism. As a result, we deduce that $\pi_C{\downarrow}_{A_K^{sy}(2m,\leq n)}$ is also an isomorphism. This proves 1). The statement 2) is an immediate consequence of the statement 1). □

Henceforth, we shall use Corollary 3.7 to identify $A_K^{sy}(2m,n)$ and $A_K^{sy}(2m,\leq n)$ as subspaces of $\widetilde{A}_{\mathscr{A}}^{sy}(2m)$ without further comments. The involution "$*$" gives rise to an $\mathscr{A}$-linear involution of $A_{\mathscr{A}}^{sy}(2m, \leq n)$, which will be still denoted by "$*$". Recall that in the paragraph below (3.5) we have introduced an order "$\prec$" on the set of partitions of a fixed integer. Now we generalize it to the case of partitions of possibly different integers. For any $\lambda, \mu \in \Lambda^+(m, \leq n)$, write $\lambda \prec \mu$ if

$$|\lambda| - |\mu| \in 2\mathbb{N} \quad \text{or} \quad |\lambda| = |\mu| \text{ and } \lambda \prec \mu.$$

Corollary 3.8. *With respect to the ordered set $(\Lambda^+(m, \leq n), \prec)$, the finite set I_{λ}^{mys} (for each $\lambda \in \Lambda^+(m, \leq n)$) and the $\mathscr{A}$-linear involution "$*$", the coalgebra $A_{\mathscr{A}}^{sy}(2m, \leq n)$ is a cellular coalgebra with cellular basis given by*

$$\Big\{ T_q^{\lambda}(\mathbf{i},\mathbf{j}) \Bigm| \lambda \in \Lambda^+(m, \leq n),\ \ \mathbf{i},\mathbf{j} \in I_{\lambda}^{mys} \Big\}.$$

Furthermore, for each commutative $\mathscr{A}$-algebra K, the canonical map

$$A_{\mathscr{A}}^{sy}(2m, \leq n) \otimes_{\mathscr{A}} K \to A_K^{sy}(2m, \leq n)$$

is an isomorphism.

Let $\Lambda^+(m):=\bigcup_{k\geq 0}\Lambda^+(m,k)$. For each $\lambda\in\Lambda^+(m,\leq n)$, we set

$$A_{\mathscr{A}}^{sy}(2m,\leq n)^{\succ\lambda}=\mathscr{A}\text{-Span}\big\{T_q^{\mu}(\mathbf{i},\mathbf{j})\bigm| \lambda\prec\mu\in\Lambda^+(m,\leq n),\ \mathbf{i},\mathbf{j}\in I_\mu^{mys}\big\},$$
$$A_{\mathscr{A}}^{sy}(2m,\leq n)^{\succeq\lambda}=\mathscr{A}\text{-Span}\big\{T_q^{\mu}(\mathbf{i},\mathbf{j})\bigm| \lambda\preceq\mu\in\Lambda^+(m,\leq n),\ \mathbf{i},\mathbf{j}\in I_\mu^{mys}\big\}.$$

By general theory of cellular coalgebra, we have that both $A_{\mathscr{A}}^{sy}(2m,\leq n)^{\succ\lambda}$ and $A_{\mathscr{A}}^{sy}(2m,\leq n)^{\succeq\lambda}$ are two-sided coideal of $A_{\mathscr{A}}^{sy}(2m,\leq n)$, and we have a two-sided $A_{\mathscr{A}}^{sy}(2m,\leq n)$-comodule isomorphism

$$A_{\mathscr{A}}^{sy}(2m,\leq n)^{\succeq\lambda}/A_{\mathscr{A}}^{sy}(2m,\leq n)^{\succ\lambda}\cong\nabla^r(\lambda)\otimes\nabla(\lambda),$$

where $\nabla(\lambda)$ (resp., $\nabla^r(\lambda)$) is the cell right (resp., left) comodule corresponding to λ. Note that $\nabla^r(\lambda)=\nabla(\lambda)$ as $\mathscr{A}$-module, while the left $A_{\mathscr{A}}^{sy}(2m,\leq n)$-coaction is obtained by twisting the right $A_{\mathscr{A}}^{sy}(2m,\leq n)$-coaction using "$*$". If we extend the base ring $\mathscr{A}$ to the rational function field $\mathbb{Q}(q)$, then $\nabla(\lambda)_{\mathbb{Q}(q)}$ is an irreducible right $A_{\mathbb{Q}(q)}^{sy}(2m,\leq n)$-comodule.

Suppose that $\lambda\in\Lambda^+(m,k)$ with $0\leq k\leq n$. We set $\underline{\lambda}=(\lambda,0)\in\Lambda_k$. Note that the $A_{\mathscr{A}}^{sy}(2m,\leq n)$-comodule $\nabla(\lambda)_{\mathscr{A}}$ is isomorphic to the restriction of the $A_{\mathscr{A}}^{sy}(2m,k)$-comodule $\nabla(\underline{\lambda})_{\mathbb{Q}(q)}$. In particular, every simple $A_{\mathbb{Q}(q)}^{sy}(2m,\leq n)$-comodule comes from the restriction of a simple $A_{\mathbb{Q}(q)}^{sy}(2m,k)$-comodule, or equivalently, of a simple $S_{\mathbb{Q}(q)}^{sy}(2m,k)$-module for some $0\leq k\leq n$. Therefore, by the surjection from $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ onto $S_{\mathbb{Q}(q)}^{sy}(2m,k)$, we see that every simple $A_{\mathbb{Q}(q)}^{sy}(2m,\leq n)$-comodule comes from the restriction of a simple $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$-module. For any field K which is an $\mathscr{A}$-algebra, we define

$$\nabla_K(\lambda):=\nabla(\lambda)\otimes_{\mathscr{A}}K,\ \ S_K^{sy}(2m,k):=S_{\mathscr{A}}^{sy}(2m,k)\otimes_{\mathscr{A}}K.$$

Then $\nabla_K(\lambda)\cong\nabla_K(\lambda,0)$ can be regarded as a $S_K^{sy}(2m,k)$-module.

Corollary 3.9. *Let K be a field which is an $\mathscr{A}$-algebra. Let $k\geq 0$ be an integer and $\lambda\in\Lambda^+(m,k)$. Then as a $S_K^{sy}(2m,k)$-module, $\nabla_K(\lambda)$ has a unique simple socle.*

Proof. By [47], $S_K^{sy}(2m,k)$ is a cellular quasi-hereditary algebra. By definition, the dual of $\nabla_K(\lambda)$ is a cell module of $S_K^{sy}(2m,k)$. Thus, the corollary follows from the general theory of cellular quasi-hereditary algebra. □

Recall that ψ_C induces a natural morphism from $\mathbb{U}_K(\mathfrak{g})$ to $S_K^{sy}(2m,k)$, via which $S_K^{sy}(2m,k)$-module $\nabla_K(\lambda)$ can be regarded as an $\mathbb{U}_K$-module. Note that the above corollary does not immediately imply that $\nabla_K(\lambda)$ has a unique simple socle as an $\mathbb{U}_K$-module because at this moment we did not know if that natural morphism from $\mathbb{U}_K$ to $S_K^{sy}(2m,k)$ is surjective or not. However, in the next section we shall show that this is indeed the case.

Corollary 3.10. *For each integer $n\geq 0$, we have a two-sided $A_{\mathscr{A}}^{sy}(2m,\leq n)$-comodule isomorphism*

$$\theta_n: A_{\mathbb{Q}(q)}^{sy}(2m,\leq n)\cong\bigoplus_{\lambda\in\Lambda^+(m,\leq n)}\nabla^r(\lambda)_{\mathbb{Q}(q)}\otimes\nabla(\lambda)_{\mathbb{Q}(q)}.$$

Furthermore, we have a two-sided $\widetilde{A}_{\mathscr{A}}^{sy}(2m)$-comodule isomorphism

$$\theta:\widetilde{A}_{\mathbb{Q}(q)}^{sy}(2m)\cong\bigoplus_{k\geq 0,\lambda\in\Lambda^+(m,k)}\nabla^r(\lambda)_{\mathbb{Q}(q)}\otimes\nabla(\lambda)_{\mathbb{Q}(q)},$$

and a commutative diagram

$$\begin{array}{ccc} A^{sy}_{\mathbb{Q}(q)}(2m, \leq n) & \xrightarrow{\theta_n} & \bigoplus_{\lambda\in\Lambda^+(m,\leq n)} \nabla^r(\lambda)_{\mathbb{Q}(q)} \otimes \nabla(\lambda)_{\mathbb{Q}(q)} \\ \downarrow & & \downarrow \\ \widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m) & \xrightarrow{\theta} & \bigoplus_{k\geq 0,\lambda\in\Lambda^+(m,k)} \nabla^r(\lambda)_{\mathbb{Q}(q)} \otimes \nabla(\lambda)_{\mathbb{Q}(q)} \end{array},$$

where the two vertical maps are natural embedding.

Proof. This follows from the cellular structure of $A^{sy}_{\mathscr{A}}(2m, \leq n)$ given in Corollary 3.8 and the fact

$$\widetilde{A}^{sy}_{\mathscr{A}}(2m) = \bigcup_{n\geq 0} A^{sy}_{\mathscr{A}}(2m, \leq n),$$

and the following commutative diagram

$$\begin{array}{ccc} A^{sy}_{\mathbb{Q}(q)}(2m, \leq n) & \xrightarrow{\theta_n} & \bigoplus_{\lambda\in\Lambda^+(m,\leq n)} \nabla^r(\lambda)_{\mathbb{Q}(q)} \otimes \nabla(\lambda)_{\mathbb{Q}(q)} \\ \downarrow & & \downarrow \\ A^{sy}_{\mathbb{Q}(q)}(2m, \leq \widehat{n}) & \xrightarrow{\theta_{\widehat{n}}} & \bigoplus_{\lambda\in\Lambda^+(m,\leq \widehat{n})} \nabla^r(\lambda)_{\mathbb{Q}(q)} \otimes \nabla(\lambda)_{\mathbb{Q}(q)} \end{array},$$

where $\widehat{n} \geq n \geq 0$ and the two vertical maps are natural embedding. □

Corollary 3.11. *For each integer $n \geq 0$ and each $\lambda \in \Lambda^+(m, n)$, let pr_λ be the natural projection from $\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)$ onto $\nabla^r(\lambda)_{\mathbb{Q}(q)} \otimes \nabla(\lambda)_{\mathbb{Q}(q)}$. Then the elements in the following set*

$$\Big\{\mathrm{pr}_\lambda\big(\pi_C(T^\lambda_q(\mathbf{i},\mathbf{j}))\big) \ \Big|\ \mathbf{i},\mathbf{j} \in I^{mys}_\lambda\Big\}$$

form a $\mathbb{Q}(q)$-basis of $\nabla^r(\lambda)_{\mathbb{Q}(q)} \otimes \nabla(\lambda)_{\mathbb{Q}(q)}$.

We end this section with the following lemma. Recall the definition of $\mathbf{i}_\lambda$, $\widehat{\mathbf{i}}_\lambda$ given in the paragraph above Lemma 3.3.

Lemma 3.12. *For each $\lambda \in \Lambda^+(m, n)$, we have*

$$\pi_C\Big(T^\lambda_q(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\Big) \equiv q^{a_\lambda}\pi_C\Big(T^\lambda_q(\widehat{\mathbf{i}}_\lambda, \widehat{\mathbf{i}}_\lambda)\Big) + \sum_{\substack{\mathbf{j},\mathbf{k}\in I^{mys}_\lambda\\ \mathbf{j},\mathbf{k}\lhd\widehat{\mathbf{i}}_\lambda}} C^\lambda_{\mathbf{j},\mathbf{k}}\pi_C\Big(T^\lambda_q(\mathbf{j},\mathbf{k})\Big)$$
$$\Big(\bmod A^{sy}_{\mathscr{A}}(2m, \leq n)^{\succ\lambda}\Big),$$

where $a_\lambda \in \mathbb{Z}$, $C^\lambda_{\mathbf{j},\mathbf{k}} \in \mathscr{A}$, and "$\lhd$" is the same as defined in [47, Proposition 8.4]. *In particular, we have that*

$$0 \neq \mathrm{pr}_\lambda\big(\pi_C(T^\lambda_q(\mathbf{i}_\lambda, \mathbf{i}_\lambda))\big) \in \nabla^r(\lambda) \otimes \nabla(\lambda).$$

Proof. This follows directly from [47, (8), Proposition 8.4, Corollary 9.12] and the facts that any quantum symplectic bideterminant in $A^{sy}_{\mathbb{Q}(q)}(2m)$ is homogeneous and the central group-like element d_q is homogeneous of degree 2. □

4. A comparison of two quantized coordinate algebras

In [34], Kashiwara introduced a version of quantized coordinate algebras (which was denoted by $A_q^{\mathbb{Z}}(\mathfrak{g})$ there) for any symmetrizable Kac–Moody Lie algebras $\mathfrak{g}$. In this section we shall first show that in the case of type C, the $\mathbb{Z}[q,q^{-1}]$ algebra $A_q^{\mathbb{Z}}(\mathfrak{g})$ and the quantized coordinate algebra $\widetilde{A}^{sy}_{\mathscr{A}}(2m)$ are isomorphic to each other. Then we shall give a proof of Theorem 1.4. Throughout this section, we set

$$\mathfrak{g}:=\mathfrak{sp}_{2m}(\mathbb{C}),\ \mathbb{U}_{\mathbb{Q}(q)}:=\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}),\ \dot{\mathbb{U}}_{\mathbb{Q}(q)}:=\dot{\mathbb{U}}_{\mathbb{Q}(q)}(\mathfrak{g}).$$

Following [34], we use $O_{int}(\mathfrak{g})$ to denote the category of $\mathbb{U}_{\mathbb{Q}(q)}$-modules M such that

(1) $M=\oplus_{\lambda\in X}M_\lambda$, where $M_\lambda:=\left\{x\in M \mid k_ix=q^{\langle\lambda,\alpha_i^\vee\rangle}x,\forall\, 1\le i\le m\right\}$,
(2) for any i, M is a union of finite dimensional $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}_i)$-modules, where $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}_i)$ denotes the $\mathbb{Q}(q)$-subalgebra generated by
$$e_i,f_i,k_1^{\pm1},k_2^{\pm1},\cdots,k_m^{\pm1}.$$
(3) for any $u\in M$, there exists $l\ge 0$ satisfying $e_{i_1}\cdots e_{i_l}u=0$ for any $i_1,\cdots,i_l\in\{1,2,\cdots,m\}$.

Then $O_{int}(\mathfrak{g})$ is semisimple and any simple object is isomorphic to the irreducible module $V(\lambda)$ with highest weight λ. Note that $\mathbb{U}_{\mathbb{Q}(q)}$ has a structure of bi-$\mathbb{U}_{\mathbb{Q}(q)}$-module. Hence $\mathbb{U}_{\mathbb{Q}(q)}^*$ was naturally endowed with a structure of bi-$\mathbb{U}_{\mathbb{Q}(q)}$-module.

Definition 4.1. ([34, (7.2.1)]) *We set*

$$\begin{aligned}A_q(\mathfrak{g})&:=\left\{u\in\mathbb{U}^*_{\mathbb{Q}(q)}\ \middle|\ \begin{matrix}\mathbb{U}_{\mathbb{Q}(q)}u \text{ belongs to } O_{int}(\mathfrak{g}) \text{ and } u\mathbb{U}_{\mathbb{Q}(q)}\\ \text{belongs to } O_{int}(\mathfrak{g}^{\mathrm{opp}})\end{matrix}\right\},\\ A_q^{\mathbb{Z}}(\mathfrak{g})&:=\left\{u\in A_q(\mathfrak{g})\ \middle|\ \langle u,\mathbb{U}_{\mathscr{A}}\rangle\subseteq\mathscr{A}\right\},\\ A_q^{\mathbb{Q}}(\mathfrak{g})&:=\left\{u\in A_q(\mathfrak{g})\ \middle|\ \langle u,\mathbb{U}_{\mathscr{A}}\rangle\subseteq\mathbb{Q}[q,q^{-1}]\right\},\end{aligned}$$

where $\langle,\rangle$ is the natural pairing.

Recall the pairing $\langle,\rangle_0$ between $A^{sy}_{\mathbb{Q}(q)}(2m)$ and $\mathbb{U}_{\mathbb{Q}(q)}$ (see Section 3, the second paragraph below Lemma 3.6).

Lemma 4.2. ([29, Theorem 6.4(2)]) *We have that*

$$\langle d_q-1,y\rangle_0=0,\quad \text{for any } y\in\mathbb{U}_{\mathbb{Q}(q)}.$$

and the pairing $\langle,\rangle_0$ induces a non-degenerate Hopf pairing $\langle,\rangle_0$ between $\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)$ and the quantized enveloping algebra $\mathbb{U}_{\mathbb{Q}(q)}$

$$\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)\times\mathbb{U}_{\mathbb{Q}(q)}\to\mathbb{Q}(q).$$

As a result, we have two Hopf algebra injections

$$\iota_A:\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)\hookrightarrow\Big(\mathbb{U}_{\mathbb{Q}(q)}\Big)^\circ,\ \iota_U:\mathbb{U}_{\mathbb{Q}(q)}\hookrightarrow\Big(\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)\Big)^\circ,$$

where for any Hopf algebra H, H° denotes the Hopf dual of H.

Lemma 4.3. *With the above notations, the pairing $\langle,\rangle_0$ naturally induces a non-degenerate Hopf pairing $\langle,\rangle_0$ between $\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)$ and the modified quantized enveloping algebra $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$*

$$\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)\times\dot{\mathbb{U}}_{\mathbb{Q}(q)}\to\mathbb{Q}(q).$$

As a result, we have two Hopf algebra injections

$$\widetilde{\iota}_A:\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)\hookrightarrow\Big(\dot{\mathbb{U}}_{\mathbb{Q}(q)}\Big)^\circ,\ \widetilde{\iota}_U:\dot{\mathbb{U}}_{\mathbb{Q}(q)}\hookrightarrow\Big(\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)\Big)^\circ,$$

Proof. This is an easy consequence of Lemma 2.3, Lemma 4.2 as well as the following two standard facts:

(a) any simple $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$-module is a submodule of $V_{\mathbb{Q}(q)}^{\otimes n}$ for some $n \in \mathbb{Z}^{\geq 0}$;
(b) if $u \in \dot{\mathbb{U}}_{\mathbb{Q}(q)}$ acts as 0 on every simple $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$-module, then $u = 0$.

□

For each $\mathbf{i} \in I(2m,n)$, we define $\mathrm{wt}(\mathbf{i}) = (\mu_1, \cdots, \mu_m)$, where

$$\mu_s := \#\{1 \leq j \leq n \mid i_j = s\} - \#\{1 \leq j \leq n \mid i_j = s'\},\ s = 1, 2, \cdots, m.$$

We identify $\mathrm{wt}(\mathbf{i}) = (\mu_1, \cdots, \mu_m)$ with the weight $\mu_1\varepsilon_1 + \cdots + \mu_m\varepsilon_m \in X$.

Lemma 4.4. *Let $\mathbf{i}, \mathbf{j} \in I(2m,n)$. Set $\mu = \mathrm{wt}(\mathbf{j})$. Then for any integer a with $1 \leq a \leq m$,*

$$\langle \pi_C\big(T_q^\lambda(\mathbf{i},\mathbf{j})\big), k_a\rangle_0 = q^{\langle \mu, \alpha_a^\vee\rangle} \varepsilon\big(\pi_C\big(T_q^\lambda(\mathbf{i},\mathbf{j})\big)\big).$$

Proof. This follows from direct verification. □

Note that $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}) \cong \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}^{opp})$ via the anti-automorphism ϕ defined on generators by:

$$e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad k_i \mapsto k_i, \quad i = 1, 2, \cdots, m.$$

We identify $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}\oplus\mathfrak{g})$ with $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})\otimes\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})$. Using ϕ, the bi-$\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})$ structure on $\big(\mathbb{U}_{\mathbb{Q}(q)}\big)^*$ can be interpreted as a left $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}\oplus\mathfrak{g})$-structure, i.e.,

$$\big((a\otimes b)f\big)(x) := f\big(\phi(b)xa\big),\ \forall\, a, b, x \in \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}), f \in \big(\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})\big)^*.$$

Let $W_m = W(C_m)$ be the Weyl group of type C_m. Let w_0 be the longest element in W_m. If $\lambda = (\lambda_1, \cdots, \lambda_m) \in X$, then $w_0\lambda = (-\lambda_1, \cdots, -\lambda_m)$. Let $k \in \mathbb{Z}^{\geq 0}$ and $\lambda \in \Lambda^+(m,k)$. Recall our definitions of $\mathbf{i}_\lambda$ given above Lemma 3.3. We have the following observation.

Corollary 4.5. *Let $k \in \mathbb{Z}^{\geq 0}$ and $\lambda \in \Lambda^+(m,k)$. Then $\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)$ is a weight vector of weight $\big(\lambda,\lambda\big)$ satisfying*

$$e_i\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big) = 0 = \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)f_i,\ \forall\, 1 \leq i \leq m.$$

Proof. Note that $\mathbf{i}_\lambda \in I_\lambda^<$. We identify $\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)$ as an element in $\big(\mathbb{U}_{\mathbb{Q}(q)}\big)^*$ via ι_A. Recall that the $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}\oplus\mathfrak{g})$-structure on $\big(\mathbb{U}_{\mathbb{Q}(q)}\big)^*$ comes from its bi-$\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})$ structure.

We first look at the left $\mathbb{U}_{\mathbb{Q}(q)}$-action on $\big(\mathbb{U}_{\mathbb{Q}(q)}\big)^*$. Recall that π_C is a bialgebra map. For each integer $j \geq 1$, by Lemma 3.4 and [47, (20)], we know that

$$\begin{aligned}&\Delta^{(j)}\Big(\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)\Big)\\ =& \sum_{\mathbf{h}^{(1)},\cdots,\mathbf{h}^{(j)}\in I_\lambda^<} \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{h}^{(1)})\big)\otimes\pi_C\big(T_q^\lambda(\mathbf{h}^{(1)},\mathbf{h}^{(2)})\big)\otimes\cdots\\ &\otimes\pi_C\big(T_q^\lambda(h^{(j)},\mathbf{i}_\lambda)\big).\end{aligned}$$

In particular, we have that

$$\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big) = \sum_{\mathbf{h}\in I_\lambda^<} \varepsilon\Big(\pi_C\big(T_q^\lambda(\mathbf{h},\mathbf{i}_\lambda)\big)\Big)\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{h})\big).$$

With these in mind and using Lemma 4.4, for any $f \in \mathbb{U}_{\mathbb{Q}(q)}$ and any integer a with $1 \leq a \leq m$, we get that

$$\begin{aligned}
&\langle k_a \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big), f\rangle_0 \\
&= \langle \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big), f k_a\rangle_0 \\
&= \sum_{\mathbf{h}\in I_\lambda^<} \langle \Delta\Big(\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)\Big), f\otimes k_a\rangle_0 \\
&= \sum_{\mathbf{h}\in I_\lambda^<} \langle \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{h})\big), f\rangle_0 \langle \pi_C\big(T_q^\lambda(\mathbf{h}, \mathbf{i}_\lambda)\big), k_a\rangle_0 \\
&= q^{\langle\lambda,\alpha_a^\vee\rangle} \sum_{\mathbf{h}\in I_\lambda^<} \langle \varepsilon\Big(\pi_C\big(T_q^\lambda(\mathbf{h}, \mathbf{i}_\lambda)\big)\Big)\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{h})\big), f\rangle_0 \\
&= q^{\langle\lambda,\alpha_a^\vee\rangle} \langle \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big), f\rangle_0,
\end{aligned}$$

which implies that $k_a\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big) = q^{\langle\lambda,\alpha_a^\vee\rangle}\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)$. In a similar way, one can prove that if we regard $\big(\mathbb{U}_{\mathbb{Q}(q)}\big)^*$ as a right $\mathbb{U}_{\mathbb{Q}(q)}$-module, then $\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)$ is a weight vector of weight λ.

It remains to show that $e_i\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big) = 0 = \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)f_i$ for any $1 \leq i \leq m$. It suffices to show that for any $f \in \mathbb{U}_{\mathbb{Q}(q)}$,

$$\langle e_i\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big), f\rangle_0 = \langle \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)f_i, f\rangle_0. \tag{4.6}$$

By definition,

$$\begin{aligned}
\langle e_i\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big), f\rangle_0 &= \langle \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big), f e_i\rangle_0, \\
\langle \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)f_i, f\rangle_0 &= \langle \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big), f_i f\rangle_0.
\end{aligned}$$

By direct verification, one can show that for any $\mathbf{h} \in I_\lambda^<$,

$$\langle \pi_C\big(T_q^\lambda(\mathbf{h}, \mathbf{i}_\lambda)\big), e_i\rangle_0 = 0 = \langle \pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{h})\big), f_i\rangle_0,$$

from which the equality (4.6) follows immediately. This completes the proof of the corollary. □

By Corollary 3.10 and the discussion above it, we see that every simple $\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)$ comodule comes from the restriction of a simple $\mathbb{U}_{\mathbb{Q}(q)}$-module. For each $\lambda \in X^+$, let $V(\lambda)$ (resp., $V^r(\lambda)$) denotes the left (resp., the right) simple module with highest weight λ. By Corollary 3.11, Lemma 3.12 and Lemma 4.5, it is easy to see that $\nabla(\lambda)_{\mathbb{Q}(q)}$ is identified with $V(\lambda)$ as left $\mathbb{U}_{\mathbb{Q}(q)}$-module, and $\nabla^r(\lambda)_{\mathbb{Q}(q)}$ is identified with $V^r(\lambda)$ as right $\mathbb{U}_{\mathbb{Q}(q)}$-module. By [34, Proposition 7.2.2], we have a Peter–Weyl decomposition

$$A_q(\mathfrak{g}) \cong \bigoplus_{\lambda\in X^+} V^r(\lambda)\otimes V(\lambda), \tag{4.7}$$

from which the following result follows easily.

Lemma 4.8. *With the above notations, we have that*

$$\iota_A\Big(\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)\Big) = A_q(\mathfrak{g}).$$

For later use, we denote by Φ_λ the canonical embedding induced from the isomorphism (4.7) from $V^r(\lambda)\otimes V(\lambda)$ into $A_q(\mathfrak{g})$ for each $\lambda \in X^+$.

In [34], Kashiwara introduced a crystal basis $B(A_q(\mathfrak{g}))$ of $A_q(\mathfrak{g})$. He proved that $B(A_q(\mathfrak{g}))$ has the crystal structure

$$B(A_q(\mathfrak{g})) = \bigsqcup_{\lambda\in X^+} \widetilde{B}(\lambda),$$

where $\widetilde{B}(\lambda) := B^r(\lambda)\otimes B(\lambda)$, and $B^r(\lambda)$ (resp., $B(\lambda)$) denotes the crystal basis of $V^r(\lambda)$ (resp., of $V(\lambda)$). For each $b\in\widetilde{B}(\lambda)$, let $\widetilde{G}(b)$ be the corresponding upper global crystal base of $A_q(\mathfrak{g})$. Recall that $\mathfrak{g}=\mathfrak{sp}_{2m}$ in this paper. By the results in [42], $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ has also a crystal base $\sqcup_{\lambda\in X^+}\widetilde{B}(\lambda)$ as well as a canonical base (or lower global crystal base) $\{\widehat{G}(b)|b\in\widetilde{B}(\lambda),\lambda\in X^+\}$. This canonical base is an $\mathscr{A}$-basis of $\dot{\mathbb{U}}_{\mathscr{A}}$. There exists a canonical coupling $\langle,\rangle_1$ between $A_q(\mathfrak{g})$ and $\dot{\mathbb{U}}_{\mathbb{Q}(q)}$ (cf. [35, (10.1.1)]) defined by

$$\langle\Phi_\lambda(u\otimes v),P\rangle_1 := (u,Pv),\ \forall\,\lambda\in X^+,u\in V^r(\lambda),v\in V(\lambda),P\in\dot{\mathbb{U}}_{\mathbb{Q}(q)},$$

where $(,)$ is the pairing between $V^r(\lambda)$ and $V(\lambda)$ introduced in [34, (7.1.2)]. By the results in [35], for each $\lambda\in X^+$, there is a bijection $\widetilde{\Psi}:\widetilde{B}(\lambda)\to\widetilde{B}(\lambda)$, such that

$$\langle\widetilde{G}(b'),\widehat{G}(b)\rangle_1=\delta_{b,\widetilde{\Psi}(b')},\ \ \forall\,b,b'\in\widetilde{B}(\lambda). \tag{4.9}$$

In [34], Kashiwara proved that the upper global crystal bases $\{\widetilde{G}(b)|b\in\widetilde{B}(\lambda),\lambda\in X^+\}$ form a $\mathbb{Q}[q,q^{-1}]$ basis of $A_q^{\mathbb{Q}}(\mathfrak{g})$, and he remarked that it is actually an $\mathscr{A}$-basis of $A_q^{\mathbb{Z}}(\mathfrak{g})$. For the reader's convenience, we include a proof here.

