

ON A GENERALISATION OF THE DIPPER–JAMES–MURPHY CONJECTURE

JUN HU

ABSTRACT. Let r, n be positive integers. Let e be 0 or an integer bigger than 1. Let $v_1, \dots, v_r \in \mathbb{Z}/e\mathbb{Z}$ and $\mathcal{K}_r(n)$ the set of Kleshchev r -partitions of n with respect to $(e; \mathbf{Q})$, where $\mathbf{Q} := (v_1, \dots, v_r)$. The Dipper–James–Murphy conjecture asserts that $\mathcal{K}_r(n)$ is the same as the set of (\mathbf{Q}, e) -restricted bipartitions of n if $r = 2$. In this paper we consider an extension of this conjecture to the case where $r > 2$. We prove that any multi-core $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ in $\mathcal{K}_r(n)$ is a (\mathbf{Q}, e) -restricted r -partition. As a consequence, we show that in the case $e = 0$, $\mathcal{K}_r(n)$ coincides with the set of (\mathbf{Q}, e) -restricted r -partitions of n and also coincides with the set of ladder r -partitions of n .

1. INTRODUCTION

A composition $\alpha = (\alpha_1, \alpha_2, \dots)$ is a finite sequence of non-negative integers; we denote by $|\alpha|$ the sum of this sequence and call α a composition of $|\alpha|$. A partition is a composition whose parts are non-increasing. Let r, n be positive integers. A multipartition, or r -partition, of n is an ordered sequence $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions such that $|\lambda^{(1)}| + \dots + |\lambda^{(r)}| = n$. The partitions $\lambda^{(1)}, \dots, \lambda^{(r)}$ are called the components of $\boldsymbol{\lambda}$. If $r = 2$, a multipartition is also called a bipartition.

Let e be 0 or an integer bigger than 1. Let $v_1, \dots, v_r \in \mathbb{Z}/e\mathbb{Z}$. If $e > 1$, then a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of k is said to be e -restricted if $\lambda_i - \lambda_{i+1} < e$ for any $i \geq 1$. We make the convention that every partition is e -restricted if $e = 0$. The notion of e -restricted partitions plays an important role in the modular representation theory of the symmetric groups \mathfrak{S}_n as well as its associated Iwahori–Hecke algebra $\mathcal{H}_q(\mathfrak{S}_n)$. For example, if $e > 1$ and the parameter q is a primitive e -th root of unity, then it is well-known that simple modules of $\mathcal{H}_q(\mathfrak{S}_n)$ are in one-to-one correspondence with the set of e -restricted partitions of n . The same is true if $e = 0$ and q is not a root of unity. Another important application (cf. [22]) is that the set of e -restricted partitions provides a combinatorial realization of the crystal graph of the integrable highest weight module of level one over the affine Lie algebra $\widehat{\mathfrak{sl}}_e$ if $e > 1$; or over the affine Lie algebra \mathfrak{gl}_∞ if $e = 0$.

In [7], Ariki and Mathas introduced a notion of Kleshchev multipartitions which provides a combinatorial realization of the crystal graph of integrable highest weight module of level r over the affine Lie algebra $\widehat{\mathfrak{sl}}_e$ if $e > 1$; or over the affine Lie algebra \mathfrak{gl}_∞ if $e = 0$. *A priori*, the notion of Kleshchev multipartition is defined with respect to the given $(r+1)$ -tuple $(e; v_1, \dots, v_r)$ and is recursively defined. It is desirable to look for a non-recursive definition. In the case $r = 1$, it coincides with the notion of e -restricted partitions. In general, by a result of Ariki [3], the notion of Kleshchev multipartitions fits nicely with the Dipper–James–Mathas Specht module theory of the cyclotomic Hecke algebra $\mathcal{H}_{r,n}(q; q^{v_1}, \dots, q^{v_r})$ and gives natural labelling of the simple modules of $\mathcal{H}_{r,n}(q; q^{v_1}, \dots, q^{v_r})$, where the parameter q is a primitive e -th root of unity if $e > 1$; or not a root of unity if $e = 0$. Thus, the notion of

2000 *Mathematics Subject Classification.* 20C08, 20C30.

Key words and phrases. Crystal basis, Fock spaces, Kleshchev multipartitions, ladder multipartitions, ladder nodes, Lakshimibai–Seshadri paths.

Kleshchev multipartitions can be regarded as a natural generalization of the notion of e -restricted partitions.

In 1995, when $r = 2$, Dipper, James and Murphy (see [13]) proposed a notion of (Q, e) -restricted bipartitions (which is non-recursively defined), where $Q = -q^m$, $q = \sqrt[r]{1}$ and $e > 1, m \in \mathbb{Z}/e\mathbb{Z}$, and they conjectured that a Kleshchev bipartition of n with respect to $(e; m, 1)$ is the same as a (Q, e) -restricted bipartition of n . This conjecture was proved only recently by Ariki–Jacon [6], using the result of another recent work of Ariki–Kreiman–Tsuchioka [8]. The paper [8] contains a new non-recursive description of Kleshchev bipartitions. In general, in the case $r > 2$, the question of finding a non-recursive characterization of Kleshchev r -partitions remains open.

The starting point of this paper is to explore this open question. We give a natural extension of the Dipper–James–Murphy notion of (Q, e) -restricted bipartitions to the case where $r > 2$, i.e., (\mathbf{Q}, e) -restricted multipartitions, where $\mathbf{Q} := (v_1, \dots, v_r)$. We also introduce a notion of ladder r -partitions. It turns out that any (\mathbf{Q}, e) -restricted multipartition of n is a Kleshchev multipartition in $\mathcal{K}_r(n)$. Our main result asserts that any multi-core $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ in $\mathcal{K}_r(n)$ is a (\mathbf{Q}, e) -restricted multipartition.

As a consequence, we show that if $e = 0$ (in that case every multipartition is a multi-core), then $\mathcal{K}_r(n)$ coincides with the set of (\mathbf{Q}, e) -restricted multipartitions of n , which gives a non-recursive description of Kleshchev r -partition in this case; and also coincides with the set of ladder r -partitions of n , which gives a new recursive description of Kleshchev r -partition in that case. The main result is a generalization of the theorem of Ariki and Jacon [6], i.e., we prove a generalization of the Dipper–James–Murphy conjecture to the case where $e = 0$ and $r > 2$. Conjecturally, everything should be still true in the case where $e > 1$.

The paper is organized as follows. In Section 2, we recall the notions of Kleshchev multipartitions and (\mathbf{Q}, e) -restricted multipartitions. In particular, we show that any (\mathbf{Q}, e) -restricted r -partition of n is a Kleshchev r -partition with respect to $(e; \mathbf{Q})$. We also recall a result of Littelmann and a related result of Kashiwara, and give some consequence of these two results. In Section 3, after introducing the notion of ladder nodes, ladder sequences, ladder multipartitions as well as strong ladder multipartitions, we give the proof of our main result Proposition 3.10. As a consequence we prove the generalized Dipper–James–Murphy conjecture when $e = 0$, where we also show that the notion of ladder r -partition coincides with the notion of strong ladder r -partition in that case.

2. PRELIMINARIES

Let r, n be positive integers. Let $\mathcal{P}_r(n)$ be the set of r -partitions of n . If $\lambda \in \mathcal{P}_r(n)$, then we write $\lambda \vdash n$ and $|\lambda| = n$. Then $\mathcal{