Lemma 4.10. *With the above notations, we have that the elements in the following set*

$$\Big\{\widetilde{G}(b)\ \Big|\ b\in\widetilde{B}(\lambda),\lambda\in X^+\Big\}$$

form an $\mathscr{A}$-basis of $A_q^{\mathbb{Z}}(\mathfrak{g})$.

Proof. First we show that for each $\lambda\in X^+$, and each $b\in\widetilde{B}(\lambda)$,

$$\widetilde{G}(b)\in A_q^{\mathbb{Z}}(\mathfrak{g}).$$

Let $P\in\mathbb{U}_{\mathscr{A}}$. By definition, for any $\mu\in X,\lambda\in X^+,u\in V^r(\lambda),v\in V(\lambda)$,

$$\langle\Phi_\lambda(u\otimes v),P1_\mu\rangle_1=(u,P1_\mu v).$$

It follows that for any given $\lambda\in X^+,u\in V^r(\lambda),v\in V(\lambda)$, there are only finitely many $\mu\in X$ such that

$$\langle\Phi_\lambda(u\otimes v),P1_\mu\rangle_1\neq 0.$$

Therefore, applying [34, Proposition 7.2.2] and (4.9), we get that

$$\langle\widetilde{G}(b),P\rangle=\sum_{\mu\in X}\langle\widetilde{G}(b),P1_\mu\rangle_1\in\mathscr{A},$$

which implies that $\widetilde{G}(b)\in A_q^{\mathbb{Z}}(\mathfrak{g})$.

Second, we want to show that

$$A_q^{\mathbb{Z}}(\mathfrak{g})\subseteq\Big\{x\in A_q(\mathfrak{g})\ \Big|\ \langle x,\dot{\mathbb{U}}_{\mathscr{A}}\rangle_1\in\mathscr{A}\Big\},$$

from which and together with (4.9) the lemma would follow immediately.

Let $f=\sum_{i=1}^t x_i\otimes y_i\in A_q^{\mathbb{Z}}(\mathfrak{g})$, where for each $1\leq i\leq t$, $x_i\in V^r(\lambda^{(i)})$, $y_i\in V(\lambda^{(i)})$, and $\lambda^{(i)}\in X^+$. Let $P\in\mathbb{U}_{\mathscr{A}}$, $\mu\in X$. Our purpose is to show that $\langle f,P1_\mu\rangle_1\in\mathscr{A}$. Let n be an integer which is large enough such that

(1) $|\mu|\leq n$, $|\mu|\equiv n\pmod{2\mathbb{Z}}$, and
(2) for each $1\leq i\leq t$, either $V(\lambda^{(i)})\subseteq V^{\otimes n-1}$ or $V(\lambda^{(i)})\subseteq V^{\otimes n}$.

We write $\{1,2,\cdots,t\}=I_1\sqcup I_2$, where

$$I_1:=\big\{1\leq i\leq t\bigm| V(\lambda^{(i)})\subseteq V^{\otimes n}\big\},\ \ I_2:=\big\{1\leq i\leq t\bigm| V(\lambda^{(i)})\subseteq V^{\otimes n-1}\big\}.$$

Let $p'_\mu\in\mathbb{U}_{\mathscr{A}}$ be as defined in the proof of Lemma 2.3 with respect to our fixed n and μ. By the definition of $A_q^{\mathbb{Z}}(\mathfrak{g})$, we have that $\langle f,Pp'_\mu\rangle\in\mathscr{A}$. Note that $p'_\mu V^{\otimes n-1}=0=1_\mu V^{\otimes n-1}$. It follows that

$$\begin{aligned}\langle f,P1_\mu\rangle_1&=\sum_{i\in I_1}(x_i,P1_\mu y_i)=\sum_{i\in I_1}(x_i,Pp'_\mu y_i)=\sum_{i\in I_1\sqcup I_2}(x_i,Pp'_\mu y_i)\\&=\langle f,Pp'_\mu\rangle\in\mathscr{A},\end{aligned}$$

as required. This completes the proof of the lemma. □

Let $\lambda\in X^+$. Let $\Delta_{\mathscr{A}}(\lambda)$ denote the standard $\mathscr{A}$-form of $V(\lambda)$, i.e., the $\mathbb{U}_{\mathscr{A}}(\mathfrak{g})$-submodule generated by the highest weight vector v_λ. Then $\Delta_{\mathscr{A}}(\lambda)$ is spanned by Lusztig's *canonical basis* as in [41, §14.4]. Note that the upper global crystal basis $\big\{G(b)\bigm| b\in B(\lambda)\big\}$ is Lusztig's dual canonical basis. The dual basis to the upper global crystal basis under the canonical contravariant form $(.,.)$ on $V(\lambda)$ is the *lower global crystal basis*, i.e., Lusztig's canonical basis, cf. [27] and [34, §3.3, 4.2.1]. Let

$$V_{\mathscr{A}}(\lambda):=\big\{v\in V(\lambda)\bigm|(v,w)\in\mathscr{A}\text{ for all }w\in\Delta_{\mathscr{A}}(\lambda)\big\}.$$

Then $V_{\mathscr{A}}(\lambda)\cong\mathrm{Hom}_{\mathscr{A}}(V_{\mathscr{A}},\mathscr{A}))$ is $\mathbb{U}_{\mathscr{A}}(\mathfrak{g})$-stable and spanned by the upper global crystal basis of $V(\lambda)$. Similar statement is true for $V^r_{\mathscr{A}}(\lambda),V^r(\lambda)$. For any field K which is an $\mathscr{A}$-algebra, we define

$$\begin{aligned}V^r_K(\lambda)\otimes V_K(\lambda)&:=(V^r_{\mathscr{A}}(\lambda)\otimes V_{\mathscr{A}}(\lambda))\otimes_{\mathscr{A}}K,\\ \Delta^r_K(\lambda)\otimes\Delta_K(\lambda)&:=(\Delta^r_{\mathscr{A}}(\lambda)\otimes\Delta_{\mathscr{A}}(\lambda))\otimes_{\mathscr{A}}K.\end{aligned}$$

Since $\Delta^r_K(\lambda)\otimes\Delta_K(\lambda)$ is a highest weight module generated by its highest weight vector (cf. [41, Theorem 14.4.11]) and has the same dimension as $V^r(\lambda)\otimes V(\lambda)$, it follows that $\Delta^r_K(\lambda)\otimes\Delta_K(\lambda)$ is isomorphic to the Weyl module of $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$ associated to (λ,λ). Note that $V^r_K(\lambda)\otimes V_K(\lambda)\cong\big(\Delta^r_K(\lambda)\otimes\Delta_K(\lambda)\big)^*$. Therefore, we have

Lemma 4.11. *Let $\lambda\in X^+$. For each field K which is an $\mathscr{A}$-algebra, $V^r_K(\lambda)\otimes V_K(\lambda)$ is isomorphic to the co-Weyl module of $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$ associated to (λ,λ).*

We recall the Bruhat order "$<$" on the set X^+ of dominant weights. Namely, $\lambda<\mu$ if and only if $\mu-\lambda\in\sum_{i=1}^m\mathbb{Z}^{\geq0}\alpha_i$. Note that $\lambda<\mu$ implies that $\lambda\lhd\mu$, where "$\lhd$" is the usual dominance order defined on the set of partitions (cf. [11]). In particular, $|\lambda|\leq|\mu|$. If $|\lambda|<|\mu|$, then $\lambda<\mu$ implies that $|\mu|-|\lambda|\in2\mathbb{N}$ and hence $\lambda\succ\mu$; if $|\lambda|=|\mu|$, then $\lambda\lhd\mu$ implies that $\lambda^t\rhd\mu^t$, which also implies that λ^t is bigger than μ^t under the lexicographical order, hence we still have $\lambda\succ\mu$. For each $\lambda\in X^+$, we define

$$\begin{aligned}A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq\lambda}&:=\mathscr{A}\text{-Span}\Big\{\widetilde{G}(b)\Bigm|\lambda\geq\mu\in X^+,b\in\widetilde{B}(\mu)\Big\},\\ A_q^{\mathbb{Z}}(\mathfrak{g})^{<\lambda}&:=\mathscr{A}\text{-Span}\Big\{\widetilde{G}(b)\Bigm|\lambda>\mu\in X^+,b\in\widetilde{B}(\mu)\Big\},\end{aligned}$$

and $A_q(\mathfrak{g})^{\leq\lambda}:=A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq\lambda}\otimes_{\mathscr{A}}\mathbb{Q}(q)$, $A_q(\mathfrak{g})^{<\lambda}:=A_q^{\mathbb{Z}}(\mathfrak{g})^{<\lambda}\otimes_{\mathscr{A}}\mathbb{Q}(q)$. For each $b\in\widetilde{B}(\lambda)$, let $G(b)$ be the corresponding upper global crystal base of $V^r(\lambda)\otimes V(\lambda)$. By construction, we know that $V^r(\lambda)\otimes V(\lambda)$ is spanned by $\big\{G(b)\bigm| b\in\widetilde{B}(\lambda)\big\}$. The following result is implicit in the proof of [34, Section 5,6]. The author is indebted to Professor Kashiwara for pointing out this to him.

Lemma 4.12. ([34]) *With the decomposition (4.7) in mind, we have that*

(1) *for each* $\lambda \in X^+, b \in \widetilde{B}(\lambda)$,

$$\widetilde{G}(b) \in G(b) + \sum_{\substack{b' \in \widetilde{B}(\mu) \\ \lambda > \mu \in X^+}} \mathbb{Q}(q) G(b');$$

(2) *for each* $\lambda \in X^+$,

$$A_q(\mathfrak{g})^{\leq \lambda} \cong \oplus_{\lambda \geq \mu \in X^+} V^r(\mu) \otimes V(\mu), \quad A_q(\mathfrak{g})^{< \lambda} \cong \oplus_{\lambda > \mu \in X^+} V^r(\mu) \otimes V(\mu).$$

In particular, $A_q(\mathfrak{g})^{\leq \lambda}/A_q(\mathfrak{g})^{<\lambda} \cong V^r(\lambda) \otimes V(\lambda)$;

(3) *both* $A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq \lambda}$ *and* $A_q^{\mathbb{Z}}(\mathfrak{g})^{<\lambda}$ *are* $\mathbb{U}_{\mathscr{A}}(\mathfrak{g} \oplus \mathfrak{g})$*-stable.*

For each integer $k \geq 0$, we define

$$A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq k} := \mathscr{A}\text{-Span}\Big\{ \widetilde{G}(b) \Bigm| b \in \widetilde{B}(\mu), \mu \in \Lambda^+(m, \leq k) \Big\}.$$

Then it is clear that

$$A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq k} = \sum_{\lambda \in \Lambda^+(m, \leq k)} A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq \lambda}.$$

In particular, $A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq k}$ is $\mathbb{U}_{\mathscr{A}}(\mathfrak{g} \oplus \mathfrak{g})$-stable. Recall that for any $\lambda \in X^+$, $w_0\lambda = (-\lambda_1, \cdots, -\lambda_m)$. Recall also our definitions of $\nabla_K^r(\lambda)$ and $\nabla_K(\lambda)$ in Section 3. When regarded as a left $\mathbb{U}_K(\mathfrak{g} \oplus \mathfrak{g})$-module, $\nabla_K^r(\lambda) \otimes \nabla_K(\lambda)$ is isomorphic to $\nabla_K(\lambda) \otimes \nabla_K(\lambda)$.

Lemma 4.13. *Let* K *be a field which is an* $\mathscr{A}$*-algebra and* $n \geq 0$ *be an integer. Let* $\lambda \in \Lambda^+(m, n)$*. If* N *is a nonzero* $\mathbb{U}_K(\mathfrak{g} \oplus \mathfrak{g})$*-submodule of* $\nabla_K^r(\lambda) \otimes \nabla_K(\lambda)$*, then* $\mathrm{pr}_\lambda\Big(\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)\Big) \otimes_{\mathscr{A}} 1_K \in N$*. In particular,* $\nabla_K^r(\lambda) \otimes \nabla_K(\lambda)$ *is isomorphic to the co-Weyl module of* $\mathbb{U}_K(\mathfrak{g} \oplus \mathfrak{g})$ *associated to* (λ, λ)*.*

Proof. We divide the proof into four steps:

Step 1. First, we claim that for any integer $t \geq 0$ and any $\mathbf{i}, \mathbf{j} \in I_\lambda^< \cup I_\lambda^{mys}$,

$$e_i^{(t)} T_q^\lambda(\mathbf{i}, \mathbf{j}) = \sum_{\substack{\mathbf{h} \in I_\lambda^< \\ \mathbf{h} \overset{(i,t)}{\rightsquigarrow} \mathbf{j}}} (-1)^{a'(\mathbf{h}, \mathbf{j})} q^{a(\mathbf{h}, \mathbf{j})} T_q^\lambda(\mathbf{i}, \mathbf{h}),$$

$$f_i^{(t)} T_q^\lambda(\mathbf{i}, \mathbf{j}) = \sum_{\substack{\mathbf{h} \in I_\lambda^< \\ \mathbf{h} \overset{(i,t)}{\rightsquigarrow} \mathbf{j}}} (-1)^{b'(\mathbf{h}, \mathbf{j})} q^{b(\mathbf{h}, \mathbf{j})} T_q^\lambda(\mathbf{i}, \mathbf{h}),$$

$$T_q^\lambda(\mathbf{i}, \mathbf{j}) f_i^{(t)} = \sum_{\substack{\mathbf{h} \in I_\lambda^< \\ \mathbf{h} \overset{(i,t)}{\rightsquigarrow} \mathbf{i}}} (-1)^{c'(\mathbf{i}, \mathbf{h})} q^{c(\mathbf{i}, \mathbf{h})} T_q^\lambda(\mathbf{h}, \mathbf{j}),$$

$$T_q^\lambda(\mathbf{i}, \mathbf{j}) e_i^{(t)} = \sum_{\substack{\mathbf{h} \in I_\lambda^< \\ \mathbf{h} \overset{(i,t)}{\rightsquigarrow} \mathbf{i}}} (-1)^{d'(\mathbf{i}, \mathbf{h})} q^{d(\mathbf{i}, \mathbf{h})} T_q^\lambda(\mathbf{h}, \mathbf{j}),$$

where $a'(\mathbf{h}, \mathbf{j}), a(\mathbf{h}, \mathbf{j}), b'(\mathbf{i}, \mathbf{h}), b(\mathbf{i}, \mathbf{h}), c'(\mathbf{h}, \mathbf{j}), c(\mathbf{h}, \mathbf{j}), d'(\mathbf{i}, \mathbf{h}), d(\mathbf{i}, \mathbf{h}) \in \mathbb{Z}$ and $\mathbf{h} \overset{(i,t)}{\rightsquigarrow} \mathbf{i}$ means that $\mathbf{h}$ differs from $\mathbf{i}$ on exactly t indices, on which each index i is changed into $i+1$ or $i-1$.

We only prove the first equality as the others can be proved in a similar way. By the same argument used in the proof of Lemma 3.4, we can assume without loss of generality that $\mathbf{i}, \mathbf{j} \in I_\lambda^<$. For any $f \in \mathbb{U}_{\mathbb{Q}(q)}$, we have that

$$\langle e_i^{(t)} T_q^\lambda(\mathbf{i}, \mathbf{j}), f \rangle_0 = \langle T_q^\lambda(\mathbf{i}, \mathbf{j}), f e_i^{(t)} \rangle_0 = \sum_{\mathbf{h} \in I_\lambda^<} \langle T_q^\lambda(\mathbf{i}, \mathbf{h}), f \rangle_0 \langle T_q^\lambda(\mathbf{h}, \mathbf{j}), e_i^{(t)} \rangle_0.$$

It suffices to show that $\langle T_q^\lambda(\mathbf{h},\mathbf{j}), e_i^{(t)}\rangle_0 \neq 0$ only if $\mathbf{h} \overset{(i,t)}{\rightsquigarrow} \mathbf{j}$, and in that case it is equal to $(-1)^{a'(\mathbf{h},\mathbf{j})}q^{a(\mathbf{h},\mathbf{j})}$ for some $a'(\mathbf{h},\mathbf{j}), a(\mathbf{h},\mathbf{j}) \in \mathbb{Z}$.

By definition,

$$\langle T_q^\lambda(\mathbf{h},\mathbf{j}), e_i^{(t)}\rangle_0 = \sum_{\substack{\mathbf{b}\in I(2m,n)\\ w\in\mathfrak{S}_{\lambda^t}}} (-q^2)^{-\ell(w)}\beta(w)_{\mathbf{h},\mathbf{b}}u_{\mathbf{b}}\Big(e_i^{(t)}v_{\mathbf{j}}\Big).$$

Note that $\mathbf{h},\mathbf{j} \in I_\lambda^<$. Recall the definition of $e_i^{(t)}$ (cf. [41, 3.1.5]) and the action of e_i given in Section 2. To calculate the above sum, it suffices to consider only those $\mathbf{b} \in I(2m,n)$ such that $\mathbf{b} \overset{(i,t)}{\rightsquigarrow} \mathbf{j}$ and the entries in each column of $T_{\mathbf{b}}^\lambda$ are weakly increasing from top to bottom. By the definition of $\beta(w)$, we know that $\beta(w)_{\mathbf{h},\mathbf{b}} \neq 0$ only if each column of $\mathbf{h}$ has the same set of entries as the corresponding column of $\mathbf{b}$. It follows that we can further restrict ourselves to those $\mathbf{b} \in I_\lambda^<$. Now applying [47, Lemma 11.8], we deduce that only the case when $w = 1, \mathbf{b} = \mathbf{h}$ can make contribution to the above sum, from which our claim follows immediately.

Step 2. We define $P(N) := \{(\mu,\nu) \in X \times X | N_{\mu,\nu} \neq 0\}$. We claim that for some $k_2,\cdots,k_m \in \mathbb{Z}$ and some $\nu \in X$, $(\nu, \lambda_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_m\varepsilon_m) \in P(N)$.

Recall that W_m is the Weyl group of type C_m. By [1, Lemma 1.13], we know that for any $\lambda \in X$,

$$\lambda \in P(N) \text{ implies that } (w_1\lambda, w_2\lambda) \in P(N) \text{ for any } w_1, w_2 \in W_m. \tag{4.14}$$

Therefore, it is equivalent to show that for some $k_1,\cdots,k_{m-1} \in \mathbb{Z}$ and some $\nu \in X$, $(\nu, k_1\varepsilon_1 + \cdots + k_{m-1}\varepsilon_{m-1} + \lambda_1\varepsilon_m) \in P(N)$. For simplicity, for each $\mathbf{i},\mathbf{j} \in I_\lambda^{mys}$, we write

$$v(\mathbf{i},\mathbf{j}) := \mathrm{pr}_\lambda\Big(\pi_C\big(T_q^\lambda(\mathbf{i},\mathbf{j})\big)\Big).$$

Then the elements in $\{v(\mathbf{i},\mathbf{j}) | \mathbf{i},\mathbf{j} \in I_\lambda^{mys}\}$ form a K-basis of $\nabla_K^r(\lambda) \otimes \nabla_K(\lambda)$.

Since N is a submodule of $\nabla_K^r(\lambda) \otimes \nabla_K(\lambda)$, any weight (ν,μ) of N satisfies $\lambda \geq \nu, \lambda \geq \mu$. In particular, $\lambda_1 \geq \nu_1, \lambda_1 \geq \mu_1$. By (4.14), we can assume without loss of generality that for some integers $k_1,\cdots,k_m$ with $0 < k_m \leq \lambda_1$ and some $\nu \in X$, $(\nu, k_1\varepsilon_1 + \cdots + k_m\varepsilon_m) \in P(N)$. Furthermore, we assume that our k_m is chosen such that k_m is as big as possible. For each weight vector $0 \neq x \in N$ with weight $(\nu, k_1\varepsilon_1 + \cdots + k_m\varepsilon_m)$, we can write

$$x = \sum_{\mathbf{i},\mathbf{j}\in I_\lambda^{mys}} C_{\mathbf{i},\mathbf{j}} v(\mathbf{i},\mathbf{j}),$$

for some $C_{\mathbf{i},\mathbf{j}} \in K$. Set

$$J(x) = \{\mathbf{j} \in I_\lambda^{mys} | C_{\mathbf{i},\mathbf{j}} \neq 0, \text{for some } \mathbf{i} \in I_\lambda^{mys}\}.$$

For each $\mathbf{j} \in J(x)$, the assumption $k_m > 0$ and the fact that $\mathbf{j} \in I_\lambda^{mys}$ imply that $j_1 = m$. Let $\mathfrak{t}_\lambda$ be the standard λ-tableau such that the numbers $1,2,\cdots,k$ entered in usual order along the successive columns of λ. We define

$$c_{\mathbf{j}} := \#\big\{1 \leq t \leq k \bigm| j_t = j_1 \text{ and } t \text{ sits in the first row of } \mathfrak{t}_\lambda\big\},$$
$$\widehat{c}_{\mathbf{j}} := \#\big\{1 \leq t \leq k \bigm| j_t = j_1' \text{ and } t \text{ sits in the first row of } \mathfrak{t}_\lambda\big\}.$$

We assert that $c_{\mathbf{j}} = k_m$ and $\widehat{c}_{\mathbf{j}} = 0$, $\forall \mathbf{j} \in J(x)$. In fact, if this is not true, then we can find some $\mathbf{j}_0 \in J(x)$ such that $0 < \widehat{c}_{\mathbf{j}_0} \geq \widehat{c}_{\mathbf{j}}$ for any $\mathbf{j} \in J(x)$. It is easy to see that $e_m^{(\widehat{c}_{\mathbf{j}_0})}x$ is a nonzero weight vector with weight $\Big(\nu, \sum_{t=1}^{m-1} k_t\varepsilon_t + (k_m + 2\widehat{c}_{\mathbf{j}_0})\varepsilon_m\Big)$, a contradiction to the maximality of k_m.

If $k_m = \lambda_1$, then we are done. Henceforth we assume $0 < k_m < \lambda_1$. For each $\mathbf{j} \in J(x)$, we define $b_{\mathbf{j}} \in \{1,2,\cdots,2m\}$ to be the least integer (with respect to the order " $\prec$ ") in the first row of $T_{\mathbf{j}}^{\lambda}$ such that $b_{\mathbf{j}} \succ m$. In particular, $b_{\mathbf{j}} \succ m'$ (because $\widehat{c}_{\mathbf{j}} = 0$). We set

$$J_1(x) := \big\{\mathbf{j} \in J(x) \bigm| b_{\mathbf{j}} \preceq b_{\mathbf{h}},\ \forall\, \mathbf{h} \in J(x)\big\}.$$

Let $b = b_{\mathbf{j}}$ for any $\mathbf{j} \in J_1(x)$. For any $\mathbf{j} \in J_1(x)$, we define

$$c''_{\mathbf{j}} := \#\big\{1 \leq t \leq n \bigm| j_t = b \text{ and } t \text{ sits in the first row of } \mathfrak{t}_{\lambda}\big\}.$$

Let

$$J_2(x) := \big\{\mathbf{j} \in J_1(x) \bigm| c''_{\mathbf{j}} \geq c''_{\mathbf{h}},\ \forall\, \mathbf{h} \in J_1(x)\big\}.$$

Let $c'' = c''_{\mathbf{j}}$ for any $\mathbf{j} \in J_2(x)$. Then $c'' > 0$. We define

$$z := \begin{cases} f_m^{(c''+k_m)} f_{m-1}^{(c'')} \cdots f_{b+1}^{(c'')} f_b^{(c'')}, & \text{if } 1 \leq b < m; \\ e_m^{(k_m+c'')} \cdots e_{b'+1}^{(c'')} e_{b'}^{(c'')} f_m^{(k_m)}, & \text{if } m < b < 2m. \end{cases}$$

Using the first two formulae we have given in Step 1 and the fact that $T_q^{\lambda}(\mathbf{i},\mathbf{h}) = 0$ whenever there are two identical indices appearing in an adjacent position in a column of $T_{\mathbf{h}}^{\lambda}$ ([47, Corollary 9.2]) as well as the definition of I_{λ}^{mys} and the action of each e_i, f_i given in Section 2, we deduce that $zx \in N$ is a nonzero weight vector with weight equal to either

$$\Bigg(\nu, \sum_{\substack{1\leq s\leq m\\ s\neq b, s\neq m}} k_s\varepsilon_s + (k_b - c'')\varepsilon_b - (k_m + c'')\varepsilon_m\Bigg),$$

or

$$\Bigg(\nu, \sum_{\substack{1\leq s\leq m\\ s\neq b', s\neq m}} k_s\varepsilon_s + (k_{b'} + c'')\varepsilon_{b'} + (k_m + c'')\varepsilon_m\Bigg).$$

In both case, applying (4.14) if necessary, we get a contradiction to the maximality of k_m. This proves our claim.

Step 3. We claim that for any integer $1 \leq t \leq m$, there exist some integers $k''_{t+1}, \cdots, k''_m \in \mathbb{Z}$ and some $\nu'' \in X$, such that $\Big(\nu'', \sum_{j=1}^{t} \lambda_j\varepsilon_j + \sum_{s=t+1}^{m} k''_s\varepsilon_s\Big) \in P(N)$.

We make induction on t. If $t = 1$, this is true by the result obtained in Step 2. Suppose that the claim is true for $t-1$. That is, for some integers $k_t, \cdots, k_m \in \mathbb{Z}$ and some $\nu \in X$, we have that

$$(\nu, \widetilde{\lambda}) := \Bigg(\nu, \sum_{j=1}^{t-1} \lambda_j\varepsilon_j + \sum_{s=t}^{m} k_s\varepsilon_s\Bigg) \in P(N).$$

Using (4.14), we can further assume that the weight $\widetilde{\lambda}$ is chosen such that

$$k_m = \max\big\{|k_s| \bigm| t \leq s \leq m\big\} \geq 0, \text{ and } k_m \text{ is as big as possible.} \tag{4.15}$$

We are going to show that for some integers $\widehat{k}_t, \cdots, \widehat{k}_{m-1} \in \mathbb{Z}$,

$$\Bigg(\nu, \sum_{j=1}^{t-1} \lambda_j\varepsilon_j + \lambda_t\varepsilon_m + \sum_{s=t}^{m-1} \widehat{k}_s\varepsilon_s\Bigg) \in P(N).$$

If this is true, then we can apply (4.14) again to get our claim.

If $\ell(\lambda) \leq t-1$, then there is nothing to prove. Henceforth, we assume that $\ell(\lambda) \geq t$. In particular, $\lambda_t > 0$. By (4.14), we know that $(\nu, \sum_{j=1}^{t-1} \lambda_j\varepsilon_j + k_m\varepsilon_t + \sum_{s=t+1}^{m} k_{s-1}\varepsilon_s) \in P(N)$, which implies that $\sum_{j=1}^{t-1} \lambda_j\varepsilon_j + k_m\varepsilon_t + \sum_{s=t+1}^{m} k_{s-1}\varepsilon_s \leq \lambda$. It follows that $0 \leq k_m \leq \lambda_t$. We assert that $k_m = \lambda_t$, from which our claim will follows immediately.

Suppose $k_m < \lambda_t$. For each nonzero weight vector x of weight $(\nu, \widetilde{\lambda})$ in N, we write

$$x = \sum_{\mathbf{i},\mathbf{j} \in I_\lambda^{mys}} C_{\mathbf{i},\mathbf{j}}(x) v(\mathbf{i}, \mathbf{j}).$$

for some $C_{\mathbf{i},\mathbf{j}}(x) \in K$. We define $J(x), c_{\mathbf{j}}, \widehat{c}_{\mathbf{j}}, b_{\mathbf{j}}, J_1(x), b, c''_{\mathbf{j}}, J_2(x), c''$ as in Step 2. We first show that $k_m > 0$. In fact, if $k_m = 0$, then by (4.15) we know that $k_i = 0$ for any $t \le i \le m$. Using the definition of I_λ^{mys}, it is easy to see that for each $\mathbf{j} \in J(x)$,

(4.16) $\quad \forall\, 1 \le l \le t-1,\ l$ appears λ_l times in $\mathbf{j}$ and l' does not appear in $\mathbf{j}$.

Since $\lambda_t > 0$, it follows that for any $\mathbf{j} \in J(x)$,

$$j_1 \in \{m, m', m-1, (m-1)', \cdots, t, t'\}.$$

If there exists $\mathbf{j} \in J(x)$ such that $j_1 = m$, then (as $k_m = 0$) $b_{\mathbf{j}} = m'$. In this case $b = m'$ and it is easy to see that $e_m^{(c'')} x \in N$ is a nonzero weight vector of weight $(\nu, \sum_{j=1}^{t-1} \lambda_j \varepsilon_j + 2c'' \varepsilon_m)$. Since $c'' > 0$, we get a contradiction to (4.15).

Henceforth we assume that $j_1 \neq m$ for any $\mathbf{j} \in J(x)$. In particular, $j_1 \succ m'$. Note that $j_1 \preceq t'$. We define j to be the unique integer such that $j = j_1$ for some $\mathbf{j} \in J(x)$ and $j \preceq h_1$, $\forall\, \mathbf{h} \in J(x)$. Set $c = \max\{c_{\mathbf{j}} | \mathbf{j} \in J(x), j_1 = j\}$. Then $c > 0$. If $t \le j < m$ then it is easy to see that $f_m^{(c)} \cdots f_{j+1}^{(c)} f_j^{(c)} x \in N$ is a nonzero weight vector of weight $(\nu, \sum_{s=1}^{t-1} \lambda_s \varepsilon_s - c\varepsilon_j - c\varepsilon_m)$; while if $m' < j \le t'$ then it is easy to see that $e_m^{(c)} \cdots e_{j'+1}^{(c)} e_{j'}^{(c)} x \in N$ is a nonzero weight vector of weight $(\nu, \sum_{s=1}^{t-1} \lambda_s \varepsilon_s + c\varepsilon_j + c\varepsilon_m)$. In both cases using (4.14), we get a contradiction to (4.15). Therefore, we must have $k_m > 0$. In particular, for any $\mathbf{j} \in J(x)$, $j_1 = m$.

The remaining argument is similar to that used in Step 2 with some slight modification. First, by the same argument used in Step 2, we can show that $j_1 = m$, $c_{\mathbf{j}} = k_m$ and $\widehat{c}_{\mathbf{j}} = 0$. In particular, $b \succ m'$. We claim that $b \preceq t'$. In fact, if $b \succeq t-1$, then for any $\mathbf{j} \in J(x)$, the first row of the remaining tableau after deleting all the entries of $T_{\mathbf{j}}^\lambda$ in $\{t-1, (t-1)', \cdots, 2, 2', 1, 1'\}$ has length k_m. On the other hand, we know that for any $\mathbf{j} \in I_\lambda^{mys}$ satisfying (4.16), the first row of the remaining tableau after deleting all of the entries of $T_{\mathbf{j}}^\lambda$ in $\{t-1, (t-1)', \cdots, 2, 2', 1, 1'\}$ must have length $\lambda_1 \ge \lambda_t$. It follows that $k_m = \lambda_t$, a contradiction to our assumption. This proves that $m' \prec b \preceq t'$.

Now we follow exactly the same argument used in Step 2 to define an element z. Note that the condition $m' \prec b \preceq t'$ ensures that $zx \neq 0$ is a weight vector in N with weight

$$\Big(\nu, \sum_{j=1}^{t-1} \lambda_j \varepsilon_j + \sum_{s=t}^{m} \widehat{k}_s \varepsilon_s\Big)$$

such that $|\widehat{k}_m| > k_m$. Applying (4.14) if necessary, we get a contradiction to (4.15). This proves our assertion that $k_m = \lambda_t$.

By induction and set $t = m$, we get that for some $\nu \in X$, $(\nu, \lambda) \in P(N)$.

Step 4. Starting from a nonzero weight vector $x \in N$ with weight (ν, λ) and Using a similar argument as used in Step 3, we can prove that $(\lambda, \lambda) \in P(N)$. Note that the (λ, λ)-weight space of $\nabla_K^r(\lambda) \otimes \nabla_K(\lambda)$ is one-dimensional and is spanned by $\mathrm{pr}_\lambda\Big(\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)\Big) \otimes_{\mathscr{A}} 1_K$. As N is a submodule of $\nabla_K^r(\lambda) \otimes \nabla_K(\lambda)$, we can deduce that

$$\mathrm{pr}_\lambda\Big(\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda, \mathbf{i}_\lambda)\big)\Big) \otimes_{\mathscr{A}} 1_K \in N.$$

This completes the proof of the lemma. □

Corollary 4.17. *Let K be a field which is an $\mathscr{A}$-algebra and $n \geq 0$ be an integer. Let $\lambda \in \Lambda^+(m,n)$. Then $\nabla_K(\lambda)$ is isomorphic to the co-Weyl module of $\mathbb{U}_K(\mathfrak{g})$ associated to λ.*

Proof. If L is a simple $\mathbb{U}_K(\mathfrak{g})$-submodule of $\nabla_K(\lambda)$, then $L \otimes L$ is a simple $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$-submodule of $\nabla^r_K(\lambda)\otimes\nabla_K(\lambda)$. By Lemma 4.13, $\nabla^r_K(\lambda)\otimes\nabla_K(\lambda)$ is isomorphic to the co-Weyl module of $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$ associated to (λ,λ). So it must have a unique simple $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$-socle. This implies that $\nabla_K(\lambda)$ also has a unique simple $\mathbb{U}_K(\mathfrak{g})$-socle. By the universal property of co-Weyl module, we see that there exists an embedding from $\nabla_K(\lambda)$ into the co-Weyl module $V_K(\lambda)$. Comparing their dimensions, we deduce that this embedding must be an isomorphism. □

Theorem 4.18. *With the above notations, we have that*

$$\iota_A\Big(\widetilde{A}^{sy}_{\mathscr{A}}(2m)\Big) = A_q^{\mathbb{Z}}(\mathfrak{g}),\ \ \iota_A\Big(\widetilde{A}^{sy}_{\mathscr{A}}(2m,\leq k)\Big) = A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq k},\ \forall\, k \geq 0.$$

In other words, the quantized coordinate algebra defined by Kashiwara and the quantized coordinate algebra $\widetilde{A}^{sy}_{\mathscr{A}}(2m)$ arising from a generalized FRT construction are isomorphic to each other as $\mathscr{A}$-algebras. Furthermore, we have the following commutative diagram

$$\begin{array}{ccc} \dot{\mathbb{U}}_{\mathbb{Q}(q)} & \xrightarrow{\ \mathrm{id}\ } & \dot{\mathbb{U}}_{\mathbb{Q}(q)} \\ \big\downarrow & & \big\downarrow \\ A_q(\mathfrak{g})^* & \xrightarrow{\ \iota_A^*\ } & (\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m))^* \end{array},$$

where the two vertical maps are induced by the two natural pairings $\langle,\rangle_0$, $\langle,\rangle_1$ respectively.

Proof. We first show that $\iota_A\Big(A^{sy}_{\mathscr{A}}(2m,\leq k)\Big) = A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq k}$ for any $k \geq 0$, from which the equality $\iota_A\Big(\widetilde{A}^{sy}_{\mathscr{A}}(2m)\Big) = A_q^{\mathbb{Z}}(\mathfrak{g})$ follows at once. We divide the proof into two steps:

Step 1. We claim that $\iota_A\Big(A^{sy}_{\mathscr{A}}(2m,\leq k)\Big) \subseteq A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq k}$.

By Lemma 4.8 and the bimodules decomposition we have discussed before, we have

$$\iota_A\Big(A^{sy}_{\mathscr{A}}(2m,\leq k)\Big) \subseteq \iota_A\Big(A^{sy}_{\mathbb{Q}(q)}(2m,\leq k)\Big) \subseteq A_q(\mathfrak{g})^{\leq k}.$$

For any integer $\lambda \in \Lambda^+(m,\leq k)$, and any $\mathbf{i},\mathbf{j} \in I_\lambda^{mys}$, it is easy to verify directly that

$$\langle \pi_C\big(T_q^\lambda(\mathbf{i},\mathbf{j})\big), \mathbb{U}_{\mathscr{A}}\rangle_0 \in \mathscr{A}.$$

It follows that $\iota_A\Big(A^{sy}_{\mathscr{A}}(2m,\leq k)\Big) \subseteq A_q^{\mathbb{Z}}(\mathfrak{g})$. Hence by Lemma 4.12,

$$\iota_A\Big(A^{sy}_{\mathscr{A}}(2m,\leq k)\Big) \subseteq A_q(\mathfrak{g})^{\leq k} \cap A_q^{\mathbb{Z}}(\mathfrak{g}) = A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq k},$$

as required.

Step 2. We now show that $\iota_A\Big(A^{sy}_{\mathscr{A}}(2m,\leq k)\Big) = A_q^{\mathbb{Z}}(\mathfrak{g})^{\leq k}$. Our strategy is to show that for any field K which is an $\mathscr{A}$-algebra, $\iota_K := \iota_A \otimes_{\mathscr{A}} 1_K$ is an injection from $A^{sy}_K(2m,\leq k)$ into $A_q^K(\mathfrak{g})^{\leq k}$.

For each $\lambda \in \Lambda^+(m,k)$, let b_λ be the unique element in $\widetilde{B}(\lambda)$ such that $G(b_\lambda) \in V^r(\lambda)\otimes V(\lambda)$ is a highest weight vector of weight (λ,λ). Note that

$$e_i G(b_\lambda) = 0,\ \ \forall\, e_i \in \mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g}\oplus\mathfrak{g}).$$

We use induction on dominant weights with respect to the order "$\prec$". If $\lambda \in X^+$ is maximal with respect to the order "$\prec$", then λ is minimal with respect to the order "$<$". Then

$$\nabla^r(\lambda)\otimes\nabla(\lambda)\subseteq A^{sy}_{\mathscr{A}}(2m,\leq k),\ \ V^r_{\mathscr{A}}(\lambda)\otimes V_{\mathscr{A}}(\lambda)\subseteq A^{\mathbb{Z}}_q(\mathfrak{g})^{\leq k}.$$

By Lemma 4.12, we also know that $\widetilde{G}(b_\lambda)=G(b_\lambda)$. In that case, as both $\widetilde{G}(b_\lambda)$ and $\iota_A\Big(\pi_C\big(T^\lambda_q(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)\Big)$ are the highest weight vectors of weight (λ,λ) in $A^{\mathbb{Z}}_q(\mathfrak{g})$, we deduce (by Corollary 4.5 and Lemma 3.12) that

$$\widetilde{G}(b_\lambda)=\pm q^a\iota_A\Big(\pi_C\big(T^\lambda_q(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)\Big),$$

for some $a\in\mathbb{Z}$. Since

$$\iota_A\Big(\nabla^r(\lambda)\otimes\nabla(\lambda)\Big)\subseteq A^{\mathbb{Z}}_q(\mathfrak{g})^{\leq k}\bigcap\big(V^r(\lambda)\otimes V(\lambda)\big)=V^r_{\mathscr{A}}(\lambda)\otimes V_{\mathscr{A}}(\lambda),$$

we deduce that for any field K which is an $\mathscr{A}$-algebra, ι induces a nonzero $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$-homomorphism $\iota_K:=\iota_A\otimes_{\mathscr{A}}1_K$ from $\nabla^r_K(\lambda)\otimes\nabla_K(\lambda)$ to $V^r_K(\lambda)\otimes V_K(\lambda)$. By Lemma 4.11 and Lemma 4.13, we know that both modules are co-Weyl modules of $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$ associated to (λ,λ). It follows that ι_K must always be an isomorphism. Hence ι_A must be an isomorphism as well. In particular,

$$\iota_A\Big(\nabla^r(\lambda)\otimes\nabla(\lambda)\Big)=V^r_{\mathscr{A}}(\lambda)\otimes V_{\mathscr{A}}(\lambda).$$

In general, let $\lambda\in X^+$, assume that for any field K which is an $\mathscr{A}$-algebra, ι_K is an injection from $A^{sy}_K(2m,\leq k)^{\succ\lambda}$ into

$$A^K_q(\mathfrak{g})^{\leq k}(\ngeq\lambda):=\sum_{\substack{\mu\in\Lambda^+(m,\leq k)\\ \mu\ngeq\lambda}}A^K_q(\mathfrak{g})^{\leq\mu}.$$

We want to prove that ι_K is also an injection from $A^{sy}_K(2m,\leq k)^{\succeq\lambda}$ into

$$A^K_q(\mathfrak{g})^{\leq k}(\ngtr\lambda):=\sum_{\substack{\mu\in\Lambda^+(m,\leq k)\\ \mu\ngtr\lambda}}A^K_q(\mathfrak{g})^{\leq\mu}.$$

By bimodules decomposition, Lemma 4.12 and definition, we know that

$$\iota_{\mathscr{A}}\Big(A^{sy}_{\mathscr{A}}(2m,\leq k)^{\succeq\lambda}\Big)\subseteq A^{\mathbb{Z}}_q(\mathfrak{g})\bigcap A_q(\mathfrak{g})^{\leq k}(\ngtr\lambda):=A^{\mathbb{Z}}_q(\mathfrak{g})^{\leq k}(\ngtr\lambda).$$

By the same argument as before, we know that

$$\widetilde{G}(b_\lambda)=\pm q^a\iota_A\Big(\pi_C\big(T^\lambda_q(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)\Big),$$

for some $a\in\mathbb{Z}$. Therefore, $\iota_{\mathscr{A}}$ induces a nonzero $\mathbb{U}_{\mathscr{A}}(\mathfrak{g}\oplus\mathfrak{g})$-homomorphism $\bar{\iota}_{\mathscr{A}}$ from

$$\nabla^r(\lambda)\otimes\nabla(\lambda)\cong A^{sy}_{\mathscr{A}}(2m,\leq k)^{\succeq\lambda}/A^{sy}_{\mathscr{A}}(2m,\leq k)^{\succ\lambda}$$

to

$$V^r_{\mathscr{A}}(\lambda)\otimes V_{\mathscr{A}}(\lambda)\cong A^{\mathbb{Z}}_q(\mathfrak{g})^{\leq k}(\ngtr\lambda)/A^{\mathbb{Z}}_q(\mathfrak{g})^{\leq k}(\ngeq\lambda).$$

For any field K which is an $\mathscr{A}$-algebra, we get by base change a nonzero $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$-homomorphism $\bar{\iota}_K$ from

$$\nabla^r_K(\lambda)\otimes\nabla_K(\lambda)\cong A^{sy}_K(2m,\leq k)^{\succeq\lambda}/A^{sy}_K(2m,\leq k)^{\succ\lambda}$$

to

$$V^r_K(\lambda)\otimes V_K(\lambda)\cong A^K_q(\mathfrak{g})^{\leq k}(\ngtr\lambda)/A^K_q(\mathfrak{g})^{\leq k}(\ngeq\lambda).$$

By Lemma 4.11 and Lemma 4.13, we know that both modules are co-Weyl modules of $\mathbb{U}_K(\mathfrak{g}\oplus\mathfrak{g})$ associated to (λ,λ). It follows that $\bar{\iota}_K$ must always be an isomorphism. It follows that ι_K must always be an injection, as required.

By induction, we know that for any field K which is an $\mathscr{A}$-algebra, ι_K is an injection from $A_K^{sy}(2m, \leq k)$ into $A_q^K(\mathfrak{g})^{\leq k}$. Comparing their dimensions, we can deduce that it must be an isomorphism, from which we can deduce that ι_A must be an isomorphism as well. This proves the first part of this Theorem.

It remains to prove the commutativity of the diagram. Note that as $\dot{\mathbb{U}}_{\mathbb{Q}(q)}(\mathfrak{g}\oplus\mathfrak{g})$-module, $\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)$ is generated by the elements $\pi_C\big(T_q^\lambda(i_\lambda, i_\lambda)\big)$, $\lambda\in\Lambda^+(m)$. Therefore, it suffices to show that for any $\lambda\in\Lambda^+(m,k)$, $k\geq 0$ and any $P\in\mathbb{U}_{\mathscr{A}}, \mu\in X$, $\langle\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big), P1_\mu\rangle_0=\langle\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big), P1_\mu\rangle_1$.

Note that there exists also a canonical coupling $\langle,\rangle_1$ between $A_q(\mathfrak{g})$ and $\mathbb{U}_{\mathbb{Q}(q)}$ (cf. [35]) defined by

$$\langle\Phi_\lambda(u\otimes v), P\rangle_1 := (u, Pv),\ \forall\,\lambda\in X^+, u\in V^r(\lambda), v\in V(\lambda), P\in\mathbb{U}_{\mathbb{Q}(q)},$$

where $(,)$ is the pairing between $V^r(\lambda)$ and $V(\lambda)$ introduced in [34, (7.1.2)]. By direct verification, we see that

$$\begin{aligned}\langle\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big), P1_\mu\rangle_0&=\delta_{\lambda,\mu}\langle\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big), P\rangle_0,\\ \langle\iota_A\big(\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)\big), P1_\mu\rangle_1&=\delta_{\lambda,\mu}\langle\iota_A\big(\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)\big), P\rangle_1.\end{aligned}$$

Therefore, it suffices to show that

$$\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big), P\rangle_0=\langle\iota_A\big(\pi_C\big(T_q^\lambda(\mathbf{i}_\lambda,\mathbf{i}_\lambda)\big)\big), P\rangle_1.$$

Using the PBW basis of $\mathbb{U}_{\mathbb{Q}(q)}$ and Lemma 4.5, we can reduce to the proof to the case where P is generated by $k_1^{\pm 1},\cdots,k_m^{\pm 1}$. In that case, the proof follows from an easy verification. This completes the proof of the Theorem. □

We remark that each integer $k\geq 0$, the dual of the $\mathbb{U}_{\mathbb{Q}(q)}\big(\mathfrak{g}\oplus\mathfrak{g}\big)$-module $A_q(\mathfrak{g})^{\leq k}$ together with the dual basis of its upper global crystal basis actually forms a based module in the sense of [41, 27.1.2].

Henceforth, we shall identify $A_q^{\mathbb{Z}}(\mathfrak{g})$ with $\widetilde{A}^{sy}_{\mathscr{A}}(2m)$ via ι_A. By Theorem 4.18, the quantized coordinate algebra $A_q^{\mathbb{Z}}(\mathfrak{g})$ was equipped with two bases. One is $\big\{\widetilde{G}(b)\big\}_{\lambda\in X^+,\ b\in\widetilde{B}(b)}$, another is $\big\{\pi_C(T_q^\lambda(\mathbf{i},\mathbf{j}))\big\}_{\lambda\in X^+,\ \mathbf{i},\mathbf{j}\in I_\lambda^{mys}}$. The transition matrix between these two bases must be invertible as a matrix over $\mathscr{A}$. Combining this with (4.9), we can find an $\mathscr{A}$-basis $\big\{\widehat{G}^\lambda_{\mathbf{i},\mathbf{j}}\big\}_{\lambda\in X^+,\ \mathbf{i},\mathbf{j}\in I_\lambda^{mys}}$ of $\dot{\mathbb{U}}_{\mathscr{A}}$, such that

$$\langle\pi_C\big(T_q^\mu(\mathbf{b},\mathbf{l})\big), \widehat{G}^\lambda_{\mathbf{i},\mathbf{j}}\rangle=\begin{cases}1, & \text{if } \lambda=\mu, \mathbf{i}=\mathbf{b}, \mathbf{j}=\mathbf{l},\\ 0, & \text{otherwise.}\end{cases}\tag{4.19}$$

for any $\lambda,\mu\in X^+$, $\mathbf{i},\mathbf{j}\in I_\lambda^{mys}$ and $\mathbf{b},\mathbf{l}\in I_\mu^{mys}$.

Proof of Theorem 1.4: For each integer $0\leq l\leq[n/2]$ and each $\lambda\in\Lambda^+(m,n-2l)$, $\mathbf{i},\mathbf{j}\in I_\lambda^{mys}$, we use $\Big(D^{\lambda,l}_{\mathbf{i},\mathbf{j}}\Big)^*$ to denote the base element of $S^{sy}_{\mathscr{A}}(2m,n)$ dual to the base element $D^{\lambda,l}_{\mathbf{i},\mathbf{j}}$ of $A^{sy}_{\mathscr{A}}(2m,n)$. We have the following commutative diagram:

$$\begin{array}{ccccc}\dot{\mathbb{U}}_{\mathbb{Q}(q)} & \xrightarrow{\widetilde{\psi}_C} & S^{sy}_{\mathbb{Q}(q)}(2m,n) & \xrightarrow{\text{id}} & S^{sy}_{\mathbb{Q}(q)}(2m,n)\\ {\scriptstyle\widetilde{\iota}_U}\big\downarrow & & & & \wr\big\downarrow\\ \Big(\widetilde{A}^{sy}_{\mathbb{Q}(q)}(2m)\Big)^* & \xrightarrow{\pi_C^*} & \Big(A^{sy}_{\mathbb{Q}(q)}(2m)\Big)^* & \longrightarrow & \Big(A^{sy}_{\mathbb{Q}(q)}(2m,n)\Big)^*\end{array}$$

By Theorem 4.18, (4.19) and the fact

$$\langle d_q f, \pi_C^*\widetilde{\iota}_U(u)\rangle_0=\langle\pi_C(f),\widetilde{\iota}_U(u)\rangle_0,\quad\forall\,u\in\dot{\mathbb{U}}_{\mathbb{Q}(q)}, f\in A^{sy}_{\mathbb{Q}(q)}(2m),$$

we deduce that

$$\widetilde{\psi}_C\big(\widehat{G}^{\lambda}_{\mathbf{i},\mathbf{j}}\big)=\begin{cases}\Big(D^{\lambda,l}_{\mathbf{i},\mathbf{j}}\Big)^*, & \text{if } |\lambda|\le n \text{ and } 2l:=n-|\lambda| \text{ is even;}\\ 0, & \text{otherwise.}\end{cases}$$

In particular, this shows that $\widetilde{\psi}_C\big(\dot{\mathbb{U}}_{\mathscr{A}}\big)=S^{sy}_{\mathscr{A}}(2m,n)$. By base change, we know that for any commutative $\mathscr{A}$-algebra K,

$$\Big(\widetilde{\psi}_C\downarrow_{\dot{\mathbb{U}}_{\mathscr{A}}}\otimes_{\mathscr{A}}K\Big)\big(\dot{\mathbb{U}}_K\big)=S^{sy}_K(2m,n).$$

This completes the proof of Theorem 1.4.

Corollary 4.20. *With the above notations, we have that*

$$\begin{aligned}\operatorname{Ker}\big(\widetilde{\psi}_C\downarrow_{\dot{\mathbb{U}}_{\mathscr{A}}}\big)&=\mathscr{A}\text{-}Span\Big\{\widehat{G}^{\lambda}_{\mathbf{i},\mathbf{j}}\;\Big|\;\begin{matrix}\mathbf{i},\mathbf{j}\in I^{mys}_{\lambda},\ \lambda\in\Lambda^+(m,k),\ k>n\\ or\ k<n\ and\ n-k\ is\ odd\end{matrix}\Big\},\\ &=\mathscr{A}\text{-}Span\Big\{\widehat{G}(b)\;\Big|\;\begin{matrix}b\in\widetilde{B}(\lambda),\lambda\in\Lambda^+(m,k),\ k>n\\ or\ k<n\ and\ n-k\ is\ odd\end{matrix}\Big\}.\end{aligned}$$

As a result, this is still true if we replace $\mathscr{A}$ by any commutative $\mathscr{A}$-algebra K.

Proof. It suffices to prove the second statement. Let π be the set of dominant weights in $V^{\otimes n}$. Let ${}_{\mathbb{Q}(q)}\mathbf{S}(\pi)$ be the generalized Schur algebra associated to π defined by Doty [16]. Then it is easy to check that the homomorphism $\widetilde{\psi}_C$ factors through ${}_{\mathbb{Q}(q)}\mathbf{S}(\pi)$. Let ${}_{\mathscr{A}}\mathbf{S}(\pi)$ be the $\mathscr{A}$-form of ${}_{\mathbb{Q}(q)}\mathbf{S}(\pi)$ defined in [16]. For any field K which is an $\mathscr{A}$-algebra, let ${}_K\mathbf{S}(\pi):={}_{\mathscr{A}}\mathbf{S}(\pi)\otimes_{\mathscr{A}}K$. Applying Theorem 1.4 and comparing dimensions we deduce that the natural homomorphism from ${}_K\mathbf{S}(\pi)$ to $\operatorname{End}_{\mathfrak{B}_n(\zeta^{2m+1},\zeta)}\big(V_K^{\otimes n}\big)$ is an isomorphism. So the same is true if we replace K by $\mathscr{A}$. Now the second statement follows directly from the definition of ${}_{\mathscr{A}}\mathbf{S}(\pi)$ given in [16]. □

Note that in the proof of the above corollary, we have also given a proof of Corollary 1.7.

5. Proof of Theorem 1.5 in the case where $m\ge n$

The purpose of this and the next section is to give a proof of Theorem 1.5. Before starting the proof, we make some reduction. By the results in [47], we know that the symplectic q-Schur algebra is stable under base change. That is, for any commutative $\mathscr{A}$-algebra K, there is a canonical isomorphism

$$S^{sy}_{\mathscr{A}}(2m,n)\otimes_{\mathscr{A}}K\cong S^{sy}_K(2m,n).$$

Furthermore, $S^{sy}_{\mathscr{A}}(2m,n)$ is an integral quasi-hereditary algebra. For any field K which is an $\mathscr{A}$-algebra, $V_K\cong\Delta_K(\varepsilon_1)\cong\nabla_K(\varepsilon_1)\cong L_K(\varepsilon_1)$ is a tilting module over $S^{sy}_K(2m,n)$. It follows that $V_K^{\otimes n}$ is also a tilting module over $S^{sy}_K(2m,n)$. Applying Theorem 1.4 and using [21, Lemma 4.4 (c)], we get that

$$\begin{aligned}\operatorname{End}_{\mathbb{U}_{\mathscr{A}}}\Big(V_{\mathscr{A}}^{\otimes n}\Big)\otimes_{\mathscr{A}}K&=\operatorname{End}_{S^{sy}_{\mathscr{A}}(2m,n)}\Big(V_{\mathscr{A}}^{\otimes n}\Big)\otimes_{\mathscr{A}}K\\ &\cong\operatorname{End}_{S^{sy}_K(2m,n)}\Big(V_K^{\otimes n}\Big)=\operatorname{End}_{\mathbb{U}_K}\Big(V_K^{\otimes n}\Big).\end{aligned}$$

In other words, the endomorphism algebra $\operatorname{End}_{\mathbb{U}_K}\Big(V_K^{\otimes n}\Big)$ is stable under base change. Therefore, to prove Theorem 1.5, it suffices to show that the natural homomorphism from $(\mathfrak{B}_n(-q^{2m+1},q)_{\mathscr{A}})^{\rm op}$ to $\operatorname{End}_{\mathbb{U}_{\mathscr{A}}}\Big(V_{\mathscr{A}}^{\otimes n}\Big)$ is surjective. Equivalently, it suffices to prove this is true with $\mathscr{A}$ replaced by any field K which is an $\mathscr{A}$-algebra.

In this section we shall give a proof of Theorem 1.5 in the case where $m\geq n$. *Henceforth, we shall assume that K is field, and ζ is the image of q in K and $m\geq n$.* Note that, in this case, by [29, Proposition 4.2],

$$\begin{aligned}&\dim\operatorname{End}_{\mathbb{U}_K}\Big(V_K^{\otimes n}\Big)=\operatorname{End}_{\mathbb{U}_{\mathbb{Q}(q)}}\Big(V_{\mathbb{Q}(q)}^{\otimes n}\Big)\\ =&\sum_{\substack{0\leq f\leq [n/2]\\ \lambda\vdash n-2f}}\Big(\dim D(\lambda^t)\Big)^2=\dim\mathfrak{B}_n(-q^{2m+1},q)=\dim\mathfrak{B}_n(-\zeta^{2m+1},\zeta).\end{aligned}$$

Therefore, in order to prove Theorem 1.5 in the case $m\geq n$, it suffices to show that φ_C is injective.

Our strategy to prove the injectivity of φ_C is similar to that used in [10, Section 3], but some extra technical difficulties do arise due to the complexity of the action on n-tensor space in this quantized case. First, we make some convention on the left and right place permutation actions. Throughout the rest of this paper, for any $\sigma,\tau\in\mathfrak{S}_n, a\in\{1,2,\cdots,n\}$, we set

$$(a)(\sigma\tau)=\big((a)\sigma\big)\tau,\quad (\sigma\tau)(a)=\sigma\big(\tau(a)\big).$$

In particular, we have $\sigma(a)=(a)\sigma^{-1}$. Therefore, for any $\mathbf{i}=(i_1,i_2,\cdots,i_n)\in I(2m,n), w\in\mathfrak{S}_n$,

$$\mathbf{i}w=(i_1,i_2,\cdots,i_n)w=(i_{w(1)},i_{w(2)},\cdots,i_{w(n)}),$$

which gives the so-called right place permutation action:

$$v_{\mathbf{i}}w=(v_{i_1}\otimes\cdots\otimes v_{i_n})w=v_{i_{w(1)}}\otimes\cdots\otimes v_{i_{w(n)}}=v_{\mathbf{i}w}.$$

For each $w\in\mathfrak{S}_n$, the element T_w (resp., $\widehat{T}_w$) is well defined in the BMW algebra $\mathfrak{B}_n(-\zeta^{2m+1},\zeta)$ (resp., in the Hecke algebra $\mathscr{H}_K(\mathfrak{S}_n)$) because of the braid relations. Precisely,

$$T_w=T_{j_1}T_{j_2}\cdots T_{j_k}\in\mathfrak{B}_n(-\zeta^{2m+1},\zeta),\quad \widehat{T}_w=\widehat{T}_{j_1}\widehat{T}_{j_2}\cdots\widehat{T}_{j_k}\in\mathscr{H}_K(\mathfrak{S}_n),$$

for any reduced expression $s_{j_1}s_{j_2}\cdots s_{j_k}$ of w.

Set

$$\begin{aligned}\widehat{\beta}:=\sum_{1\leq i\leq 2m}\Big(qE_{i,i}\otimes E_{i,i}\Big)+\sum_{\substack{1\leq i,j\leq 2m\\ i\neq j}}E_{i,j}\otimes E_{j,i}+\\ (q-q^{-1})\sum_{1\leq i<j\leq 2m}\Big(E_{i,i}\otimes E_{j,j}\Big).\end{aligned}$$

For $i=1,2,\cdots,n-1$, we set

$$\widehat{\beta}_i:=\mathrm{id}_{V^{\otimes i-1}}\otimes\widehat{\beta}\otimes\mathrm{id}_{V^{\otimes n-i-1}}.$$

By [33], the map $\widehat{\varphi}$ which sends each $\widehat{T}_i$ to $\widehat{\beta}_i$ for $i=1,2,\cdots.n-1$ can be naturally extended to a representation of $\mathscr{H}_{\mathscr{A}}(\mathfrak{S}_n)$ on $V_{\mathscr{A}}^{\otimes n}$.

Lemma 5.1. *Let $\mathbf{i}=(i_1,i_2,\cdots,i_n)\in I(2m,n)$. Suppose that $i_j\neq i'_k$ for any $1\leq j,k\leq n$. Then for any $w\in\mathfrak{S}_n$,*

$$v_{\mathbf{i}}T_w=v_{\mathbf{i}}\widehat{T}_w;$$

if furthermore $i_1>i_2>\cdots>i_n$, then $v_{\mathbf{i}}T_w=v_{\mathbf{i}w}$.

Proof. This follows directly from the definition of action (see the formulae given above (3.2)). □

Let q be an indeterminate over $\mathbb{Z}$. Let $\widetilde{R}$ be the ring

$$\widetilde{R} := \mathbb{Z}[r, r,^{-1}, q, q^{-1}, x]/\big((1-x)(q-q^{-1}) + (r - r^{-1})\big).$$

$\widetilde{R}$ naturally becomes an R-algebra (with z acting as $q - q^{-1}$). We regard $\mathscr{A}$ as an $\widetilde{R}$-algebra by sending r to $-q^{2m+1}$ and x to $1 - \sum_{i=-m}^{m} q^{2i}$. The resulting $\mathscr{A}$-algebra is exactly $\mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}}$. We refer the reader to the beginning of Section 3 to understand how $\mathscr{A}$ is an R-algebra. Let $\mathfrak{B}_n(r,q) := \mathfrak{B}_n(r,x,z) \otimes_R \widetilde{R}$. In [22], a cellular basis for $\mathfrak{B}_n(r,q)$ indexed by certain bitableaux was constructed by Enyang. The advantage of that basis is that it is explicitly described in terms of generators and amenable to computation. In the remaining part of this section we shall use Enyang's results in [22]. We first recall some notations and notions.

For each natural number n and each integer f with $0 \le f \le [n/2]$, we set $\nu = \nu_f := ((2^f), (n-2f))$, where $(2^f) := (\underbrace{2, 2, \cdots, 2}_{f \text{ copies}})$ and $(n-2f)$ are considered as partitions of $2f$ and $n - 2f$ respectively. So ν is a bipartition of n. Let $\mathfrak{t}^\nu$ be the standard ν-bitableau in which the numbers $1, 2, \cdots, n$ appear in order along successive rows of the first component tableau, and then in order along successive rows of the second component tableau. We define

$$\mathcal{D}_\nu := \left\{ d \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \mathfrak{t}^\nu d \text{ is row standard and the first column of } \mathfrak{t}^{(1)} \text{ is an} \\ \text{increasing sequence when read from top to bottom} \end{array} \right\}.$$

For each partition λ of $n - 2f$, we denote by $\mathrm{Std}(\lambda)$ the set of all the standard λ-tableaux with entries in $\{2f+1, \cdots, n\}$. The initial tableau $\mathfrak{t}^\lambda$ in this case has the numbers $2f+1, \cdots, n$ in order along successive rows.

Lemma 5.2. *Let $d \in \mathcal{D}_{\nu_f}$. Assume that $d = d's_j$ with $\ell(d) = \ell(d') + 1$, where $1 \le j \le n-1$. Then $d' \in \mathcal{D}_{\nu_f}$.*

Proof. Since $d = d's_j$ and $\ell(d) = \ell(d') + 1$, we get $(j)(d')^{-1} < (j+1)(d')^{-1}$. It follows that $j, j+1$ can not both sit in the second component of $\mathfrak{t}^\nu d'$. If $j, j+1$ sits in different components of $\mathfrak{t}^\nu d'$, then the lemma follows immediately. So it suffices to consider the case where both $j, j+1$ sits in the first component of $\mathfrak{t}^\nu d'$. But $d \in \mathcal{D}_f$, we deduce that j, $j+1$ must be located in different rows and can not be both located in the first column of $\mathfrak{t}^{(2^f)} d'$, which implies that $d' \in \mathcal{D}_{\nu_f}$ (as $\mathfrak{t}^\nu d'$ and $\mathfrak{t}^\nu d$ differ only in the positions of $j, j+1$). □

Recall that (cf. [22]) the map $T_i \mapsto T_i, E_i \mapsto E_i, \forall\, 1 \le i \le n-1$ extends naturally to an algebra anti-automorphism of $\mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}}$. We denote this anti-automorphism by "$*$".

Lemma 5.3. ([22]) *For each $\lambda \vdash n-2f$, $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$, let $m_{\mathfrak{s}\mathfrak{t}}$ denote the canonical image in $\mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}}$ of the corresponding Murphy basis element (cf.* [45]*) of the Hecke algebra $\mathscr{H}_{\mathscr{A}}(\mathfrak{S}_{\{2f+1,\cdots,n\}})$. Then the set*

$$\left\{ T_{d_1}^* E_1 E_3 \cdots E_{2f-1} m_{\mathfrak{s}\mathfrak{t}} T_{d_2} \;\middle|\; \begin{array}{l} 0 \le f \le [n/2],\ \lambda \vdash n-2f,\ \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda), \\ d_1, d_2 \in \mathcal{D}_\nu, \text{ where } \nu := ((2^f), (n-2f)) \end{array} \right\}$$

is a cellular basis of the BMW algebra $\mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}}$.

As a consequence, by combining Lemma 5.3 and [22, (3.3)], we have

Corollary 5.4. *With the above notations, the set*

$$\left\{ T_{d_1}^* E_1 E_3 \cdots E_{2f-1} T_\sigma T_{d_2} \;\middle|\; \begin{array}{c} 0 \le f \le [n/2],\ \sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}},\ d_1, d_2 \in \mathcal{D}_\nu, \\ \text{where } \nu := ((2^f), (n-2f)) \end{array} \right\}$$

is a basis of the BMW algebra $\mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}}$.

By base change, we can apply the previous results to the specialized algebra $\mathfrak{B}_n(-\zeta^{2m+1},\zeta)$. The main result in this section is

Theorem 5.5. *Suppose $m \geq n$. Then the natural homomorphism*

$$\varphi_C:\ \mathfrak{B}_n(-\zeta^{2m+1},\zeta)\to \operatorname{End}_K\big(V^{\otimes n}\big)$$

is injective. In particular, φ_C maps $\mathfrak{B}_n(-\zeta^{2m+1},\zeta)$ isomorphically onto

$$\operatorname{End}_{\mathbb{U}_K(\mathfrak{sp}_{2m})}\big(V^{\otimes n}\big).$$

To prove the theorem, it suffices to show the annihilator $\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n})$ is (0). Note that

$$\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(V^{\otimes n})=\bigcap_{v\in V^{\otimes n}}\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(v).$$

Thus it is enough to calculate $\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(v)$ for some set of chosen vectors $v\in V^{\otimes n}$ such that the intersection of annihilators is (0). We write

$$\operatorname{ann}(v)=\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}(v):=\big\{x\in \mathfrak{B}_n(-\zeta^{2m+1},\zeta)\ \big|\ vx=0\big\}.$$

For each integer f with $0\leq f\leq [n/2]$, we denote by $B^{(f)}$ the two-sided ideal of $\mathfrak{B}_n(-q^{2m+1},q)_{\mathscr{A}}$ generated by $E_1E_3\cdots E_{2f-1}$. Note that $B^{(f)}$ is spanned by all the basis elements whose indexing diagrams contain at least $2f$ horizontal edges (f edges in each of the top and the bottom rows in the diagrams). We recall a notion introduced in [10]. For $\mathbf{i}\in I(2m,n)$, an ordered pair (s,t) $(1\leq s<t\leq n)$ is called a *symplectic pair* in $\mathbf{i}$ if $i_s=(i_t)'$. Two ordered pairs (s,t) and (u,v) are called disjoint if $\big\{s,t\big\}\cap\big\{u,v\big\}=\emptyset$. We define the *symplectic length* $\ell_s(v_{\mathbf{i}})=\ell_s(\mathbf{i})$ to be the maximal number of disjoint symplectic pairs (s,t) in $\mathbf{i}$. Note that if $f>\ell_s(v_{\mathbf{i}})$, then clearly $B^{(f)}\subseteq \operatorname{ann}(v_{\mathbf{i}})$.

Lemma 5.6. $\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V^{\otimes n}\big)\subseteq B^{(1)}$.

Proof. Since $m\geq n$, the tensor $v:=v_n\otimes v_{n-1}\otimes\cdots\otimes v_1$ is defined. Note that $i\neq j'$ for any $i,j\in\{1,2,\cdots,n\}$. Applying Lemma 5.1, we deduce that $vT_w=v\widehat{T}_w$ for any $w\in\mathfrak{S}_n$. Now $B^{(1)}$ is contained in the annihilator of $v\widehat{T}_w$, hence is contained in the intersection of all annihilators of $v\widehat{T}_w$, as w ranges over $\mathfrak{S}_n$. Hence $B^{(1)}$ annihilates the subspace S spanned by the $vT_w=v\widehat{T}_w$, where w runs through $\mathfrak{S}_n$.

On the other hand, since $m\geq n$, it is well known (cf. [21]) that the annihilator of v in the Hecke algebra $\mathscr{H}_K(\mathfrak{S}_n)$ is $\{0\}$. Therefore, we conclude that $\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V^{\otimes n}\big)\subseteq B^{(1)}$. □

Suppose that we have already shown $\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V^{\otimes n}\big)\subseteq B^{(f)}$ for some natural number $f\geq 1$. We want to show $\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V^{\otimes n}\big)\subseteq B^{(f+1)}$. If $f>[n/2]$ then $\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V^{\otimes n}\big)\subseteq B^{(f)}=0$ implies that $\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V^{\otimes n}\big)=0\subseteq B^{(f+1)}$ and we are done. Thus we may assume $f\leq[n/2]$.

For $\mathbf{i}:=(i_1,\cdots,i_n)\in I(2m,n)$, we define $\mathrm{WT}(\mathbf{i})=\lambda=(\lambda_1,\cdots,\lambda_{2m})$, where λ_j is the number of times that v_j occurs as tensor factor in $v_{\mathbf{i}}$ for each $1\leq j\leq 2m$. We call $\mathrm{WT}(\mathbf{i})$ the GL_{2m}-weight of $v_{\mathbf{i}}$. Note that for a given composition λ of n, the simple tensors of GL_{2m}-weight λ span a $\mathscr{H}_K(\mathfrak{S}_n)$-submodule M^{λ} of $V^{\otimes n}$, thus

$$V^{\otimes n}=\bigoplus_{\lambda\in\Lambda(2m,n)}M^{\lambda}$$

as $\mathscr{H}_K(\mathfrak{S}_n)$-module, where $\Lambda(2m,n)$ denotes the set of compositions of n into not more than $(2m)$ parts. It is well-known that M^{λ} is isomorphic to the permutation representation of $\mathscr{H}_K(\mathfrak{S}_n)$ corresponding to λ.

As a consequence, each element $v \in V^{\otimes n}$ can be written as a sum

$$v = \sum_{\lambda\in\Lambda(2m,n)} v_\lambda$$

for uniquely determined $v_\lambda \in M^\lambda$.

Following [10], we consider the subgroup Π of $\mathfrak{S}_{\{1,\cdots,2f\}} \leq \mathfrak{S}_n$ permuting the rows of $\mathfrak{t}^{\nu^{(1)}}$ but keeping the entries in the rows fixed. The group Π normalizes the stabilizer $\mathfrak{S}_{(2^f)}$ of $\mathfrak{t}^{\nu^{(1)}}$ in $\mathfrak{S}_{2f}$. We set $\Psi := \mathfrak{S}_{(2^f)} \rtimes \Pi$. By [10, Lemma 3.7], we have

$$\mathfrak{S}_{2f} = \bigsqcup_{d\in\mathcal{D}_f} \Psi d,$$

where "$\sqcup$" means a disjoint union. We set $\mathcal{D}_f := \mathcal{D}_{\nu_f} \bigcap \mathfrak{S}_{2f}$.

Lemma 5.7. *Let $d \in \mathcal{D}_f$. Then for any $w \in \Psi$, $\ell(wd) \geq \ell(d)$.*

Proof. Let $w \in \Psi$. By definition, we can write $w = w''w'$, where $w'' \in \mathfrak{S}_{(2^f)}, w' \in \Pi$. Note that $\mathfrak{S}_{(2^f)}$ is generated by $s_1, s_3, \cdots, s_{2f-1}$, and Π is generated $\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_{f-1}$, where $\widetilde{s}_i := s_{2i}s_{2i-1}s_{2i+1}s_{2i}$ for $i = 1, 2, \cdots, f-1$.

We claim that $\ell(w) = \ell(w''w'd) \geq \ell(w'd)$. In fact, this follows easily from the counting of the number of inversions and the fact that for any $\sigma \in \mathfrak{S}_n$,

$$\ell(\sigma) = \big\{(i,j) \bigm| 1 \leq i < j \leq n, (i)\sigma > (j)\sigma\big\}.$$

Therefore, it remains to show that $\ell(w'd) \geq \ell(d)$.

Note that the subgroup generated by $\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_{f-1}$ is isomorphic to the symmetric group $\mathfrak{S}_f$. We use $\widetilde{\ell}$ to denote the length function of $\mathfrak{S}_f$ with respect to the generators $\widetilde{s}_i, i = 1, \cdots, f-1$. We use induction on $\widetilde{\ell}(w')$. If $\widetilde{\ell}(w') = 1$, then $w' = \widetilde{s}_i = s_{2i}s_{2i-1}s_{2i+1}s_{2i}$. In this case, our claim $\ell(w'd) \geq \ell(d)$ follows directly from the counting of the number of inversions. Suppose that for any $w' \in \Pi$ with $\widetilde{\ell}(w') = k-1$, we have $\ell(w'd) \geq \ell(d)$. Let $w' \in \Pi$ with $\widetilde{\ell}(w') = k$. We can write $w' = \widetilde{s}_j u'$, where $1 \leq j \leq f-1$, such that $\widetilde{\ell}(w') = \widetilde{\ell}(u') + 1$. Now counting the number of inversions, it is easy to see that $\ell(w'd) = \ell(\widetilde{s}_j u'd) \geq \ell(u'd)$. On the other hand, by induction hypothesis, $\ell(u'd) \geq \ell(d)$. Therefore, $\ell(w'd) \geq \ell(d)$, as required. This completes the proof of the lemma. □

Let $\mathcal{P}_f := \{(i_1, \cdots, i_{2f}) | 1 \leq i_1 < \cdots < i_{2f} \leq n\}$. For each $J \in \mathcal{P}_f$, we use d_J to denote the unique element in $\mathcal{D}_{\nu_f}$ such that the first component of $\mathfrak{t}^\nu d_J$ is the tableau obtained by inserting the integers in J in increasing order along successive rows in $\mathfrak{t}^{\nu^{(1)}}$. Let $\widetilde{\mathcal{D}}_{(2f,n-2f)}$ be the set of distinguished right coset representatives of $\mathfrak{S}_{(2f,n-2f)}$ in $\mathfrak{S}_n$. Clearly $d_J \in \widetilde{\mathcal{D}}_{(2f,n-2f)}$, and every element of $\widetilde{\mathcal{D}}_{(2f,n-2f)}$ is of the form d_J for some $J \in \mathcal{P}_f$. The following result is well known.

Lemma 5.8. *Let $J = (i_1, i_2, \cdots, i_{2f}) \in \mathcal{P}_f$. Then*

$$(s_{2f}s_{2f+1}\cdots s_{i_{2f}-1})(s_{2f-1}s_{2f}\cdots s_{i_{2f-1}-1})\cdots(s_2s_3\cdots s_{i_2-1})(s_1s_2\cdots s_{i_1-1})$$

is a reduced expression of d_J.

By [10, Lemma 3.8], $\mathcal{D}_{\nu_f} = \bigsqcup_{J\in\mathcal{P}_f} \mathcal{D}_f d_J$.

Definition 5.9. *We define*

$$\begin{aligned}\mathbf{c} &= (c_1, c_2, \cdots, c_{2f})\\ &= \Big((m-f+1)', \cdots, (m-1)', m', m, m-1, \cdots, m-f+1\Big).\end{aligned}$$

Note that $m-f+1 < m-f+2 < \cdots < m < m' < \cdots < (m-f+2)' < (m-f+1)'$. Let d_0 be the unique element in $\mathfrak{S}_{2f}$ such that

$$(a)d_0 = \begin{cases} (a+1)/2, & \text{if } a \in \{1,3,\cdots,2f-1\},\\ 2f+1-a/2, & \text{if } a \in \{2,4,\cdots,2f\}.\end{cases}$$

Then $d_0 \in \mathcal{D}_f$. Counting the number of inversions, we deduce that $\ell(d_0) = f(f-1)$. On the other hand, by direct verification, we know that

$$d_0 = (s_{2f-2}s_{2f-1})(s_{2f-4}s_{2f-3}s_{2f-2}s_{2f-1})\cdots(s_2s_3\cdots s_{2f-2}s_{2f-1}).$$

It follows that

$$(s_{2f-2}s_{2f-1})(s_{2f-4}s_{2f-3}s_{2f-2}s_{2f-1})\cdots(s_2s_3\cdots s_{2f-2}s_{2f-1}) \tag{5.10}$$

is a reduced expression of d_0.

Definition 5.11. *We define*

$$v_{\mathbf{c}_0} = v_{\mathbf{c}}d_0^{-1} = v_{(m-f+1)'} \otimes v_{m-f+1} \otimes \cdots \otimes v_{(m-1)'} \otimes v_{m-1} \otimes v_{m'} \otimes v_m.$$

Lemma 5.12. *Let $d \in \mathcal{D}_f$, $J_0 := \{n-2f+1, n-2f+2, \cdots, n\}$.*

1. *There exists $w \in \mathfrak{S}_n$, such that $d_0 = dw$ and $\ell(d_0) = \ell(d) + \ell(w)$;*
2. *For any $J \in \mathcal{P}_f$, there exists $w' \in \mathfrak{S}_n$, such that $d_{J_0} = d_Jw'$ with $\ell(d_{J_0}) = \ell(d_J) + \ell(w')$;*
3. *for any $d \in \mathcal{D}_{\nu_f}$ with $d \neq d_0d_{J_0}$, there exists integer $1 \leq j \leq n-1$ such that $ds_j \in \mathcal{D}_{\nu_f}$ and $\ell(ds_j) = \ell(d)+1$.*

Proof. The statement (2) is a well-known result, see e.g., [11]. We only give the proof of the statements (1) and (3).

First, we claim that there exists an element $w_1 \in \mathfrak{S}_{2f}$, such that $dw_1 \in \mathcal{D}_f$, $\ell(dw_1) = \ell(d) + \ell(w_1)$ and the numbers $1,2,\cdots,f$ are located in the first column of $\mathfrak{t}^{(2^f)}$. In fact, let $1 \leq a \leq f$ be the smallest integer which is not located in the first column of $\mathfrak{t}^{(2^f)}d$, then for any integer b which is located in the first column of $\mathfrak{t}^{(2^f)}d$, we must have $b \geq a+1$. Furthermore, any integer between a and $b-1$ can only be located in the first $a-1$ rows of the second column of $\mathfrak{t}^{(2^f)}$. Now let $w_1 := s_{b-1}s_{b-2}\cdots s_a$. It is easy to see that $dw_1 \in \mathcal{D}_f$, $\ell(dw_1) = \ell(d)+\ell(w)$, and $1,2,\cdots,a$ are located in the first column of $\mathfrak{t}^{(2^f)}$. Using induction on a, we can find an element $w' \in \mathfrak{S}_{2f}$, such that $dw' \in \mathcal{D}_f$, $\ell(dw') = \ell(d)+\ell(w')$ and the numbers $1,2,\cdots,f$ are located in the first column of $\mathfrak{t}^{(2^f)}$. Let $w_{0,f}$ be the unique element in $\mathfrak{S}_{2f}$ such that

$$(a)w_{0,f} = \begin{cases} (a+1)/2, & \text{if } a \in \{1,3,\cdots,2f-1\},\\ f+a/2, & \text{if } a \in \{2,4,\cdots,2f\}.\end{cases}$$

Then, $dw' = w_{0,f}w_1'$ for some $w_1' \in \mathfrak{S}_{\{f+1,f+2,\cdots,2f\}}$.

Let $w_0' \in \mathfrak{S}_{\{f+1,f+2,\cdots,2f\}}$ be defined by

$$(f+1,f+2,\cdots,2f)w_0' = (2f,2f-1,\cdots,f+1).$$

Then w_0' is the unique longest element in $\mathfrak{S}_{\{f+1,f+2,\cdots,2f\}}$. It is well known that there exists $w'' \in \mathfrak{S}_{\{f+1,f+2,\cdots,2f\}}$ such that $w_0' = w_1'w''$ and $\ell(w_0') = \ell(w_1')+\ell(w'')$. It is clear that

$$\begin{aligned}\ell(dw'w'') &= \ell(w_{0,f}w_0') = \ell(w_{0,f}) + \ell(w_0') = \ell(w_{0,f}) + \ell(w_1') + \ell(w'')\\ &= \ell(w_{0,f}w_1') + \ell(w'') = \ell(dw') + \ell(w'') = \ell(d) + \ell(w') + \ell(w'').\end{aligned}$$

Therefore,

$$\ell(d) + \ell(w') + \ell(w'') = \ell(dw'w'') \leq \ell(d) + \ell(w'w'') \leq \ell(d) + \ell(w') + \ell(w''),$$

which forces $\ell(w'w'') = \ell(w') + \ell(w'')$. Since $d_0 = dw_{0,f}w'_0 = dw_{0,f}w'_1w'' = d(w'w'')$. The statement (1) is proved.

Let $d \in \mathcal{D}_{\nu_f}$ with $d \neq d_0 d_{J_0}$. We can write $d = d_1 d_{J_1}$, where $d_1 \in \mathcal{D}_f, J_1 \in \mathcal{P}_f$. For any $d' \in \widetilde{\mathcal{D}}_{(2f,n-2f)}$ and any integer $1 \leq j \leq n-1$ satisfying $d' = d''s_j$ and $\ell(d') = \ell(d'') + 1$, it is well known that $d'' \in \widetilde{\mathcal{D}}_{(2f,n-2f)}$. If $J_1 \neq J_0$, then the statement (3) follows directly from this well-known fact and the statement (2). Now we assume $J_1 = J_0$. Then $d_1 \neq d_0$. By statement (1), we can find $s_l \in \mathfrak{S}_{2f}$ such that $d_1 s_l \in \mathcal{D}_f$ and $\ell(d_1 s_l) = \ell(d_1) + 1$. Then $d_{J_0}^{-1} s_l d_{J_0} = s_j$ for some $s_j \in \mathfrak{S}_{(n-2f,2f)}$. Note that

$$\begin{aligned}\ell(ds_j) &= \ell(d_1 d_{J_0} s_j) = \ell(d_1 s_l d_{J_0}) = \ell(d_1 s_l) + \ell(d_{J_0}) = \ell(d_1) + 1 + \ell(d_{J_0})\\ &= \ell(d) + 1,\end{aligned}$$

as required. □

We define

$$I_f := \Big\{ \mathbf{b} = (b_1, \cdots, b_{n-2f}) \;\Big|\; 1 \leq b_{n-2f} < \cdots < b_2 < b_1 \leq m - f \Big\}.$$

It is clear that $\ell_s(v_{\mathbf{c}} \otimes v_{\mathbf{b}}) = f$ for all $\mathbf{b} \in I_f$.

For an arbitrary element $v \in V^{\otimes n}$, we say the simple tensor $v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_n}$ is involved in v, if $v_{\mathbf{i}}$ has nonzero coefficient in writing v as linear combination of the basis $\{v_{\mathbf{j}} \mid \mathbf{j} \in I(2m, n)\}$ of $V^{\otimes n}$. For later use, we note the following very useful fact: for any $(i_1, i_2), (j_1, j_2) \in I(2m, 2)$,

(5.13) $\quad v_{j_1} \otimes v_{j_2}$ is involved in $(v_{i_1} \otimes v_{i_2})\beta'$ only if $j_1 \leq i_2$ and $j_2 \geq i_1$.

Lemma 5.14. *Let $s, i_1, \cdots, i_a$ be integers such that*

(1) $1 \leq s \leq f$*;*
(2) $\ell_s(i_1, \cdots, i_a) = 0$*;*
(3) *for each integer $1 \leq t \leq a$, either $1 \leq i_t < m - f + 1$ or $m' \leq i_t \leq (m - f + s + 1)'$.*

Let d be a distinguished right coset representative of

$$\mathfrak{S}_{(1,2,\cdots,2s)} \times \mathfrak{S}_{(2s+1,\cdots,2s+a)}$$

in $\mathfrak{S}_{2s+a}$. Let $J := \{a+1, a+2, \cdots, a+2s\}$. Let

$$\widetilde{v} = v_{i_1} \otimes \cdots \otimes v_{i_a},$$
$$\widetilde{w} = v_{(m-f+1)'} \otimes v_{(m-f+2)'} \otimes \cdots \otimes v_{(m-f+s)'} \otimes v_{m-f+s} \otimes \cdots \otimes v_{m-f+1}.$$

Then

$$\big(\widetilde{v} \otimes \widetilde{w}\big) T_{d^{-1}} = \zeta^z \delta_{d,d_J} \widetilde{w} \otimes \widetilde{v} + \sum_{\mathbf{u} \in I(2m,2s+a)} A_{\mathbf{u}} v_{u_1} \otimes \cdots \otimes v_{u_{2s+a}},$$

for some $z \in \mathbb{Z}$, and $A_{\mathbf{u}} \neq 0$ only if

(4) $\ell_s(u_1, \cdots, u_{2s}) < s$*; and*
(5) *any integer x with $(m - f + 1)' < x \leq 2m$ or $m - f + s + 1 \leq x \leq m$ does not appear in $(u_1, \cdots, u_{2s})$.*

Proof. We write

$$j_1 = (1)d,\ j_2 = (2)d,\ \cdots j_{2s} = (2s)d.$$

Then $1 \leq j_1 < j_2 < \cdots < j_{2s} \leq 2s + a$, and $d = d_J$ if and only if $j_t = a + t$ for each integer $1 \leq t \leq 2s$. Note that

$$(s_{j_1-1} \cdots s_2 s_1)(s_{j_2-1} \cdots s_3 s_2) \cdots (s_{j_{2s}-1} \cdots s_{2s+1} s_{2s})$$

is a reduced expression of d^{-1}.

If $\ell(d)=1$, i.e., $d=1$, then there is nothing to prove. In general, let
$$d'^{-1}=(s_{j_2-1}\cdots s_3s_2)(s_{j_3-1}\cdots s_4s_3)\cdots(s_{j_{2s-1}-1}\cdots s_{2s}s_{2s-1}).$$
Then
$$d^{-1}=(s_{j_1-1}\cdots s_2s_1)d'^{-1}(s_{j_{2s}-1}\cdots s_{2s+1}s_{2s}),\ \ell(d)=\ell(d')+j_1+j_{2s}-2s-1,$$
and d' is a distinguished right coset representative of
$$\mathfrak{S}_{\{2,3,\cdots,2s-1\}}\times\mathfrak{S}_{\{2s,\cdots,2s+a-1\}}$$
in $\mathfrak{S}_{\{2,3,\cdots,2s+a-1\}}$. Note that since $j_1\leq a+1$, any simple tensor involved in
$$\big(\widetilde{v}\otimes\widetilde{w}\big)T_{j_1-1}\cdots T_2T_1$$
is of the form
$$v_{\widehat{i}_1}\otimes v_{\widehat{i}_2}\otimes\cdots\otimes v_{\widehat{i}_a}\otimes v_{\widehat{i}_{a+1}}\otimes\widetilde{w}'\otimes v_{(m-f+s)'},$$
where
$$\widetilde{w}'=v_{(m-f+1)'}\otimes\cdots\otimes v_{(m-f+s)'}\otimes v_{m-f+s}\otimes\cdots\otimes v_{m-f+1}.$$
It suffices to consider the following three cases:

Case 1. $1\leq j_1\leq a$. Then $\widehat{i}_{a+1}=m-f+s$. Since $\ell_s(i_1,\cdots,i_a)=0$, by the definition of β', it is easy to see that
$$\mathrm{WT}(i_1,\cdots,i_a)=\mathrm{WT}(\widehat{i}_1,\cdots,\widehat{i}_a).$$
In particular, either $1\leq\widehat{i}_1<m-f+1$ or $m'<\widehat{i}_1\leq(m-f+s+1)'$. We define
$$\widetilde{v}''=v_{\widehat{i}_2}\otimes v_{\widehat{i}_3}\otimes\cdots\otimes v_{\widehat{i}_a},\ \ \widetilde{w}''=\widetilde{w}.$$
Then our conclusion follows easily from induction on a.

Case 2. $j_1=a+1$ and $\widehat{i}_1\neq(m-f+1)'$. Then we must have $j_t=a+t$ for each integer $1\leq t\leq 2s$. By the definition of β', it is easy to see that $\widehat{i}_1\neq(m-f+1)'$ implies that either $1\leq\widehat{i}_1<m-f+1$ or $m'<\widehat{i}_1\leq(m-f+s+1)'$. We define
$$\widetilde{v}''=v_{\widehat{i}_2}\otimes v_{\widehat{i}_3}\otimes\cdots\otimes v_{\widehat{i}_a}\otimes v_{\widehat{i}_{a+1}},\ \ \widetilde{w}''=\widetilde{w}'.$$
By induction on $\ell(d)$, we deduce that
$$\big(\widetilde{v}''\otimes\widetilde{w}''\big)T_{(d')^{-1}}=\zeta^z\widetilde{w}''\otimes\widetilde{v}''+\sum_{\mathbf{u}\in I(2m,2s+a-2)}A'_{\mathbf{u}}v_{u_1}\otimes\cdots\otimes v_{u_{2s+a-2}},$$
for some $z\in\mathbb{Z}$, and $A_{\mathbf{u}}\neq 0$ only if

(a1) $\ell_s(u_1,\cdots,u_{2s-2})<s-1$; and

(a2) any integer x with $1\leq x<m-f+1$ or $m'\leq x\leq(m-f+s)'$ does not appear in $(u_1,\cdots,u_{2s-2})$.

It remains to consider
$$\Big(v_{\widehat{i}_1}\otimes\widetilde{w}''\otimes\widetilde{v}''\otimes v_{m-f+1}\Big)T_{a+2s-1}\cdots T_{2s+1}T_{2s},$$
$$\Big(v_{\widehat{i}_1}\otimes v_{u_1}\otimes\cdots\otimes v_{u_{2s+a-2}}\otimes v_{m-f+1}\Big)T_{a+2s-1}\cdots T_{2s+1}T_{2s},$$
where $u_1,\cdots,u_{2s-2}$ satisfy the conditions (a1), (a2) above. Note that under the action of $T_{a+2s-1}\cdots T_{2s+1}T_{2s}$, the first $(2s-1)$ parts do not change, while by (5.13) the $2s$ position will be replaced by a vector of the form v_p with $p\leq m-f+1$. Now using the condition (a2), our conclusion follows immediately.

Case 3. $j_1=a+1$ and $\widehat{i}_1=(m-f+1)'$. Then we also must have $\widehat{i}_{2s+1}=i_{2s}$, $j_t=a+t$ and $\widehat{i}_t=i_{t-1}$ for each integer $2\leq t\leq 2s$. In this case, our conclusion follows from the same argument used in the proof of Case 2. □

Lemma 5.15. *Let $\mathbf{b}\in I_f$, $v=v_{\mathbf{b}}\otimes v_{\mathbf{c}}\in V^{\otimes n}$. Let $w\in\mathcal{D}_{\nu_f}$. If $w\neq d_0d_{J_0}$, then*

$$T_w^*E_1E_3\cdots E_{2f-1}\in\operatorname{ann}(v).$$

Proof. Let $w\in\mathcal{D}_{\nu_f}$. We write $w=d_1d_J$, where $d_1\in\mathcal{D}_f, J\in\mathcal{P}_f$. Then

$$vT_w^*=(v_{\mathbf{b}}\otimes v_{\mathbf{c}})T_{d_J^{-1}}T_{d_1^{-1}}.$$

Let $J=(i_1,i_2,\cdots,i_{2f})$. By Lemma 5.8,

$$(s_{i_1-1}\cdots s_2s_1)(s_{i_2-1}\cdots s_3s_2)\cdots(s_{i_{2f-1}-1}\cdots s_{2f}s_{2f-1})(s_{i_{2f}-1}\cdots s_{2f+1}s_{2f})$$

is a reduced expression of d_J^{-1}. Using Lemma 5.14, we get that

$$(v_{\mathbf{b}}\otimes v_{\mathbf{c}})T_{d_J^{-1}}=\zeta^z\delta_{J,J_0}(v_{\mathbf{c}}\otimes v_{\mathbf{b}})+\sum_{v'}A_{v'}v', \tag{5.16}$$

for some integer z and some $A_{v'}\in K$, where the subscript v' runs over all the simple n-tensor such that its first $(2f)$-parts have symplectic length less than f. It follows that if $J\neq J_0$, then we are done. Henceforth, we assume that $J=J_0$, then $d_1\neq d_0$ and

$$vT_w^*E_1E_3\cdots E_{2f-1}=\zeta^a(v_{\mathbf{c}}\otimes v_{\mathbf{b}})T_{d_1^{-1}}E_1E_3\cdots E_{2f-1},$$

for some integer a.

By (5.10),

$$\begin{aligned}\sigma=(s_{2f-1}s_{2f-2}\cdots s_3s_2)(s_{2f-1}s_{2f-2}\cdots s_5s_4)\cdots\\ (s_{2f-1}s_{2f-2}s_{2f-3}s_{2f-4})(s_{2f-1}s_{2f-2})\end{aligned}$$

is a reduced expressed expression of d_0^{-1}. By Lemma 5.12, for any $d\in\mathcal{D}_f$, d is less or equal than d_0 in the Bruhat order. It follows that there is a subexpression of σ which is equal to a reduced expression of d^{-1}. Combining this with the definitions of the operator β' and the indices $\mathbf{b},\mathbf{c}$, it is easy to see that

$$(v_{\mathbf{c}}\otimes v_{\mathbf{b}})T_{d^{-1}}=v_{\mathbf{c}}T_{d^{-1}}\otimes v_{\mathbf{b}}=\zeta^z\delta_{d,d_0}v_{\mathbf{c}}d_0^{-1}\otimes v_{\mathbf{b}}+\sum_{v'}B_{v'}v', \tag{5.17}$$

for some integer $z\in\mathbb{Z}$ and some $B_{v'}\in K$, where the subscript v' runs over all the simple n-tensor $v_{j_1}\otimes\cdots\otimes v_{j_n}$ such that there exists $1\leq s\leq f$ satisfying $j_{2s-1}\neq(j_{2s})'$. Now using the fact that $d_1\neq d_0$, it is easy to see that $(v_{\mathbf{c}}\otimes v_{\mathbf{b}})T_{d_1^{-1}}E_1E_3\cdots E_{2f-1}=0$. Hence, $vT_w^*E_1E_3\cdots E_{2f-1}=0$, as required. □

We are now ready to prove the key lemma from which our main result in this section will follow easily.

Lemma 5.18. *Let S be the subset*

$$\left\{T_{d_1}^*E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2}\;\middle|\;\begin{array}{c}d_1,d_2\in\mathcal{D}_{\nu_f},\,d_1\neq d_0d_{J_0},\\ \sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}\end{array}\right\}$$

of the basis (5.4) of $\mathfrak{B}_n(-\zeta^{2m+1},\zeta)$, and let U be the subspace spanned by S. Then

$$B^{(f)}\bigcap\Big(\bigcap_{\mathbf{b}\in I_f}\operatorname{ann}(v_{\mathbf{b}}\otimes v_{\mathbf{c}})\Big)=B^{(f+1)}\oplus U.$$

Proof. By definition of I_f, $\ell_s(v_{\mathbf{b}})=0$. Hence $\ell_s(v_{\mathbf{b}}\otimes v_{\mathbf{c}})=f$. It follows that $B^{(f+1)}\subseteq\operatorname{ann}(v_{\mathbf{b}}\otimes v_{\mathbf{c}})$. This, together with the Lemma 5.15, shows that the right-hand side is contained in the left-hand side.

Now let $x\in B^{(f)}\bigcap\bigl(\bigcap_{\mathbf{b}\in I_f}\operatorname{ann}(v_{\mathbf{b}}\otimes v_{\mathbf{c}})\bigr)$. We want to show that $x\in B^{(f+1)}\oplus U$. Using the basis (5.4) of the BMW algebra $\mathfrak{B}_n(-\zeta^{2m+1},\zeta)$, we may assume that

$$x=T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}\biggl(\sum_{d\in\mathcal{D}_{\nu_f}}z_dT_d\biggr),$$

where $\nu=\nu_f=((2^f),(n-2f))$ and the elements z_d, $d\in\mathcal{D}_{\nu_f}$ are taken from the K-linear subspace spanned by $\bigl\{T_w\bigm| w\in\mathfrak{S}_{\{2f+1,\cdots,n\}}\bigr\}$. We then have to show $x=0$, or equivalently, to show that $z_d=0$ for each $d\in\mathcal{D}_{\nu_f}$.

Let $d\in\mathcal{D}_{\nu_f}$. We write $z_d=\sum_{\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}}B_\sigma T_\sigma$, where $B_\sigma\in K$ for each σ. Suppose that $v_{\mathbf{b}}z_d=0$ for any $\mathbf{b}\in I_f$. By the definition of I_f, it is easy to see that

$$\begin{aligned}0=v_{\mathbf{b}}z_d&=\sum_{\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}}B_\sigma v_{\mathbf{b}}T_\sigma\\&=\sum_{\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}}B_\sigma v_{\mathbf{b}}\widehat{T}_\sigma\\&=v_{\mathbf{b}}\sum_{\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}}B_\sigma \widehat{T}_\sigma.\end{aligned}$$

However, since $m-2f\geq n-2f$, the Hecke algebra $\mathscr{H}_K(\mathfrak{S}_{\{2f+1,\cdots,n\}})$ acts faithfully on $v_{\mathbf{b}}$. This implies $B_\sigma=0$ for each $\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$. Thus $z_d=0$, as required. Therefore, to show that $z_d=0$, it suffices to show that $v_{\mathbf{b}}z_d=0$ for any $\mathbf{b}\in I_f$. We divide the proof into two steps:

Step 1. We first prove that $z_{d_0d_{J_0}}=0$, equivalently, $v_{\mathbf{b}}z_{d_0d_{J_0}}=0$ for any $\mathbf{b}\in I_f$. Let $\mathbf{b}\in I_f$.

$$\begin{aligned}0=(v_{\mathbf{b}}\otimes v_{\mathbf{c}})x&=\sum_{d\in\mathcal{D}_{\nu_f}}(v_{\mathbf{b}}\otimes v_{\mathbf{c}})T_{d_{J_0}^{-1}}T_{d_0^{-1}}E_1E_3\cdots E_{2f-1}z_dT_d\\&=\zeta^z\sum_{d\in\mathcal{D}_{\nu_f}}\Bigl(v_{\mathbf{c}_0}E_1E_3\cdots E_{2f-1}\otimes v_{\mathbf{b}}\Bigr)z_dT_d\\&=\zeta^z\sum_{d\in\mathcal{D}_{\nu_f}}\Bigl(v_{\mathbf{c}_0}E_1E_3\cdots E_{2f-1}\otimes v_{\mathbf{b}}z_d\Bigr)T_d,\end{aligned}$$

for some integer $z\in\mathbb{Z}$, where the third equality follows from (5.16) and (5.17).

By [10, Lemma 3.8], for each $d\in\mathcal{D}_\nu$, we can write $d=d_1d_J$, where $d_1\in\mathcal{D}_f$, $J\in\mathcal{P}_f$, and $\ell(d)=\ell(d_1)+\ell(d_J)$. Hence $T_d=T_{d_1}T_{d_J}$. Therefore

$$\begin{aligned}0&=\sum_{d\in\mathcal{D}_\nu}\Bigl(v_{\mathbf{c}_0}E_1E_3\cdots E_{2f-1}\otimes v_{\mathbf{b}}z_d\Bigr)T_d\\&=\sum_{J\in\mathcal{P}_f}\sum_{d_1\in\mathcal{D}_f}\Bigl(v_{\mathbf{c}_0}E_1E_3\cdots E_{2f-1}\otimes v_{\mathbf{b}}z_{d_1d_J}\Bigr)T_{d_1}T_{d_J}\\&=\sum_{J\in\mathcal{P}_f}\sum_{d_1\in\mathcal{D}_f}\Bigl(v_{\mathbf{c}_0}E_1E_3\cdots E_{2f-1}T_{d_1}\otimes v_{\mathbf{b}}z_{d_1d_J}\Bigr)T_{d_J}.\end{aligned}$$

We want to show that $v_{\mathbf{b}}z_{d_0d_{J_0}}=0$. Note that

$$v_{\mathbf{c}_0}E_1E_3\cdots E_{2f-1}=\sum_{1\leq i_1,\cdots,i_f\leq 2m}\pm\zeta^{a_{\mathbf{i}}}v_{i_1}\otimes v_{i_1'}\otimes v_{i_2}\otimes v_{i_2'}\otimes\cdots\otimes v_{i_f}\otimes v_{i_f'},$$

for some $a_{\mathbf{i}}\in\mathbb{Z}$. Note also that each simple tensor $v_{\widehat{\mathbf{b}}}$ involved $v_{\mathbf{b}}z_{d_0d_{J_0}}$ has the same GL_{2m}-weight as $v_{\mathbf{b}}$. Let $(i_1,\cdots,i_f)\in I(2m,f)$. We claim that

(a) for any $\widehat{\mathbf{b}}, \widetilde{\mathbf{b}} \in I(2m, n-2f), \widehat{\mathbf{j}} \in I(2m, 2f)$ with $\mathrm{WT}(\widehat{\mathbf{b}}) = \mathrm{WT}(\widetilde{\mathbf{b}}) = \mathrm{WT}(\mathbf{b})$ and $\ell_s(\mathbf{j}) = f$, the simple n-tensor $(v_{\mathbf{c}_0 d_0} \otimes v_{\widehat{\mathbf{b}}})d_{J_0}$ is involved in

$$(v_{\widehat{j}_1} \otimes v_{\widehat{j}_2} \otimes \cdots \otimes v_{\widehat{j}_{2f}} \otimes v_{\widetilde{\mathbf{b}}})T_{d_J}$$

if and only if $(\widehat{j}_1, \widehat{j}_2, \cdots, \widehat{j}_{2f}) = \mathbf{c}_0 d_0$, $J = J_0$ and $\widehat{\mathbf{b}} = \widetilde{\mathbf{b}}$;

(b) for any $d_1 \in \mathfrak{S}_{2f}$ with $d_1 \le d_0$ (in the Bruhat order), the simple $(2f)$-tensor $v_{\mathbf{c}_0 d_0}$ is involved in

$$(v_{i_1} \otimes v_{i_1'} \otimes v_{i_2} \otimes v_{i_2'} \otimes \cdots \otimes v_{i_f} \otimes v_{i_f'})T_{d_1}$$

if and only if $(i_1, i_2, \cdots, i_f) = ((m-f+1)', \cdots, (m-1)', m')$ and $d_1 = d_0$.

Once these two claims are proved to be true, it is easy to see that the identity $v_{\mathbf{b}} z_{d_0 d_{J_0}} = 0$ follows at once. Therefore, it remains to prove the claims (a) and (b).

Suppose that $(v_{\mathbf{c}_0 d_0} \otimes v_{\widehat{\mathbf{b}}})d_{J_0}$ is involved in

$$(v_{\widehat{j}_1} \otimes v_{\widehat{j}_2} \otimes \cdots \otimes v_{\widehat{j}_{2f}} \otimes v_{\widetilde{\mathbf{b}}})T_{d_J}.$$

By definition,

$$(v_{\mathbf{c}_0 d_0} \otimes v_{\widehat{\mathbf{b}}})d_{J_0} = v_{\widehat{\mathbf{b}}} \otimes v_{\mathbf{c}_0 d_0}.$$

Let $J = (j_1, j_2, \cdots, j_{2f})$. Then

$$(s_{2f}s_{2f+1}\cdots s_{j_{2f}-1})(s_{2f-1}s_{2f}\cdots s_{j_{2f-1}-1})\cdots(s_1 s_2 \cdots s_{j_1-1})$$

is a reduced expression of d_J. Note that $1 \le j_1 < \cdots < j_{2f} \le 2f$. If $j_{2f} \ne n$, then the rightmost vector of any simple tensor involved in $(v_{\widehat{j}_1} \otimes v_{\widehat{j}_2} \otimes \cdots \otimes v_{\widehat{j}_{2f}} \otimes v_{\widetilde{\mathbf{b}}})T_{d_J}$ must be $v_{\widetilde{b}_{n-2f}}$, which is impossible (because $\widetilde{b}_{n-2f} \le m-f$). Therefore, we deduce that $j_{2f} = n$. Let Σ be the set of all the simple n-tensor v which is involved in

$$(v_{\widehat{j}_1} \otimes v_{\widehat{j}_2} \otimes \cdots \otimes v_{\widehat{j}_{2f}} \otimes v_{\widetilde{\mathbf{b}}})T_{2f}T_{2f+1}\cdots T_{n-1}.$$

Note that $\widetilde{b}_t \le m-f$ for each t. We claim that for each integer t with $1 \le t \le n-2f$, $\widetilde{b}_t \ne (\widehat{j}_{2f})'$.

In fact, if $1 \le t \le n-2f$ is the smallest integer such that $\widetilde{b}_t = (\widehat{j}_{2f})'$, then the $(2f+t)$th position of any simple tensor involved in

$$(v_{\widehat{j}_1} \otimes v_{\widehat{j}_2} \otimes \cdots \otimes v_{\widehat{j}_{2f}} \otimes v_{\widetilde{\mathbf{b}}})(T_{2f}T_{2f+1}\cdots T_{2f+t-1})$$

is a vector v_a with either $a > (m-f+1)'$ or $a < (m-f+1)$. It follows (from the definition of β' and (5.13)) that the nth position of any simple tensor involved in

$$(v_{\widehat{j}_1} \otimes v_{\widehat{j}_2} \otimes \cdots \otimes v_{\widehat{j}_{2f}} \otimes v_{\widetilde{\mathbf{b}}})(T_{2f}T_{2f+1}\cdots T_{n-1})$$

is a vector v_a with either $a \le m-f$ or $a \ge (m-f)'$. Since the action of $(T_{2f-1}T_{2f}\cdots T_{j_{2f-1}-1})\cdots(T_1T_2\cdots T_{j_1-1})$ on any simple n-tensor does not change its rightmost vector, we deduce that $v_{\widehat{\mathbf{b}}} \otimes v_{\mathbf{c}_0 d_0}$ can not be involved in

$$(v_{\widehat{j}_1} \otimes v_{\widehat{j}_2} \otimes \cdots \otimes v_{\widehat{j}_{2f}} \otimes v_{\widetilde{\mathbf{b}}})(T_{2f}T_{2f+1}\cdots T_{n-1})(T_{2f-1}T_{2f}\cdots T_{j_{2f-1}-1}) \cdots (T_1T_2\cdots T_{j_1-1}),$$

a contradiction.

Therefore, $\widetilde{b}_t \ne (\widehat{j}_{2f})'$ for any $1 \le t \le n-2f$. It follows that $v = v_{\widehat{j}_1} \otimes v_{\widehat{j}_2} \otimes \cdots \otimes v_{\widehat{j}_{2f-1}} \otimes v_{\widetilde{\mathbf{b}}} \otimes v_{\widehat{j}_{2f}}$ is the unique simple n-tensor in Σ such that $v_{\widehat{\mathbf{b}}} \otimes v_{\mathbf{c}_0 d_0}$ is involved in

$$v(T_{2f-1}T_{2f}\cdots T_{j_{2f-1}-1})\cdots(T_1T_2\cdots T_{j_1-1}).$$

In particular, we deduce that $\widehat{j}_{2f}=(\mathbf{c}_0d_0)_{2f}=m-f+1$. Now we are in a position to use induction on n. It follows easily that

$$(j_1,j_2,\cdots,j_{2f-1})=(n-2f+1,n-2f+2,\cdots,n-1),$$
$$(\widehat{j}_1,\cdots,\widehat{j}_{2f-1})=((\mathbf{c}_0d_0)_1,\cdots,(\mathbf{c}_0d_0)_{2f-1}),\quad \widetilde{\mathbf{b}}=\widehat{\mathbf{b}}.$$

Conversely, by the definition of β', $(v_{\mathbf{c}_0d_0}\otimes v_{\widehat{\mathbf{b}}})T_{d_{J_0}}=v_{\widehat{\mathbf{b}}}\otimes v_{\mathbf{c}_0d_0}$. This proves the claim (a).

We now turn to the claim (b). By Lemma 5.1 and direct verification, it is easy to see that the simple $(2f)$-tensor $v_{\mathbf{c}_0d_0}$ is involved in

$$(v_{(m-f+1)'}\otimes v_{m-f+1}\otimes v_{(m-f+2)'}\otimes v_{m-f+2}\otimes\cdots\otimes v_{m'}\otimes v_m)T_{d_0}.$$

This proves one direction of the claim (b). Now suppose that the simple $(2f)$-tensor $v_{\mathbf{c}_0d_0}$ is involved in

$$(v_{i_1}\otimes v_{i'_1}\otimes v_{i_2}\otimes v_{i'_2}\otimes\cdots\otimes v_{i_f}\otimes v_{i'_f})T_{d_1},$$

where $(i_1,\cdots,i_f)\in I(2m,f)$, $d_1\in\mathfrak{S}_{2f}$ with $d_1\leq d_0$ (in the Bruhat order). Then d_1 has a reduced expression which is a subexpression of (5.10). Hence we can write $d_1=d'_1d''_1$, where d'_1 is a subexpression of

$$(s_{2f-2}s_{2f-1})(s_{2f-4}s_{2f-3}s_{2f-2}s_{2f-1})\cdots(s_4s_5\cdots s_{2f-2}s_{2f-1}),$$

d''_1 is a subexpression of $s_2s_3\cdots s_{2f-2}s_{2f-1}$, such that $\ell(d_1)=\ell(d'_1)+\ell(d''_1)$. Then $T_{d_1}=T_{d'_1}T_{d''_1}$.

By definition of d'_1, any simple tensor involved in

$$(v_{i_1}\otimes v_{i'_1}\otimes v_{i_2}\otimes v_{i'_2}\otimes\cdots\otimes v_{i_f}\otimes v_{i'_f})T_{d'_1}$$

is of the form

$$v_{i_1}\otimes v_{i'_1}\otimes v_{l_1}\otimes v_{l_2}\otimes\cdots\otimes v_{l_{2f-2}},$$

where $\mathbf{l}=(l_1,l_2,\cdots,l_{2f-2})\in I(2m,2f-2)$ with $\ell_s(\mathbf{l})=f-1$. By assumption, we can choose one such simple n-tensor, say

$$v^{[1]}:=v_{i_1}\otimes v_{i'_1}\otimes v_{l_1}\otimes v_{l_2}\otimes\cdots\otimes v_{l_{2f-2}},$$

such that $v_{\mathbf{c}_0d_0}$ is involved in $vT_{d''_1}$. By definition of d''_1, it is easy to see that $i_1=(m-f+1)'$. We claim that

(b1) $(l_1,l_2,\cdots,l_{2f-2})=((m-f+2)',(m-f+3)',\cdots,m',m,\cdots,m-f+3,m-f+2)$;
(b2) $d''_1=s_2s_3\cdots s_{2f-2}s_{2f-1}$.

If both (b1) and (b2) are true, then the claim (b) follows easily from induction on f. Therefore, it suffices to prove the two claims (b1) and (b2).

Recall that $\mathbf{c}_0d_0=((m-f+1)',\cdots,(m-1)',m',m,m-1,\cdots,m-f+1)$. If $l_1<(m-f+2)'$, then (by (5.13)) the second position of any simple $(2f)$-tensor involved in $v^{[1]}T_{d''_1}$ is always occupied by a vector v_a with $a<(m-f+2)'$, which is impossible (because $v_{\mathbf{c}_0d_0}$ is involved in $vT_{d''_1}$). Hence $l_1\geq(m-f+2)'$. By similar reason, we can deduce that l_1 can not be strictly bigger than $(m-f+1)'$. Therefore, $l_1\in\{(m-f+1)',(m-f+2)'\}$. Assume that $l_1=(m-f+1)'$. Then by (5.13) and the fact that $\ell(\mathbf{l})=f-1$, it is easy to see for any simple $(2f)$-tensor $v_{k_1}\otimes\cdots\otimes v_{k_{2f}}$ involved in $v^{[1]}T_{d''_1}$, we have $k_t\leq m-f+1$ for some $t\geq 3$, a contradiction. This forces $l_1=(m-f+2)'$. Repeating the same argument, we deduce that for any integer $1\leq t\leq f-1$, $l_t=(m-f+t+1)'$. Now since $\ell_s(\mathbf{l})=f-1$, it follows that $\{l_f,l_{f+1},\cdots,l_{2f-2}\}=\{(m-f+1)',(m-f+2)',\cdots,(m-1)'\}$. In particular, $l_t\neq m'$ for any t. Using the same arguments as before, we easily deduce that for each integer $f\leq t\leq 2f-2$, $l_t=m-t+f$. This proves (b1). Now (b2) follows immediately from (b1) and the fact that $v_{\mathbf{c}_0 d_0}$ is involved in $vT_{d_1''}$. This proves another direction of the claim (b), hence completes the proof of the claim (b).

Step 2. Let S' be the subset

$$\left\{T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} T_\sigma T_{d_2} \;\middle|\; \begin{array}{c} d_2 \in \mathcal{D}_{\nu_f},\, d_2 \neq d_0 d_{J_0}, \\ \sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}} \end{array}\right\}$$

of the basis (5.4) of $\mathfrak{B}_n(-\zeta^{2m+1}, \zeta)$, and let U' be the subspace spanned by S'. By the main results we obtained in Step 1, we know that

$$B^{(f)} \bigcap \Big(\bigcap_{\mathbf{b} \in I_f} \mathrm{ann}(v_{\mathbf{b}} \otimes v_{\mathbf{c}}) \Big) \subseteq B^{(f+1)} \oplus U \oplus U'.$$

We want to prove that

$$B^{(f)} \bigcap \Big(\bigcap_{\mathbf{b} \in I_f} \mathrm{ann}(v_{\mathbf{b}} \otimes v_{\mathbf{c}}) \Big) \subseteq B^{(f+1)} \oplus U.$$

If this is not the case, then by Lemma 5.15,

$$U' \bigcap B^{(f)} \bigcap \Big(\bigcap_{\mathbf{b} \in I_f} \mathrm{ann}(v_{\mathbf{b}} \otimes v_{\mathbf{c}}) \Big) \neq 0.$$

Let $\widetilde{\Sigma}$ be the set of those $d_2 \in \mathcal{D}_{\nu_f}$, such that there exist some

$$x' \in U' \bigcap B^{(f)} \bigcap \mathrm{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V^{\otimes n}\big),\ d_2 \in \mathcal{D}_\nu,\ \sigma_2 \in \mathfrak{S}_{\{2f+1,\cdots,n\}}$$

satisfying

$$T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_2}$$

is involved in x'. We choose $d_2 \in \widetilde{\Sigma}$ such that $\ell(d_2)$ is as big as possible.

By the definition of U', $d_2 \neq d_0 d_{J_0}$. It follows from Lemma 5.12 that we can find an integer j with $1 \leq j \leq n-1$, such that $d_2 s_j \in \mathcal{D}_{\nu_f}$ and $\ell(d_2 s_j) = \ell(d_2) + 1$. Let

$$x' \in U' \bigcap B^{(f)} \bigcap \mathrm{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\big(V^{\otimes n}\big), d_2' \in \mathcal{D}_{\nu_f}, \sigma_2 \in \mathfrak{S}_{\{2f+1,\cdots,n\}}$$

such that $T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_2}$ is involved in x'. We claim that there exist $\tau \in \mathfrak{S}_{\{2f+1,\cdots,n\}}, d_3 \in \mathcal{D}_{\nu_f}$ with $\ell(d_3) > \ell(d_2)$, such that

$$T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} T_\tau T_{d_3}$$

is involved in $x' T_j$.

We write

$$x' = A_0 T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} z_{d_2} T_{d_2} + \sum_{\substack{\sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}} \\ d_2 \neq d_2' \in \mathcal{D}_{\nu_f} \\ \ell(d_2') \leq \ell(d_2)}} A_{d_2',\sigma} T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} T_\sigma T_{d_2'},$$

where $0 \neq z_{d_2} \in K$-Span $\{T_w | w \in \mathfrak{S}_{\{2f+1,\cdots,n\}}\}$, T_{σ_2} is involved in z_{d_2}, $0 \neq A_0 \in K$, $A_{d_2',\sigma} \in K$ for each d_2', σ.

Note that $T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} z_{d_2} T_{d_2} T_j = T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} z_{d_2} T_{d_2 s_j}$. It remains to analyze how each $T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} T_\sigma T_{d_2'} T_j$ is expressed as a linear combination of the basis elements given in Corollary 5.4. Our purpose is to show that $T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} T_{\sigma_2} T_{d_2 s_j}$ is not involved in each

$$T^*_{d_0 d_{J_0}} E_1 E_3 \cdots E_{2f-1} T_\sigma T_{d_2'} T_j.$$

We divide the discussion into cases:

Case 1. $\ell(d_2's_j)=\ell(d_2')+1$. In this case, $T_{d_2'}T_j=T_{d_2's_j}$. We write $d_2's_j=w_4d_4$, where $w_4\in\mathfrak{S}_{\nu_f}$, and $\mathfrak{t}^{\nu}d_4$ is row standard. Then $\ell(w_4d_4)=\ell(w_4)+\ell(d_4)$. Furthermore, we can write $T_{w_4}=x_4T_{w_4'}$, where

$$x_4\in\langle T_1,T_3,\cdots,T_{2f-1}\rangle,\ \ w_4'\in\mathfrak{S}_{\nu_f^{(2)}}.$$

We have

$$\begin{aligned}T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2'}T_j&=T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{w_4}T_{d_4}\\&\equiv\sum_{\widehat{\sigma}\in\mathfrak{S}_{\{2f+1,\cdots,n\}}}A_{\widehat{\sigma}}T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_{\widehat{\sigma}}T_{d_4}\pmod{B^{(f+1)}},\end{aligned}$$

for some $A_{\widehat{\sigma}}\in K$, where the second equality follows from the fact that $E_{2i-1}T_{2i-1}=r^{-1}E_{2i-1}$ for each i and [22, (3.2)]. Now we use [22, Proposition 3.7] to express $T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_{\widehat{\sigma}}T_{d_4}$ as a linear combination of the basis elements given in Corollary 5.4. Note that in the notation of [22, Proposition 3.7], it is easy to check (in all the three cases listed in [22, Proposition 3.7]) that $u=s_{2j}s_{2j+1}s_{2j-1}s_{2j}w$, $\ell(u)\leq\ell(w),\ell(v)=\ell(w)-1,\ell(v')\leq\ell(w)-1$. It follows that each $T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_{\widehat{\sigma}}T_{d_4}$ can be expressed as a linear combination of the basis elements of the form

$$T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_{\sigma''}T_{d_2''},$$

where $\sigma''\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$, $d_2''\in\mathcal{D}_{\nu_f}$, with either

(1) $\ell(d_2'')<\ell(d_4)$; or
(2) $\ell(d_2'')=\ell(d_4)$ and $d_4=zd_2''$ for some $z\in\Pi$.

Note that $\ell(d_4)\leq\ell(d_2')+1$ with equality holds only if $d_2's_j=d_4$. As a result, $\ell(d_2'')\leq\ell(d_2)+1$. If $\ell(d_2'')<\ell(d_2)+1$, then $d_2''\neq d_2s_j$ and we are done. If $\ell(d_2'')=\ell(d_2)+1$, then $\ell(d_2')=\ell(d_2)$, $\ell(d_4)=\ell(d_2'')=\ell(d_2')+1$ and $d_2's_j=d_4=zd_2''$ for some $z\in\Pi$. In this case we claim that $d_2''\neq d_2s_j$. This is true because otherwise we would deduce that $d_2'=zd_2$, which is impossible (since d_2',d_2 are different elements in $\mathcal{D}_{\nu_f}$). This completes the proof in Case 1.

Case 2. $\ell(d_2's_j)=\ell(d_2')-1$. Then by Lemma 5.2, $d_2's_j\in\mathcal{D}_{\nu_f}$. In this case (note that our T_j is $\zeta^{-1}T_j$ in [22]'s notation), by [22, Lemma 2.1],

$$T_{d_2'}T_j=T_{d_2's_j}+(\zeta-\zeta^{-1})(T_{d_2'}+\zeta^{-2m-1}T_{d_2's_j}E_j).$$

Therefore,

$$\begin{aligned}&T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2'}T_j\\&=T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2's_j}+(\zeta-\zeta^{-1})T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2'}\\&\qquad+(\zeta-\zeta^{-1})\zeta^{-2m-1}T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2's_j}E_j.\end{aligned}$$

By comparing their length, we see that $d_2s_j\notin\{d_2's_j,d_2'\}$. Hence

$$T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_{\sigma_2}T_{d_2s_j}$$

is not involved in

$$T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2's_j}+(\zeta-\zeta^{-1})T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2'}.$$

Note that $d_2s_j,d_2\in\mathcal{D}_{\nu_f}$ imply that $j,j+1$ are not in the same row of $\mathfrak{t}^{\nu}d_2s_j$. Combing this with [22, Lemma 3.4, 3.5, Proposition 3.3, 3.4], we deduce that $T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_{\sigma_2}T_{d_2s_j}$ is not involved in

$$(\zeta-\zeta^{-1})\zeta^{-2m-1}T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2's_j}E_j,$$

as required. This completes the proof in Case 2.

As a consequence, we can deduce that $T^*_{d_0d_{J_0}}E_1E_3\cdots E_{2f-1}T_{\sigma_2}T_{d_2s_j}$ is always involved in $x'T_j$. Note that $x'T_j\in B^{(f)}\bigcap\Bigl(\bigcap_{\mathbf{b}\in I_f}\operatorname{ann}(v_{\mathbf{b}}\otimes v_{\mathbf{c}})\Bigr)$ and $\ell(d_2s_j)>\ell(d_2)$. We get a contradiction to our choice of d_2. This completes the proof of the lemma. □

Theorem 5.19. *With the above notations, we have that*

$$B^{(f)}\bigcap\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\bigl(V^{\otimes n}\bigr)\subseteq B^{(f+1)}.$$

Proof. Suppose that

$$B^{(f)}\bigcap\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\bigl(V^{\otimes n}\bigr)\not\subseteq B^{(f+1)}.$$

By Lemma 5.18, we can find an element x in $B^{(f)}\bigcap\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\bigl(V^{\otimes n}\bigr)$ of the following form:

$$x=z+y,$$

where $z\in B^{(f+1)}, 0\neq y\in U$.

We write

$$y=\sum_{d_1,d_2,\sigma}A_{d_1,d_2,\sigma}T^*_{d_1}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2},$$

where the subscripts run over all elements d_1,d_2,σ such that $d_1,d_2\in\mathcal{D}_\nu$ and $d_1\neq d_0d_{J_0}$. Let Σ be the set of those $d_1\in\mathcal{D}_{\nu_f}$, such that there exist some $x\in B^{(f)}\bigcap\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\bigl(V^{\otimes n}\bigr), d_2\in\mathcal{D}_\nu,\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$ satisfying

$$T^*_{d_1}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2}\text{ is involved in } y.$$

We choose $d_1'\in\Sigma$ such that $\ell(d_1')$ is as big as possible.

By definition of U, $d_1'\neq d_0d_{J_0}$. It follows from Lemma 5.12 that we can find an integer j with $1\leq j\leq n-1$, such that $d_1's_j\in\mathcal{D}_\nu$ and $\ell(d_1's_j)=\ell(d_1')+1$. Let $x\in B^{(f)}\bigcap\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\bigl(V^{\otimes n}\bigr), d_2'\in\mathcal{D}_\nu,\sigma'\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$ be such that $T^*_{d_1'}E_1E_3\cdots E_{2f-1}T_{\sigma'}T_{d_2'}$ is involved in y. Note that

$$T_j^*z+T_j^*y=T_j^*x\in B^{(f)}\bigcap\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\bigl(V^{\otimes n}\bigr),\ T_j^*z\in B^{(f+1)}.$$

Our purpose is to show that there exist some $d_3\in\mathcal{D}_\nu,\tau\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$, such that $T^*_{d_1's_j}E_1E_3\cdots E_{2f-1}T_\tau T_{d_3}$ is involved in T_j^*y. If this true, then we get a contradiction to our choice of d_1', and we are done.

We write

$$\begin{aligned}y={}&A_0T^*_{d_1'}E_1E_3\cdots E_{2f-1}T_{\sigma'}T_{d_2'}+\\&\sum_{\substack{d_1,d_2\in\mathcal{D}_\nu,d_1\neq d_1',\\ \sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}},\\ \ell(d_1)\leq\ell(d_1')}}A_{d_1,d_2,\sigma}T^*_{d_1}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2},\end{aligned}$$

where $0\neq A_0\in K$, $A_{d_1,d_2,\sigma}\in K$ for each $d_1,d_2\in\mathcal{D}_{\nu_f},\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$.

Note that $T_j^*T^*_{d_1'}E_1E_3\cdots E_{2f-1}T_{\sigma'}T_{d_2'}=T^*_{d_1's_j}E_1E_3\cdots E_{2f-1}T_{\sigma'}T_{d_2'}$. Using the same argument as in the proof of Step 2 in Lemma 5.18, we can show that

$$T^*_{d_1's_j}E_1E_3\cdots E_{2f-1}T_{\sigma'}T_{d_2'}$$

is not involved in $T_j^*T^*_{d_1}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2}$, as required. This completes the proof of the theorem. □

Proof of Theorem 1.5 in the case where $m\geq n$: It follows easily from Lemma 5.6, Theorem 5.19 and induction on f that

$$\operatorname{ann}_{\mathfrak{B}_n(-\zeta^{2m+1},\zeta)}\bigl(V^{\otimes n}\bigr)=0.$$

This proves the injectivity of φ_C, and hence φ_C must map $\mathfrak{B}_n(-\zeta^{2m+1},\zeta)$ isomorphically onto $\mathrm{End}_{\mathbb{U}_K(\mathfrak{sp}_{2m})}\big(V_K^{\otimes n}\big)$ in this case.

6. Proof of Theorem 1.5 in the case where $m<n$

The purpose of this section is to give a proof of Theorem 1.5 in the case where $m<n$. Our strategy is similar to but technically more difficult than [10, Section 4].

Let m_0 be a natural number with $m_0\geq n$. Let $\widetilde{V}_{\mathscr{A}}$ be a free $\mathscr{A}$-module of rank $2m_0$. Assume that $\widetilde{V}_{\mathscr{A}}$ is equipped with a skew bilinear form $(,)$ as well as an ordered basis $\big\{v_1,v_2,\cdots,v_{2m_0}\big\}$ satisfying

$$(v_i,v_j)=\begin{cases}1, & \text{if } i+j=2m_0+1 \text{ and } i<j;\\ -1, & \text{if } i+j=2m_0+1 \text{ and } i>j;\\ 0 & \text{otherwise.}\end{cases}$$

For any $\mathscr{A}$-algebra K, we set $\widetilde{V}_K:=\widetilde{V}_{\mathscr{A}}\otimes_{\mathscr{A}}K$. Let ζ be the image of q in K. Let ι be the K-linear injection from V_K into $\widetilde{V}_K$ defined by

$$\sum_{i=1}^{2m}k_iv_i\mapsto\sum_{i=1}^{2m}k_iv_{i+m_0-m},\quad \forall\, k_1,\cdots,k_{2m}\in K.$$

Let π be the K-linear surjection from $\widetilde{V}_K$ into V_K defined by

$$\sum_{i=1}^{2m_0}k_iv_i\mapsto\sum_{i=1}^{2m}k_{i+m_0-m}v_i,\quad \forall\, k_1,\cdots,k_{2m_0}\in K.$$

We set $\widetilde{\iota}:=\zeta^m\iota$, $\widetilde{\pi}:=\zeta^{m_0}\pi$. We regard $\mathbb{C}$ as an $\mathscr{A}$-algebra by specializing q to 1. As before, we identify $\mathfrak{sp}(V_{\mathbb{C}})$ with $\mathfrak{sp}_{2m}(\mathbb{C})$ and $\mathfrak{sp}(\widetilde{V}_{\mathbb{C}})$ with $\mathfrak{sp}_{2m_0}(\mathbb{C})$. Then, ι induces an identification of $\mathfrak{sp}_{2m}(\mathbb{C})$ with the Lie subalgebra of $\mathfrak{sp}_{2m_0}(\mathbb{C})$ consisting of the following block diagonal matrices:

$$\Big\{\mathrm{diag}(\underbrace{0,\cdots,0,}_{(m_0-m)\text{ copies}}\ A,\ \underbrace{0,\cdots,0}_{(m_0-m)\text{ copies}})\ \Big|\ A\in\mathfrak{sp}_{2m}(\mathbb{C})\Big\}.$$

Henceforth let K be a field which is an $\mathscr{A}$-algebra, we set

$$\widetilde{\mathfrak{g}}:=\mathfrak{sp}_{2m_0}(\mathbb{C}),\ \mathfrak{g}:=\mathfrak{sp}_{2m}(\mathbb{C}),\ V=V_K,\ \widetilde{V}=\widetilde{V}_K.$$

The inclusion $\mathfrak{g}\subset\widetilde{\mathfrak{g}}$ naturally induces an injection

$$\begin{aligned}&\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})\hookrightarrow\mathbb{U}_{\mathbb{Q}(q)}(\widetilde{\mathfrak{g}})\\ &e_i\mapsto e_{i+m_0-m},\ e_i\mapsto e_{i+m_0-m},\ k_i\mapsto k_{i+m_0-m},\ i=1,2,\cdots,m.\end{aligned}$$

By restriction, we get an injection $\mathbb{U}_{\mathscr{A}}(\mathfrak{g})\hookrightarrow\mathbb{U}_{\mathscr{A}}(\widetilde{\mathfrak{g}})$. By base change, we get a natural map $\mathbb{U}_K(\mathfrak{g})\to\mathbb{U}_K(\widetilde{\mathfrak{g}})$. It is easy to see that

$$\widetilde{\pi}^{\otimes 2n}\Big(\big(\widetilde{V}^{\otimes 2n}\big)^{\mathbb{U}_K(\widetilde{\mathfrak{g}})}\Big)\subseteq\big(V^{\otimes 2n}\big)^{\mathbb{U}_K(\mathfrak{g})}.$$

For each integer i with $1\leq i\leq 2m$, we define

$$w_i=\begin{cases}v_i, & \text{if } 1\leq i\leq m;\\ (-1)^{i-m-1}v_i, & \text{if } m+1\leq i\leq 2m.\end{cases}$$

Then the natural representation of $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})$ on $V_{\mathbb{Q}(q)}$ is given by

$$e_i w_j = \begin{cases} w_i, & \text{if } j = i+1, \\ w_{2m+1-(i+1)}, & \text{if } j = 2m+1-i, \\ 0, & \text{otherwise;} \end{cases} \quad e_m w_j = \begin{cases} w_m, & \text{if } j = m+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_i w_j = \begin{cases} w_{i+1}, & \text{if } j = i, \\ w_{2m+1-i}, & \text{if } j = 2m+1-(i+1), \\ 0, & \text{otherwise;} \end{cases} \quad f_m w_j = \begin{cases} w_{m+1}, & \text{if } j = m, \\ 0, & \text{otherwise,} \end{cases}$$

$$k_i w_j = \begin{cases} q w_j, & \text{if } j = i \text{ or } j = 2m+1-(i+1), \\ q^{-1} w_j, & \text{if } j = i+1 \text{ or } j = 2m+1-i, \\ w_j, & \text{otherwise,} \end{cases}$$

$$k_m w_j = \begin{cases} q w_j, & \text{if } j = m, \\ q^{-1} w_j, & \text{if } j = m+1, \\ w_j, & \text{otherwise,} \end{cases}$$

where $1 \le i \le m-1$, $1 \le j \le 2m$. By definition (cf. [30, (8.18)]), $\{w_i\}_{1 \le i \le 2m}$ is a canonical basis of the $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})$-module $V_{\mathbb{Q}(q)}$ in the sense of [41]. Similarly, the natural $\mathbb{U}_{\mathbb{Q}(q)}(\widetilde{\mathfrak{g}})$-module $\widetilde{V}_{\mathbb{Q}(q)}$ has a canonical basis $\{\widetilde{w}_i\}_{1 \le i \le 2m_0}$ such that

$$e_i \widetilde{w}_j = \begin{cases} \widetilde{w}_i, & \text{if } j = i+1, \\ \widetilde{w}_{2m_0+1-(i+1)}, & \text{if } j = 2m_0+1-i, \\ 0, & \text{otherwise;} \end{cases}$$

$$f_i \widetilde{w}_j = \begin{cases} \widetilde{w}_{i+1}, & \text{if } j = i, \\ \widetilde{w}_{2m_0+1-i}, & \text{if } j = 2m_0+1-(i+1), \\ 0, & \text{otherwise;} \end{cases}$$

$$e_{m_0} \widetilde{w}_j = \begin{cases} \widetilde{w}_{m_0}, & \text{if } j = m_0+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{m_0} \widetilde{w}_j = \begin{cases} \widetilde{w}_{m_0+1}, & \text{if } j = m_0, \\ 0, & \text{otherwise,} \end{cases}$$

$$k_i \widetilde{w}_j = \begin{cases} q \widetilde{w}_j, & \text{if } j = i \text{ or } j = 2m_0+1-(i+1), \\ q^{-1} \widetilde{w}_j, & \text{if } j = i+1 \text{ or } j = 2m_0+1-i, \\ \widetilde{w}_j, & \text{otherwise,} \end{cases}$$

$$k_{m_0} \widetilde{w}_j = \begin{cases} q \widetilde{w}_j, & \text{if } j = m_0, \\ q^{-1} \widetilde{w}_j, & \text{if } j = m_0+1, \\ \widetilde{w}_j, & \text{otherwise,} \end{cases}$$

where $1 \le i \le m_0 - 1$, $1 \le j \le 2m_0$. Note that the subspace $\widehat{V}_{\mathbb{Q}(q)}$ spanned by $\{\widetilde{w}_i\}_{m_0-m+1 \le i \le m_0+m}$ is stable under the action of the subalgebra $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})$, and it is canonically isomorphic to $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})$-module $V_{\mathbb{Q}(q)}$.

For each integer i with $1 \le i \le 2m$, we define $w_i^* \in V_{\mathscr{A}}^* := \mathrm{Hom}_{\mathscr{A}}(V_{\mathscr{A}}, \mathscr{A})$ by $w_i^*(v) = (w_i, v)$, $\forall\, v \in V_{\mathscr{A}}$. Then $\{w_i^*\}_{i=1}^{2m}$ is an $\mathscr{A}$-basis of $V_{\mathscr{A}}^*$, and w_1^* is a highest weight vector of weight ε_1. Furthermore, the map $w_1^* \mapsto w_1$ can be naturally extended to an $\mathbb{U}_{\mathscr{A}}(\mathfrak{g})$-module isomorphism $\tau : V_{\mathscr{A}}^* \cong V_{\mathscr{A}}$ such that

$$\tau(w_i^*) = \begin{cases} q^{1-i}w_i, & \text{if } 1 \le i \le m;\\ q^{-i}w_i, & \text{if } m+1 \le i \le 2m,\end{cases}$$

where the $\mathbb{U}_{\mathscr{A}}(\mathfrak{g})$-structure on $V_{\mathscr{A}}^*$ is defined via the antipode S. In a similar way, we can define an $\mathscr{A}$-basis $\{\widetilde{w}_i^*\}_{i=1}^{2m_0}$ of $\widetilde{V}_{\mathscr{A}}^*$, and an $\mathbb{U}_{\mathscr{A}}(\widetilde{\mathfrak{g}})$-module isomorphism $\widetilde{\tau} : \widetilde{V}_{\mathscr{A}}^* \cong \widetilde{V}_{\mathscr{A}}$. Let $\widehat{V}_{\mathscr{A}}$ be the free $\mathscr{A}$-submodule generated by

$$\bigl\{\widetilde{w}_{m_0-m+1}, \widetilde{w}_{m_0-m+2}, \cdots, \widetilde{w}_{m_0+m}\bigr\}.$$

Set $\widehat{V} := \widehat{V}_{\mathscr{A}} \otimes_{\mathscr{A}} K$. Note that the algebra $\mathbb{U}_K(\mathfrak{g})$ acts on $\widehat{V}$ via the natural map $\mathbb{U}_K(\mathfrak{g}) \to \mathbb{U}_K(\widetilde{\mathfrak{g}})$. The resulting $\mathbb{U}_K(\mathfrak{g})$-module $\widehat{V}$ is naturally isomorphic to the natural $\mathbb{U}_K(\mathfrak{g})$-module V via the correspondence

$$\widetilde{w}_i \mapsto w_{i-m_0+m}, \quad \text{for } i = m_0-m+1, m_0-m+2, \cdots, m_0+m.$$

Recall our definitions of $\widetilde{\iota}, \widetilde{\pi}$ at the beginning of this section. We define a linear map Θ_0 as follows:

$$\begin{aligned}\Theta_0 :\ &\mathrm{End}\bigl(\widetilde{V}^{\otimes n}\bigr) \to \mathrm{End}\bigl(V^{\otimes n}\bigr),\\ &f \mapsto \bigl(\widetilde{\pi}^{\otimes n}\bigr) \circ f \circ \bigl(\widetilde{\iota}^{\otimes n}\bigr).\end{aligned}$$

One can verify directly that

$$\Theta_0\Bigl(\mathrm{End}_{\mathbb{U}_K(\widetilde{\mathfrak{g}})}\bigl(\widetilde{V}^{\otimes n}\bigr)\Bigr) \subseteq \mathrm{End}_{\mathbb{U}_K(\mathfrak{g})}\bigl(\widehat{V}^{\otimes n}\bigr) \cong \mathrm{End}_{\mathbb{U}_K(\mathfrak{g})}\bigl(V^{\otimes n}\bigr),$$

where the last isomorphism comes from the natural $\mathbb{U}_K(\mathfrak{g})$-module isomorphism $\widehat{V} \cong V$.

By Corollary 5.4, the BMW algebra $\mathfrak{B}_n(-q^{2m+1}, q)$ has a basis

$$\left\{T_{d_1}^* E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2} \;\middle|\; \begin{array}{c} 0 \le f \le [n/2],\ \lambda \vdash n-2f,\ \sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}},\\ d_1, d_2 \in \mathcal{D}_\nu, \text{ where } \nu := ((2^f), (n-2f))\end{array}\right\}.$$

The same is true for the BMW algebra $\mathfrak{B}_n(-q^{2m_0+1}, q)$. To distinguish its basis elements with those of $\mathfrak{B}_n(-q^{2m+1}, q)$, we denote them by

$$\widetilde{T}_{d_1}^* \widetilde{E}_1\widetilde{E}_3\cdots \widetilde{E}_{2f-1}\widetilde{T}_\sigma \widetilde{T}_{d_2},$$

where $\widetilde{T}_i, \widetilde{E}_i$ are standard generators of $\mathfrak{B}_n(-q^{2m_0+1}, q)$. We define an $\mathscr{A}$-linear isomorphism Θ_1 from the BMW algebra $\mathfrak{B}_n(-q^{2m_0+1}, q)_{\mathscr{A}}$ to the BMW algebra $\mathfrak{B}_n(-q^{2m+1}, q)_{\mathscr{A}}$ as follows:

$$\Theta_1\Bigl(\widetilde{T}_{d_1}^* \widetilde{E}_1\widetilde{E}_3\cdots \widetilde{E}_{2f-1}\widetilde{T}_\sigma \widetilde{T}_{d_2}\Bigr) = q^{(m_0+m)n} T_{d_1}^* E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2},$$

for each $0 \le f \le [n/2]$, $\lambda \vdash n-2f$, $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ and $d_1, d_2 \in \mathcal{D}_{\nu_f}$. By base change, we get a K-linear isomorphism $\mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta) \cong \mathfrak{B}_n(-\zeta^{2m+1}, \zeta)$, which will be still denoted by Θ_1.

By the main result in last section, we know that the natural homomorphism φ_C from $\mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta)$ to $\mathrm{End}_{\mathbb{U}_K(\mathfrak{sp}_{2m_0})}\bigl((\widetilde{V})^{\otimes n}\bigr)$ is always an isomorphism. Therefore, in order to prove Theorem 1.5 (in the case where $m < n$), it suffices to prove the following lemma.

Lemma 6.1. *With the notations as above,*

(1) *the following diagram of maps*

$$\begin{array}{ccc} \mathfrak{B}_n(-\zeta^{2m_0+1}, \zeta) & \xrightarrow{\varphi_C} & \mathrm{End}_{\mathbb{U}_K(\widetilde{\mathfrak{g}})}\bigl(\widetilde{V}^{\otimes n}\bigr)\\ \Theta_1\big\downarrow & & \Theta_0\big\downarrow\\ \mathfrak{B}_n(-\zeta^{2m+1}, \zeta) & \xrightarrow{\varphi_C} & \mathrm{End}_{\mathbb{U}_K(\mathfrak{g})}\bigl(V^{\otimes n}\bigr)\end{array}$$

is commutative;

(2) *the map*

$$\Theta_0 : \mathrm{End}_{\mathbb{U}_K(\widetilde{\mathfrak{g}})}\big(\widetilde{V}^{\otimes n}\big) \to \mathrm{End}_{\mathbb{U}_K(\mathfrak{g})}\big(V^{\otimes n}\big)$$

is surjective.

The remaining part of this section is devoted to the proof of Lemma 6.1. The proof of Lemma 6.1 (2) is almost the same as [10, Section 4], which we just sketch here. First, we note that the following diagram of maps

$$\begin{array}{ccccc} \mathrm{End}_{\mathbb{U}_K(\widetilde{\mathfrak{g}})}\big(\widetilde{V}^{\otimes n}\big) & \xrightarrow{\ \sim\ } & \Big(\widetilde{V}^{\otimes n}\otimes\big(\widetilde{V}^*\big)^{\otimes n}\Big)^{\mathbb{U}_K(\widetilde{\mathfrak{g}})} & \xrightarrow{\mathrm{id}^{\otimes n}\otimes\widetilde{\tau}^{\otimes n}} & \Big(\widetilde{V}^{\otimes 2n}\Big)^{\mathbb{U}_K(\widetilde{\mathfrak{g}})} \\ {\scriptstyle\Theta_0}\big\downarrow & & & & \big\downarrow{\scriptstyle\widetilde{\pi}^{\otimes 2n}} \\ \mathrm{End}_{\mathbb{U}_K(\mathfrak{g})}\big(V^{\otimes n}\big) & \xrightarrow{\ \sim\ } & \Big(V^{\otimes n}\otimes\big(V^*\big)^{\otimes n}\Big)^{\mathbb{U}_K(\mathfrak{g})} & \xrightarrow{\mathrm{id}^{\otimes n}\otimes\tau^{\otimes n}} & \Big(V^{\otimes 2n}\Big)^{\mathbb{U}_K(\mathfrak{g})} \end{array}$$

is commutative[3]. Note that (by the theory of tilting modules)

$$\begin{aligned}\Big(V^{\otimes 2n}\Big)^{\mathbb{U}_K(\mathfrak{g})} &\cong \Big(V^{\otimes n}\otimes\big(V^*\big)^{\otimes n}\Big)^{\mathbb{U}_K(\mathfrak{g})} \cong \mathrm{End}_{\mathbb{U}_K(\mathfrak{g})}\Big(V^{\otimes n}\Big)\\ &\cong \mathrm{End}_{\mathbb{U}_{\mathscr{A}}(\mathfrak{g})}\Big(V_{\mathscr{A}}^{\otimes n}\Big)\otimes_{\mathscr{A}} K \cong \Big(V_{\mathscr{A}}^{\otimes 2n}\Big)^{\mathbb{U}_{\mathscr{A}}(\mathfrak{g})}\otimes_{\mathscr{A}} K.\end{aligned}$$

Therefore, to prove Lemma 6.1(2), it suffices to show that

$$\widetilde{\pi}^{\otimes 2n}\Big(\big(\widetilde{V}^{\otimes 2n}\big)^{\mathbb{U}_K(\widetilde{\mathfrak{g}})}\Big) = \big(V^{\otimes 2n}\big)^{\mathbb{U}_K(\mathfrak{g})},$$

equivalently, to show that

$$\widetilde{\pi}^{\otimes 2n}\Big(\big(\widetilde{V}_{\mathscr{A}}^{\otimes 2n}\big)^{\mathbb{U}_{\mathscr{A}}(\widetilde{\mathfrak{g}})}\Big) = \big(V_{\mathscr{A}}^{\otimes 2n}\big)^{\mathbb{U}_{\mathscr{A}}(\mathfrak{g})}. \tag{6.2}$$

Let $M := (V_{\mathbb{Q}(q)})^{\otimes 2n}$. By [41, (27.3)], the $\mathbb{U}_{\mathbb{Q}(q)}(\mathfrak{g})$-module M is a based module. There is a canonical basis B of M, in Lusztig's notation ([41, (27.3.2)]), where each element in B is of the form $w_{i_1}\diamond w_{i_2}\diamond\cdots\diamond w_{i_{2n}}$, and $w_{i_1}\diamond\cdots\diamond w_{i_{2n}}$ is equal to $w_{i_1}\otimes\cdots\otimes w_{i_{2n}}$ plus a linear combination of elements $w_{j_1}\otimes\cdots\otimes w_{j_{2n}}$ with $(w_{j_1},\cdots,w_{j_{2n}}) < (w_{i_1},\cdots,w_{i_{2n}})$ and with coefficients in $v^{-1}\mathbb{Z}[v^{-1}]$, where " $<$ " is a partial order defined in [41, (27.3.1)]. In particular, B is an $\mathscr{A}$-basis of $V_{\mathscr{A}}^{\otimes n}$. By [41, (27.2.1)], there is a partition

$$B = \bigsqcup_{\lambda\in X^+} B[\lambda].$$

Let

$$B[\neq 0] := \bigsqcup_{0\neq\lambda\in X^+} B[\lambda], \quad M[\neq 0]_{\mathscr{A}} := \sum_{b\in B[\neq 0]} \mathscr{A}b.$$

By [41, (27.1),(27.2)] and the discussion in [10, Section 4], we know that the isomorphism $\big(\tau^{-1}\big)^{\otimes 2n} : V_{\mathscr{A}}^{\otimes 2n} \to \big(V_{\mathscr{A}}^*\big)^{\otimes 2n} \cong \big(V_{\mathscr{A}}^{\otimes 2n}\big)^*$ induces an isomorphism

$$\big(V_{\mathscr{A}}^{\otimes 2n}\big)^{\mathbb{U}_{\mathscr{A}}(\mathfrak{g})} \cong \Big(V_{\mathscr{A}}^{\otimes 2n}/M[\neq 0]_{\mathscr{A}}\Big)^*.$$

All the above have a counterpart with respect to $\widetilde{V}$, which we will just put the symbol "$\sim$". Therefore, we have the notations $\widetilde{M} := (\widetilde{V}_{\mathbb{Q}(q)})^{\otimes 2n}$, $\widetilde{B}$, $\tilde{w}_{i_1}\tilde{\diamond}\tilde{w}_{i_2}\tilde{\diamond}\cdots\tilde{\diamond}\tilde{w}_{i_{2n}}$, $\widetilde{M}[\neq 0]_{\mathscr{A}}$, and we also have an isomorphism

$$\big(\widetilde{V}_{\mathscr{A}}^{\otimes 2n}\big)^{\mathbb{U}_{\mathscr{A}}(\widetilde{\mathfrak{g}})} \cong \Big(\widetilde{V}_{\mathscr{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathscr{A}}\Big)^*.$$

[3] This is the point where we have to use the isomorphisms $\widetilde{\pi},\widetilde{\iota}$ instead of π,ι.

Lemma 6.3. *With the notations as above, the following diagram of maps*

$$\begin{array}{ccccc}
\big(\widetilde{V}_{\mathscr{A}}^{\otimes 2n}\big)^{\mathbb{U}_{\mathscr{A}}(\widetilde{\mathfrak{g}})} & \xrightarrow{\ \sim\ } & \Big(\widetilde{V}_{\mathscr{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathscr{A}}\Big)^{*} & \longrightarrow & \Big(\widetilde{V}_{\mathscr{A}}^{\otimes 2n}\Big)^{*} \\
\widetilde{\pi}^{\otimes 2n}\Big\downarrow & & & & \big(\widetilde{\iota}^{\otimes 2n}\big)^{*}\Big\downarrow \\
\big(V_{\mathscr{A}}^{\otimes 2n}\big)^{\mathbb{U}_{\mathscr{A}}(\mathfrak{g})} & \xrightarrow{\ \sim\ } & \Big(V_{\mathscr{A}}^{\otimes 2n}/M[\neq 0]_{\mathscr{A}}\Big)^{*} & \longrightarrow & \Big(V_{\mathscr{A}}^{\otimes 2n}\Big)^{*}
\end{array}$$

is commutative. In particular, we have

$$\widetilde{\iota}^{\otimes 2n}\big(M[\neq 0]_{\mathscr{A}}\big) \subseteq \widetilde{M}[\neq 0]_{\mathscr{A}}.$$

Proof. This follows from direct verification. □

Therefore, to prove (6.2), it suffices to show that

$$\big(\widetilde{\iota}^{\otimes 2n}\big)^{*}\Big(\big(\widetilde{V}_{\mathscr{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathscr{A}}\big)^{*}\Big) = \big(V_{\mathscr{A}}^{\otimes 2n}/M[\neq 0]_{\mathscr{A}}\big)^{*}. \tag{6.4}$$

Let

$$J_0 := \Big\{(i_1, \cdots, i_{2n}) \in I(2m, 2n) \,\Big|\, w_{i_1} \diamond \cdots \diamond w_{i_{2n}} \in B[0]\Big\},$$
$$\widetilde{J}_0 := \Big\{(i_1, \cdots, i_{2n}) \in I(2m_0, 2n) \,\Big|\, \tilde{w}_{i_1} \tilde{\diamond} \cdots \tilde{\diamond} \tilde{w}_{i_{2n}} \in \widetilde{B}[0]\Big\}.$$

Lemma 6.5. ([10, Corollary 4.5]) *With the above notation, the set*

$$\Big\{w_{i_1} \otimes \cdots \otimes w_{i_{2n}} + M[\neq 0]_{\mathscr{A}} \,\Big|\, (i_1, \cdots, i_{2n}) \in J_0\Big\}$$

forms an $\mathscr{A}$-basis of $V_{\mathscr{A}}^{\otimes 2n}/M[\neq 0]_{\mathscr{A}}$, and the set

$$\Big\{\tilde{w}_{i_1} \otimes \cdots \otimes \tilde{w}_{i_{2n}} + \widetilde{M}[\neq 0]_{\mathscr{A}} \,\Big|\, (i_1, \cdots, i_{2n}) \in \widetilde{J}_0\Big\}$$

forms an $\mathscr{A}$-basis of $\widetilde{V}_{\mathscr{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathscr{A}}$.

We set

$$J_0[m_0 - m] := \Big\{(m_0 - m + i_1, \cdots, m_0 - m + i_{2n}) \,\Big|\, (i_1, \cdots, i_{2n}) \in J_0\Big\}.$$

Theorem 6.6. *With the above notation, $J_0[m_0 - m] \subseteq \widetilde{J}_0$.*

Proof. This is proved by using the same argument as in the proof of [10, Theorem 4.7]). □

Now Lemma 6.5 and Theorem 6.6 imply that $\widetilde{\iota}^{\otimes 2n}$ maps $V_{\mathscr{A}}^{\otimes 2n}/M[\neq 0]_{\mathscr{A}}$ onto an $\mathscr{A}$-direct summand of $\widetilde{V}_{\mathscr{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathscr{A}}$. It follows that

$$\big(\widetilde{\iota}^{\otimes 2n}\big)^{*}\Big(\big(\widetilde{V}_{\mathscr{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathscr{A}}\big)^{*}\Big) = \big(V_{\mathscr{A}}^{\otimes 2n}/M[\neq 0]_{\mathscr{A}}\big)^{*},$$

which proves (6.4). This completes the proof of Lemma 6.1 (2).

It remains to prove Lemma 6.1 (1). *From now on and until the end of this paper we shall set, for any integer i with $1 \leq i \leq 2m_0$,*

$$i' := 2m_0 + 1 - i.$$

Note that both Θ_0 and Θ_1 are in general not algebra maps. Let f be an integer with $0 \leq f \leq [n/2]$. By definition,

$$\Theta_1\Big(\widetilde{E}_1\widetilde{E}_3 \cdots \widetilde{E}_{2f-1}\Big) = \zeta^{(m_0+m)n} E_1 E_3 \cdots E_{2f-1}.$$

Therefore,

$$\begin{aligned}\varphi_C\Theta_1\Big(\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\Big)&=\zeta^{(m_0+m)n}\varphi_C\Big(E_1E_3\cdots E_{2f-1}\Big)\\&=\zeta^{(m_0+m)n}\varphi_C(E_1)\varphi_C(E_3)\cdots\varphi_C(E_{2f-1})\\&=\Theta_0\Big(\varphi_C(\widetilde{E}_1)\varphi_C(\widetilde{E}_3)\cdots\varphi_C(\widetilde{E}_{2f-1})\Big)\\&=\Theta_0\varphi_C\Big(\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\Big),\end{aligned}$$

where the third equality follows from the fact that different $\varphi_C(\widetilde{E}_{2i-1})$ acts on pairwise non-intersected positions.

Let $\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$, $\nu:=((2^f),(n-2f))$, $d_1,d_2\in\mathcal{D}_\nu$. Our purpose is to show that

$$\varphi_C\Theta_1\Big(\widetilde{T}^*_{d_1}\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\Big)=\Theta_0\varphi_C\Big(\widetilde{T}^*_{d_1}\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\Big).$$

Equivalently,

(6.7)

$$\varphi_C\big(T^*_{d_1}E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2}\big)=\zeta^{-(m_0+m)n}\Theta_0\varphi_C\big(\widetilde{T}^*_{d_1}\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\big).$$

We divide the proof into four steps:

Step 1. We want to prove that $\varphi_C\Theta_1\Big(\widetilde{T}_w\Big)=\Theta_0\varphi_C\Big(\widetilde{T}_w\Big)$ for any $w\in\mathfrak{S}_n$.

We set

$$\widehat{I}(2m,n)=\big\{\mathbf{i}\in I(2m,n)\bigm| m_0-m+1\le i_t\le m_0+m\text{ for each }t\big\}.$$

Note that, by definition, $\Theta_1\Big(\widetilde{T}_w\Big)=\zeta^{(m_0+m)n}T_w$ for each $w\in\mathfrak{S}_n$. Hence it suffices to show that for any $w\in\mathfrak{S}_n$,

$$\varphi_C\Big(T_w\Big)=\zeta^{-(m_0+m)n}\Theta_0\varphi_C\Big(\widetilde{T}_w\Big).\tag{6.8}$$

Let $w\in\mathfrak{S}_n$, $1\le k\le n-1$, such that $\ell(ws_k)=\ell(w)+1$. Let $\mathbf{i}\in\widehat{I}(2m,n)$, $\mathbf{j}\in I(2m_0,n)$, such that $v_{\mathbf{j}}$ is involved in $v_{\mathbf{i}}\widetilde{T}_{ws_k}$. We claim that

(A) for any $\mathbf{l}\in I(2m_0,n)$ such that $v_{\mathbf{l}}$ is involved in $v_{\mathbf{i}}\widetilde{T}_w$ and $v_{\mathbf{j}}$ is involved in $v_{\mathbf{l}}\widetilde{T}_k$, if $l_k=(l_{k+1})'$, and $m_0-m+1\le l_b\le m_0+m$ whenever $b\neq k,k+1$, then $l_k\ge m_0-m+1$.

If $\ell(w)=0$, there is nothing to prove. Assume $\ell(w)=a\ge 1$. We fix a reduced expression of w as follows:

$$w=s_{q_1}s_{q_2}\cdots s_{q_a}.$$

We set $q_{a+1}=k$. By assumption, $v_{\mathbf{l}}$ is involved in $v_{\mathbf{i}}\widetilde{T}_{q_1}\cdots\widetilde{T}_{q_a}$. It follows that there exist

$$\mathbf{j}^{[0]},\mathbf{j}^{[1]},\cdots,\mathbf{j}^{[a+1]}\in I(2m_0,n)$$

such that

(1) $\mathbf{j}^{[0]}=\mathbf{i}$, $\mathbf{j}^{[a]}=\mathbf{l}$, $\mathbf{j}^{[a+1]}=\mathbf{j}$;

(2) for each integer $1\le t\le a+1$, $v_{\mathbf{j}^{[t]}}$ is involved in $v_{\mathbf{j}^{[t-1]}}\widetilde{T}_{q_t}$.

For each integer $2\le t\le a+1$, we set

$$w_t:=s_{q_1}s_{q_2}\cdots s_{q_{t-1}}.$$

For any integer t with $2\le t\le a+1$, we define

$$A_t:=\sum_{\substack{1\le b\le n\\1\le j_b^{[t-1]}<m_0-m+1}}(b)w_t^{-1},\quad B_t:=\sum_{\substack{1\le b\le n\\m_0+m<j_b^{[t-1]}\le 2m_0}}(b)w_t^{-1},$$

$$C_t:=A_t-B_t.$$

We claim that

$$0 \le C_2 \le C_3 \le \cdots \le C_{a+1}. \tag{6.9}$$

Recall by definition that $\widetilde{T}_{s_{q_t}}$ acts only on the (q_t, q_t+1) positions of the simple tensor $v_{\mathbf{j}^{[t-1]}}$ in the same way as the operator β' acts on $v_{j^{[t-1]}_{q_t}} \otimes v_{j^{[t-1]}_{q_t+1}}$. To compare C_t with C_{t+1}, it suffices to check the simple tensors involved in

$$\Big(v_{j^{[t-1]}_{q_t}} \otimes v_{j^{[t-1]}_{q_t+1}}\Big)\beta'.$$

By the explicit definition of the operator β', we need only consider the following six possibilities:

Case 1. $\big\{j^{[t-1]}_{q_t}, j^{[t-1]}_{q_t+1}, j^{[t]}_{q_t}, j^{[t]}_{q_t+1}\big\} \subseteq \big\{m_0-m+1, m_0-m+2, \cdots, m_0+m\big\}$. In this case, it is clear that $C_t = C_{t+1}$.

Case 2. $j^{[t-1]}_{q_t} \neq \big(j^{[t-1]}_{q_t+1}\big)'$. Then we have either $\big(j^{[t-1]}_{q_t}, j^{[t-1]}_{q_t+1}\big) = \big(j^{[t]}_{q_t+1}, j^{[t]}_{q_t}\big)$ or $\big(j^{[t-1]}_{q_t}, j^{[t-1]}_{q_t+1}\big) = \big(j^{[t]}_{q_t}, j^{[t]}_{q_t+1}\big)$. In both cases, we still get that $C_t = C_{t+1}$.

Case 3. $j^{[t-1]}_{q_t} = \big(j^{[t-1]}_{q_t+1}\big)' > m_0+m$. Then we must have

$$j^{[t]}_{q_t} = \big(j^{[t]}_{q_t+1}\big)' \le j^{[t-1]}_{q_t+1} < m_0-m+1.$$

In this case, since

$$\begin{aligned} &\big(q_t\big)s_{q_t}^{-1} = q_t+1, \ \ \big(q_t+1\big)s_{q_t}^{-1} = q_t, \\ &j^{[t-1]}_{q_t+1},\, j^{[t]}_{q_t} < m_0-m+1, \ \ j^{[t-1]}_{q_t}, j^{[t]}_{q_t+1} > m_0+m. \end{aligned}$$

and $\big(b\big)s_{q_t}^{-1} = b$ for any $b \notin \{q_t, q_t+1\}$, it follows easily that $C_t = C_{t+1}$.

Case 4. $m_0-m+1 \le j^{[t-1]}_{q_t} = \big(j^{[t-1]}_{q_t+1}\big)' \le m_0+m$, and

$$j^{[t]}_{q_t} = \big(j^{[t]}_{q_t+1}\big)' < m_0-m+1.$$

In this case, since $\ell(w_t s_{q_t}) = \ell(w_t)+1$, it follows that

$$\big(q_t\big)w_{t+1}^{-1} = \big(q_t+1\big)w_t^{-1} > \big(q_t\big)w_t^{-1} = \big(q_t+1\big)w_{t+1}^{-1}.$$

and $\big(b\big)s_{q_t}^{-1} = b$ for any $b \notin \{q_t, q_t+1\}$, it follows that

$$C_{t+1} = C_t + \Big(\big(q_t\big)w_{t+1}^{-1} - \big(q_t+1\big)w_{t+1}^{-1}\Big) > C_t.$$

Case 5. $j^{[t-1]}_{q_t} = \big(j^{[t-1]}_{q_t+1}\big)' < m_0-m+1$, and $\big(j^{[t-1]}_{q_t}, j^{[t-1]}_{q_t+1}\big) \neq \big(j^{[t]}_{q_t}, j^{[t]}_{q_t+1}\big)$. In this case, we must have $j^{[t]}_{q_t} = \big(j^{[t]}_{q_t+1}\big)'$. Since $\ell(w_t s_{q_t}) = \ell(w_t)+1$, it follows that

$$\big(q_t\big)w_{t+1}^{-1} = \big(q_t+1\big)w_t^{-1} > \big(q_t\big)w_t^{-1} = \big(q_t+1\big)w_{t+1}^{-1}.$$

and $\big(b\big)s_{q_t}^{-1} = b$ for any $b \notin \{q_t, q_t+1\}$. If $j^{[t]}_{q_t} > m_0+m$, then it is clear that $C_t = C_{t+1}$; if $j^{[t]}_{q_t} < m_0-m+1$, then

$$C_{t+1} = C_t - \Big(\big(q_t\big)w_t^{-1} - \big(q_t+1\big)w_t^{-1}\Big) + \Big(\big(q_t\big)w_{t+1}^{-1} - \big(q_t+1\big)w_{t+1}^{-1}\Big) > C_t;$$

if $m_0-m+1 \le j^{[t]}_{q_t} \le m_0+m$, then

$$C_{t+1} = C_t - \Big(\big(q_t\big)w_t^{-1} - \big(q_t+1\big)w_t^{-1}\Big) > C_t.$$

Case 6. $j^{[t-1]}_{q_t} = \big(j^{[t-1]}_{q_t+1}\big)' < m_0-m+1$, and $\big(j^{[t-1]}_{q_t}, j^{[t-1]}_{q_t+1}\big) = \big(j^{[t]}_{q_t}, j^{[t]}_{q_t+1}\big)$. Since $\ell(w_t s_{q_t}) = \ell(w_t)+1$, it follows that

$$\big(q_t\big)w_{t+1}^{-1} = \big(q_t+1\big)w_t^{-1} > \big(q_t\big)w_t^{-1} = \big(q_t+1\big)w_{t+1}^{-1}.$$

and $\big(b\big)s_{q_t}^{-1}=b$ for any $b\notin\{q_t,q_t+1\}$. Therefore

$$C_{t+1}=C_t-\Big(\big(q_t\big)w_t^{-1}-\big(q_t+1\big)w_t^{-1}\Big)+\Big(\big(q_t\big)w_{t+1}^{-1}-\big(q_t+1\big)w_{t+1}^{-1}\Big)>C_t,$$

as required. This proves our claim (6.9).

Since $\ell(w_{a+1}s_k)=\ell(w_{a+1})+1$, it follows that $\big(k\big)w_{a+1}^{-1}<\big(k+1\big)w_{a+1}^{-1}$. Now suppose that $l_k<m_0-m+1$. Then by our assumption on $\mathbf{l}$, it is easy to see that

$$C_{a+1}=(k)w_{a+1}^{-1}-(k+1)w_{a+1}^{-1}<0,$$

which contradicts to (6.9). It follows that $l_k\geq m_0-m+1$. This completes the proof of our claim (A).

Now we use induction on $\ell(w)$ and the results (A) to prove our claim (6.8). If $\ell(w)=0$, there is nothing to prove. Let $w=us_k$ with $\ell(w)=\ell(u)+1$. Suppose $\varphi_C\Big(T_u\Big)=\zeta^{-(m_0+m)n}\Theta_0\varphi_C\Big(\widetilde{T}_u\Big)$. Then for each $\mathbf{i}\in\widehat{I}(2m,n)$, we can write

$$v_{\mathbf{i}}\widetilde{T}_u=\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_u\big)+\sum_{\mathbf{l}\in I(2m_0,n)\setminus\widehat{I}(2m,n)}A_{\mathbf{i},\mathbf{l}}v_{\mathbf{l}},$$

where $A_{\mathbf{i},\mathbf{l}}\in K$ for each $\mathbf{l}$, and

$$\mathbf{i}-m_0+m:=(i_1-m_0+m,\cdots,i_n-m_0+m).$$

Therefore,

$$v_{\mathbf{i}}\widetilde{T}_w=\Big(\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_u\big)\Big)\widetilde{T}_k+\sum_{\mathbf{l}\in I(2m_0,n)\setminus\widehat{I}(2m,n)}A_{\mathbf{i},\mathbf{l}}v_{\mathbf{l}}\widetilde{T}_k.$$

Note that $A_{\mathbf{i},\mathbf{l}}\neq 0$ implies that $v_{\mathbf{l}}$ is involved in $v_{\mathbf{i}}\widetilde{T}_u$.

We claim that $\pi^{\otimes n}\big(v_{\mathbf{l}}\widetilde{T}_k\big)=0$ whenever $A_{\mathbf{i},\mathbf{l}}\neq 0$. In fact, by the definition of β' and the fact that $\mathbf{l}\in I(2m_0,n)\setminus\widehat{I}(2m,n)$, it is easy to see that $\pi^{\otimes n}\big(v_{\mathbf{l}}\widetilde{T}_k\big)\neq 0$ only if $l_k=(l_{k+1})'<m_0-m+1$ and $m_0-m+1\leq l_b\leq m_0+m$ whenever $b\neq k,k+1$. Applying our result (A), we know that this is impossible. This proves our claim.

Note also that

$$\Big(\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_u\big)\Big)\widetilde{T}_k=\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_uT_k\big)+\sum_{\mathbf{j}\in I(2m_0,n)\setminus\widehat{I}(2m,n)}A'_{\mathbf{i},\mathbf{j}}v_{\mathbf{j}},$$

where $A'_{\mathbf{i},\mathbf{j}}\in K$ for each $\mathbf{j}$. As a consequence, we get that

$$\pi^{\otimes n}\Big(v_{\mathbf{i}}\widetilde{T}_w\Big)=v_{\mathbf{i}-m_0+m}T_w.$$

Equivalently, $\varphi_C\Big(T_w\Big)=\zeta^{-(m_0+m)n}\Theta_0\varphi_C\Big(\widetilde{T}_w\Big)$, as required. This completes the proof of our claim (6.8). As a direct consequence, it is easy to see that for any $\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$,

$$\varphi_C\Theta_1\Big(\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\Big)=\Theta_0\varphi_C\Big(E_1E_3\cdots E_{2f-1}T_\sigma\Big).\tag{6.10}$$

Step 2. We claim that for any $d_1\in\mathcal{D}_{\nu_f}$,

$$\varphi_C\Theta_1\Big(\widetilde{T}^*_{d_1}\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\Big)=\Theta_0\varphi_C\Big(\widetilde{T}^*_{d_1}\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\Big).$$

By definition, $\Theta_1\Big(\widetilde{T}^*_{d_1}\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\Big)=\zeta^{(m_0+m)n}T^*_{d_1}E_1E_3\cdots E_{2f-1}T_\sigma$. Therefore, our claim is equivalent to

$$\varphi_C\Big(T^*_{d_1}E_1E_3\cdots E_{2f-1}T_\sigma\Big)=\zeta^{-(m_0+m)n}\Theta_0\varphi_C\Big(\widetilde{T}^*_{d_1}\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\Big).\tag{6.11}$$

By (6.10), for any $\mathbf{i}\in\widehat{I}(2m,n)$,

$$v_{\mathbf{i}}\widetilde{T}^*_{d_1}=\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T^*_{d_1}\big)+\sum_{\mathbf{l}\in I(2m_0,n)\setminus\widehat{I}(2m,n)}\widehat{A}_{\mathbf{i},\mathbf{l}}v_{\mathbf{l}},$$

where $\widehat{A}_{\mathbf{i},\mathbf{l}}\in K$ for each $\mathbf{l}$.

To prove (6.11), it suffices to show that

$$\ell_s(l_1,\cdots,l_{2f})<f \text{ whenever } \widehat{A}_{\mathbf{i},\mathbf{l}}\neq 0. \tag{6.12}$$

It remains to prove (6.12). By [10, Lemma 3.8], we can write $d_1=d_{11}d_J$, where $d_{11}\in\mathcal{D}_f$, $J\in\mathcal{P}_f$. Then,

$$\widetilde{T}^*_{d_1}=\widetilde{T}^*_{d_J}\widetilde{T}^*_{d_{11}}=\widetilde{T}_{d_J^{-1}}\widetilde{T}_{d_{11}^{-1}}.$$

Since $d_{11}^{-1}\in\mathfrak{S}_{2f}$, the action of $\widetilde{T}_{d_{11}^{-1}}$ does not change the symplectic length of the first $(2f)$ parts of any simple n-tensor. Therefore, using (6.8), we can assume without loss of generality that $d_{11}=1$, and hence $d_1=d_J\in\widetilde{\mathcal{D}}_{(2f,n-2f)}$. With this assumption, we claim that

$$v_{\mathbf{i}}\widetilde{T}^*_{d_J}=\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T^*_{d_J}\big)+\sum_{\mathbf{l}\in I(2m_0,n)\setminus\widehat{I}(2m,n)}\widehat{A}_{\mathbf{i},\mathbf{l}}v_{\mathbf{l}}, \tag{6.13}$$

where $\widehat{A}_{\mathbf{i},\mathbf{l}}\in K$ for each $\mathbf{l}$, and $\widehat{A}_{\mathbf{i},\mathbf{l}}\neq 0$ only if

$$\big\{l_1,\cdots,l_{2f}\big\}\bigcap\big\{m_0+m+1,m_0+m+2,\cdots,2m_0\big\}=\emptyset,$$

and either l_1 or l_2 belongs to $\big\{1,2,\cdots,m_0-m\big\}$. If this is true, then it is clear that $\widehat{A}_{\mathbf{i},\mathbf{l}}\neq 0$ only if $\ell_s(\mathbf{l})<f$ and hence (6.12) follows.

Let $J=(j_1,j_2,\cdots,j_{2f})$. Then $1\leq j_1<j_2<\cdots<j_{2f}\leq n$. By Lemma 5.8,

$$(s_{j_1-1}\cdots s_2s_1)(s_{j_2-1}\cdots s_3s_2)\cdots(s_{j_{2f-1}-1}\cdots s_{2f}s_{2f-1})(s_{j_{2f}-1}\cdots s_{2f+1}s_{2f})$$

is a reduced expression of d_J^{-1}. If $f=1$, then

$$\widetilde{T}^*_{d_J}=(\widetilde{T}_{j_1-1}\cdots\widetilde{T}_2\widetilde{T}_1)(\widetilde{T}_{j_2-1}\cdots\widetilde{T}_3\widetilde{T}_2).$$

In this case, suppose that $v_{\mathbf{l}}$ is involved in $v_{\mathbf{i}}\widetilde{T}^*_{d_J}$, where $\mathbf{i}\in\widehat{I}(2m,n)$. Then, there must exist

$$\mathbf{l}^{[0]},\mathbf{l}^{[1]},\cdots,\mathbf{l}^{[j_1+j_2-3]}\in I(2m_0,n)$$

such that

(1) $\mathbf{l}^{[0]}=\mathbf{i}$, $\mathbf{l}^{[j_1+j_2-3]}=\mathbf{l}$;
(2) for each $1\leq t\leq j_1-1$, $v_{\mathbf{l}^{[t]}}$ is involved in $v_{\mathbf{i}}(\widetilde{T}_{j_1-1}\cdots\widetilde{T}_{j_1-t+1}\widetilde{T}_{j_1-t})$;
(3) for each $j_1\leq t\leq j_1+j_2-3$, $v_{\mathbf{l}^{[t]}}$ is involved in
$$v_{\mathbf{i}}(\widetilde{T}_{j_1-1}\cdots\widetilde{T}_2\widetilde{T}_1)(\widetilde{T}_{j_2-1}\cdots\widetilde{T}_{j_2-t+j_1}\widetilde{T}_{j_2-t+j_1-1});$$
(4) for each $1\leq t\leq j_1-1$, $v_{\mathbf{l}^{[t]}}$ is involved in $v_{\mathbf{l}^{[t-1]}}\widetilde{T}_{j_1-t}$;
(5) for each $j_1\leq t\leq j_1+j_2-3$, $v_{\mathbf{l}^{[t]}}$ is involved in $v_{\mathbf{l}^{[t-1]}}\widetilde{T}_{j_1+j_2-t-1}$.

Now suppose $\mathbf{l}\in I(2m_0,n)\setminus\widehat{I}(2m,n)$. If there exists an integer $1\leq b\leq j_1$ such that

$$m_0-m+1\leq l^{[b-1]}_{j_1-b}=\big(l^{[b-1]}_{j_1-b+1}\big)'\leq m_0+m,\ \ l^{[b]}_{j_1-b}=\big(l^{[b]}_{j_1-b+1}\big)'<m_0-m+1,$$

then we choose such a b which is maximal. By (5.13), we have $l_1=l_1^{[j_1-1]}<m_0-m+1$ and $l_2\leq l_{j_2}^{[j_1-1]}\leq m_0+m$. If there does not exist such a b, then there must exist an integer $j_1\leq c\leq j_1+j_2-3$ such that

$$\begin{aligned}m_0-m+1\leq l^{[c-1]}_{j_1+j_2-c-1}&=\big(l^{[c-1]}_{j_1+j_2-c}\big)'\leq m_0+m,\\ l^{[c]}_{j_1+j_2-c-1}&=\big(l^{[c]}_{j_1+j_2-c}\big)'<m_0-m+1.\end{aligned}$$

We choose such a c which is maximal. By (5.13), $l_2 < m_0 - m + 1$. The non-existence of b implies that $m_0 - m + 1 \leq l_1 \leq m_0 + m$. This proves our claim in the case $f = 1$.

Now we assume that $f \geq 2$. We use induction on $n - 2f$. If $n - 2f = 0$, then $d_J = 1$, there is nothing to prove. We set

$$\widehat{d}_J = (s_{2f-1}s_{2f}\cdots s_{j_{2f-1}-1})(s_{2f-2}s_{2f-1}\cdots s_{j_{2f-2}-1})\cdots(s_1s_2\cdots s_{j_1-1}).$$

Then $d_J^{-1} = \widehat{d}_J^{-1}(s_{j_{2f}-1}\cdots s_{2f+1}s_{2f})$ and

$$\ell(d_J^{-1}) = \ell(\widehat{d}_J^{-1}) + j_{2f} - 2f.$$

If $j_{2f} \leq n-1$, then $d_J \in \mathcal{D}_{(2f,n-1-2f)}$, and we are done by induction hypothesis. If $j_{2f} = n$, then by induction hypothesis, we have

$$v_{\mathbf{i}}\widetilde{T}^*_{\widehat{d}_J} = \iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T^*_{\widehat{d}_J}\big) + \sum_{\mathbf{l}\in I(2m_0,n)\setminus\widehat{I}(2m,n)} \widehat{B}_{\mathbf{i},\mathbf{l}}v_{\mathbf{l}},$$

where $\widehat{B}_{\mathbf{i},\mathbf{l}} \in K$ for each $\mathbf{l}$, and $\widehat{B}_{\mathbf{i},\mathbf{l}} \neq 0$ only if

(1) $m_0 - m + 1 \leq l_n \leq m_0 + m$; and
(2) $\big\{l_1, \cdots, l_{2f}\big\}\bigcap\big\{m_0+m+1, m_0+m+2, \cdots, 2m_0\big\} = \emptyset$; and
(3) either l_1 or l_2 belongs to $\big\{1, 2, \cdots, m_0 - m\big\}$.

It remains to check the simple tensors involved in $v_{\mathbf{l}}\widetilde{T}_{n-1}\widetilde{T}_{n-2}\cdots\widetilde{T}_{2f}$ as well as in $\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T^*_{\widehat{d}_J}\big)\widetilde{T}_{n-1}\widetilde{T}_{n-2}\cdots\widetilde{T}_{2f}$.

Since the action of $\widetilde{T}_{n-1}\widetilde{T}_{n-2}\cdots\widetilde{T}_{2f}$ does not change the first $(2f-1)$ positions, it follows from (5.13) and the fact $m_0 - m + 1 \leq l_n \leq m_0 + m$ that

$$\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T^*_{\widehat{d}_J}\big)\widetilde{T}_{n-1}\widetilde{T}_{n-2}\cdots\widetilde{T}_{2f} = \iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T^*_{d_J}\big) + \sum_{\mathbf{u}\in I(2m_0,n)} \widehat{B}'_{\mathbf{i},\mathbf{u}}v_{\mathbf{u}},$$

where $\widehat{B}'_{\mathbf{i},\mathbf{u}} \in K$ for each $\mathbf{u}$, and $\widehat{B}_{\mathbf{i},\mathbf{u}} \neq 0$ only if

$$\big\{u_1, \cdots, u_{2f}\big\}\bigcap\big\{m_0+m+1, m_0+m+2, \cdots, 2m_0\big\} = \emptyset,$$

and either u_1 or u_2 belongs to $\big\{1, 2, \cdots, m_0 - m\big\}$. By the same reason, we can deduce that

$$v_{\mathbf{l}}\widetilde{T}_{n-1}\widetilde{T}_{n-2}\cdots\widetilde{T}_{2f-1} = \sum_{\mathbf{u}\in I(2m_0,n)} \widehat{B}''_{\mathbf{i},\mathbf{u}}v_{\mathbf{u}},$$

where $\widehat{B}''_{\mathbf{i},\mathbf{u}} \in K$ for each $\mathbf{u}$, and $\widehat{B}'_{\mathbf{i},\mathbf{u}} \neq 0$ only if

$$\big\{u_1, \cdots, u_{2f}\big\}\bigcap\big\{m_0+m+1, m_0+m+2, \cdots, 2m_0\big\} = \emptyset,$$

and either u_1 or u_2 belongs to $\big\{1, 2, \cdots, m_0 - m\big\}$. This completes the proof of (6.13), and hence the proof of (6.11).

Step 3. We want to show that for any $d_2 \in \mathcal{D}_{\nu_f}$,

$$\varphi_C\Theta_1\Big(\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\Big) = \Theta_0\varphi_C\Big(\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\Big).$$

As before, it is equivalent to show that

$$\varphi_C\Big(E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2}\Big) = \zeta^{-(m_0+m)n}\Theta_0\varphi_C\Big(\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\Big). \tag{6.14}$$

Recall that $\widetilde{V}^{\otimes n}$ has a basis consists of all the simple n-tensors $v_{\mathbf{i}}$, where $\mathbf{i} \in I(2m_0, n)$. We ordered the elements of this basis as $X_1, X_2, \cdots, X_{(2m_0)^n}$ such that the subset $\{X_1, X_2, \cdots, X_{(2m)^n}\}$ is a basis of $\iota^{\otimes n}(V^{\otimes n})$. With this ordered basis $\{X_1, \cdots, X_{(2m_0)^n}\}$ in mind, we identify $\mathrm{End}_K(\widetilde{V}^{\otimes n})$ with full matrix algebra $\mathrm{M}_{(2m_0)^n\times(2m_0)^n}(K)$, and identify the homomorphism

$$\varphi_C:\ (\mathfrak{B}_n(-\zeta^{2m_0+1},\zeta))^{\mathrm{op}} \to \mathrm{End}_K(\widetilde{V}^{\otimes n})$$

with a homomorphism

$$\varphi_C:\ (\mathfrak{B}_n(-\zeta^{2m_0+1},\zeta))^{\mathrm{op}} \to \mathrm{M}_{(2m_0)^n\times(2m_0)^n}(K).$$

We claim that for any $x\in\mathfrak{B}_n(-\zeta^{2m_0+1},\zeta)$),

$$\varphi_C(x^*) = (\varphi_C(x))^t, \tag{6.15}$$

where $(\varphi_C(x))^t$ means the transpose of the matrix $\varphi_C(x)$, and " $*$ " denotes the algebra anti-automorphism of $\mathfrak{B}_n(-\zeta^{2m_0+1},\zeta)$) defined in Section 5.

In fact, by direct verification, it is easy to see that for each $1\le i\le n-1$, both $\varphi_C(T_i)$ and $\varphi_C(E_i)$ are symmetric matrices (with respect to the above ordered basis). Since $\mathfrak{B}_n(-\zeta^{2m_0+1},\zeta)$ is generated by $T_i, E_i, i=1,2,\cdots,n-1$ and φ_C is an algebra homomorphism, the claim (6.15) follows immediately.

The above argument applies equally well to $V^{\otimes n}$. For each integer i with $1\le i\le(2m)^n$, let $Y_i:=\pi^{\otimes n}(X_i)$. Then, $\{Y_1,\cdots,Y_{(2m)^n}\}$ is a basis of $V^{\otimes n}$. With the ordered basis $\{Y_1,\cdots,Y_{(2m)^n}\}$ in mind, we identify $\mathrm{End}_K(V^{\otimes n})$ with full matrix algebra $\mathrm{M}_{(2m)^n\times(2m)^n}(K)$, and identify the homomorphism

$$\varphi_C:\ (\mathfrak{B}_n(-\zeta^{2m+1},\zeta))^{\mathrm{op}} \to \mathrm{End}_K(V^{\otimes n})$$

with a homomorphism

$$\varphi_C:\ (\mathfrak{B}_n(-\zeta^{2m+1},\zeta))^{\mathrm{op}} \to \mathrm{M}_{(2m)^n\times(2m)^n}(K).$$

As before, for any $x\in\mathfrak{B}_n(-\zeta^{2m_0+1},\zeta)$), we have $\varphi_C(x^*) = (\varphi_C(x))^t$.

We define a linear map Θ_0' from $\mathrm{M}_{(2m_0)^n\times(2m_0)^n}(K)$ to $\mathrm{M}_{(2m)^n\times(2m)^n}(K)$ as follows: for any $M\in\mathrm{M}_{(2m_0)^n\times(2m_0)^n}(K)$, $\Theta_0'(M)$ is the submatrix of M in the upper-left corner, obtained from M by deleting the last $(2m_0)^n-(2m)^n$ rows and the last $(2m_0)^n-(2m)^n$ columns. Then, it is clear that

$$\Theta_0'(M^t) = (\Theta_0'(M))^t. \tag{6.16}$$

With the ordered bases $\{X_1,X_2,\cdots,X_{(2m_0)^n}\}$ and $\{Y_1,\cdots,Y_{(2m)^n}\}$ in mind, it is easy to check that we can identify the linear map Θ_0 as the restriction of $\zeta^{(m_0+m)n}\Theta_0'$.

Applying (6.15), (6.16) and (6.11), we get that

$$\begin{aligned}&\zeta^{-(m_0+m)n}\Theta_0\varphi_C\Big(\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\Big)\\
&=\Theta_0'\varphi_C\Big(\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\Big)\\
&=\Theta_0'\varphi_C\Big(\big(\widetilde{T}_{d_2}^*\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_{\sigma^{-1}}\big)^*\Big)\\
&=\Theta_0'\Big(\big(\varphi_C(\widetilde{T}_{d_2}^*\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_{\sigma^{-1}})\big)^t\Big)\\
&=\Big(\Theta_0'\varphi_C\Big(\widetilde{T}_{d_2}^*\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_{\sigma^{-1}}\Big)\Big)^t\\
&=\Big(\varphi_C\Big(T_{d_2}^*E_1E_3\cdots E_{2f-1}T_{\sigma^{-1}}\Big)\Big)^t\\
&=\varphi_C\Big(\big(T_{d_2}^*E_1E_3\cdots E_{2f-1}T_{\sigma^{-1}}\big)^*\Big)\\
&=\varphi_C\Big(E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2}\Big),\end{aligned}$$

as required. This completes the proof of (6.14).

Step 4. We are now in a position to prove (6.7). Let $\mathbf{i}\in\widehat{I}(2m,n)$, $\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$, $\nu:=((2^f),(n-2f))$, $d_1,d_2\in\mathcal{D}_\nu$. It suffices to show that

$$\begin{aligned}v_{\mathbf{i}}\widetilde{T}_{d_1}^*\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}&=\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_{d_1}^*E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2}\big)\\
&\quad+\sum_{\mathbf{j}\in I(2m_0,n)\setminus\widehat{I}(2m,n)}\widehat{A}'_{\mathbf{i},\mathbf{j}}v_{\mathbf{j}},\end{aligned}$$

where $\widehat{A}'_{\mathbf{i},\mathbf{j}}\in K$ for each $\mathbf{j}$.

By (6.12),

$$v_{\mathbf{i}}\widetilde{T}_{d_1}^*=\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_{d_1}^*\big)+\sum_{\substack{\mathbf{l}\in I(2m_0,n)\setminus\widehat{I}(2m,n)\\ \ell_s(l_1,\cdots,l_{2f})<f}}\widehat{A}_{\mathbf{i},\mathbf{l}}v_{\mathbf{l}},$$

where $\widehat{A}_{\mathbf{i},\mathbf{l}}\in K$ for each $\mathbf{l}$. Therefore,

$$v_{\mathbf{i}}\widetilde{T}_{d_1}^*\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}=\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_{d_1}^*\big)\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}.$$

Now we use (6.14), it follows that

$$\begin{aligned}&\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_{d_1}^*\big)\widetilde{E}_1\widetilde{E}_3\cdots\widetilde{E}_{2f-1}\widetilde{T}_\sigma\widetilde{T}_{d_2}\\
&=\iota^{\otimes n}\big(v_{\mathbf{i}-m_0+m}T_{d_1}^*E_1E_3\cdots E_{2f-1}T_\sigma T_{d_2}\big)+\sum_{\mathbf{j}\in I(2m_0,n)\setminus\widehat{I}(2m,n)}\widehat{A}'_{\mathbf{i},\mathbf{j}}v_{\mathbf{j}},\end{aligned}$$

where $\widehat{A}'_{\mathbf{i},\mathbf{j}}\in K$ for each $\mathbf{j}$, as required. This completes the proof of (6.7). Hence we complete the proof of Lemma 6.1 (1).

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References

1. H. H. Andersen, P. Polo and K. X Wen, Representations of quantum algebras, *Invent. Math.* **104** (1991), 1–59.
2. A.A. Beilinson, G. Lusztig and R. Macpherson, A geometric setting for the quantum deformation of GL_n, *Duke Math. J.* **61** (1990) 655–677.
3. R. Brauer, On algebras which are connected with semisimple continuous groups, *Ann. of Math.* **38** (1937), 857–872.
4. W. P. Brown, An algebra related to the orthogonal group, *Michigan Math. J.* **3** (1955–1956), 1–22.
5. W. P. Brown, The semisimplicity of ω_f^n, *Ann. of Math.* **63** (1956), 324–335.
6. J. Birman and H. Wenzl, Braids, link polynomials and a new algebra, *Trans. Amer. Math. Soc.* (1) **313** (1989), 249–273.
7. R. W. Carter and G. Lusztig, On the modular representations of general linear and symmetric groups, *Math. Z.* **136** (1974), 193–242.
8. V. Chari and A. Pressley, "A guide to quantum groups," Cambridge University Press, Cambridge, 1994.
9. C. De Concini and C. Procesi, A characteristic free approach to invariant theory, *Adv. Math.* **21** (1976), 330–354.
10. R. Dipper, S. Doty and J. Hu, Brauer algebras, symplectic Schur algebras and Schur–Weyl duality, *Trans. Amer. Math. Soc.* **360** (2008), 189–213.
11. R. Dipper and G. D. James, Representations of Hecke algebras of general linear groups, *Proc. London. Math. Soc.* (3) **52** (1986), 20–52.
12. R. Dipper and G. D. James, The q-Schur algebras, *Proc. London Math. Soc.* (3) **59** (1989), 23–50.
13. S. Donkin, Good filtrations of rational modules for reductive groups, Arcata Conf. on Repr. of Finite Groups, Proceedings of Symp. in Pure Math., **47** (1987), 69–80.
14. S. Donkin, Invariants of several matrices, *Invent. Math.* **110** (1992), 389–401.
15. S. Doty, Polynomial representations, algebraic monoids, and Schur algebras of classic type, *J. Pure Appl. Algebra* **123** (1998), 165–199.
16. S. Doty, Presenting generalized q-Schur algebras, *Representation Theory (an electronic journal of the AMS)* **7** (2003), 196–213.
17. S. Doty, A. Giaquinto and J. Sullivan, Presenting generalized Schur algebras in types B, C, D, *Adv. Math.* (2) **206** (2006), 434–454.
18. S. Doty and J. Hu, Schur–Weyl duality for orthogonal groups, *Proc. London Math. Soc.* **98** (2009), 679–713.
19. V.G. Drinfel'd, Hopf algebras and the quantum Yang-Baxter equations, *Soviet math. Dokl.* **32** (1985), 254–258.
20. J. Du, A note on quantized Weyl reciprocity at roots of unity, *Algebra Colloq.* **2** (1995) 363–372.
21. J. Du, B. Parshall and L. Scott, Quantum Weyl reciprocity and tilting modules, *Commun. Math. Phys.* **195** (1998), 321–352.
22. J. Enyang, Cellular bases for the Brauer and Birman–Murakami–Wenzl algebras, *J. Alg.* **281** (2004), 413–449.
23. L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, Quantization of Lie groups and Lie algebras, *Algebra and Analysis* **1** (1989) 178–206 (in Russian).
24. J. Gruber and G. Hiss, Decomposition numbers of finite classical groups for linear primes, *J. Reine Angew. Math.* **485** (1997), 55–91.
25. J. J. Graham and G. I. Lehrer, Cellular algebras, *Invent. Math.* **123** (1996), 1–34.
26. J. A. Green, "Polynomial representations of GL_n," Lect. Notes in Math. Vol. 830, Springer-Verlag, 1980.
27. I. Grojnowski and G. Lusztig, A comparison of bases of quantized enveloping algebras, Contemp. Math., **153** (1993), 11–19.
28. D. Ju. Grigor'ev, An analogue of the Bruhat decomposition for the closure of the cone of a Chevalley group of the classical series, *Sov. Math. Doklady* **23** (1981), 393–397.
29. T. Hayashi, Quantum deformation of classical groups, *Publ. RIMS. Kyoto Uni.* **28** (1992), 57–81.
30. J. Hong and S. J. Kang, "Introduction to Quantum Groups and Crystal Bases", Graduate Studies in Mathematics, **42**, American Mathematical Society, Providence, RI, 2002.
31. J. Hu, Specht filtrations and tensor spaces for the Brauer algebra, *J. Algebraic Combinatorics* **28** (2008), 281–312.
32. M. Jimbo, A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation, *Lett. Math. Physics* **10** (1985), 63–69.

33. M. Jimbo, A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebras, and Yang-Baxter equation, *Lett. Math. Physics* **11** (1986), 247–252.
34. M. Kashiwara, Global crystal bases of quantum groups, *Duke Math. J.* (2) **69**, (1993), 455–485.
35. M. Kashiwara, Crystal bases of modified quantized enveloping algebra, *Duke Math. J.* (2) **73**, (1994), 383–413.
36. L.H. Kauffman, An invariant of regular isotopy, *Trans. Amer. Math. Soc.* **318** (1990), 417–471.
37. B. Kostant, Groups over $\mathbb{Z}$, Proceedings of Symp. in Pure Math., **9** (1966), 90–98.
38. G.I. Lehrer and R.B. Zhang, Strongly multiplicity free modules for Lie algebras and quantum groups, *J. Algebra* (1) **306** (2006), 138–174.
39. G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, *Adv. in Math.* **70** (1988), 237–249.
40. G. Lusztig, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, *J. Amer. Math. Soc.* **3** (1990), 257–296.
41. G. Lusztig, "Introduction to Quantum Groups," Progress in Math., **110** Birkhäuser, Boston, 1990.
42. G. Lusztig, Canonical bases in tensor product, *Proc. Natl. Acad. Sci. USA.,* **89** (1992), 8177–8179.
43. H.R. Morton and A.J. Wassermann, A basis for the Birman–Murakami–Wenzl algebra, unpublished paper, 2000, http://www.liv.ac.uk/ su14/knotprints.html.
44. J. Murakami, The Kauffman polynomial of links and representation theory, *Osaka J. Math.* (4) **26** (1987), 745–758.
45. E. Murphy, The representations of Hecke algebras of type A_n, *J. Alg.* **173** (1995), 97–121.
46. S. Oehms, Centralizer coalgebras, FRT-construction, and symplectic monoids, *J. Alg.* (1) **244** (2001), 19–44.
47. S. Oehms, Symplectic q-Schur algebras, *J. Alg.* (2) **304** (2006), 851–905.
48. I. Schur, Über die rationalen Darstellungen der allgemeinen linearen Gruppe, (1927). Reprinted in I. Schur, Gesammelte Abhandlungen, Vol. III, pp. 68–85, Springer-Verlag, Berlin, 1973.
49. H. Weyl, "The classical groups, their invariants and representations," Princeton University Press, 1946.

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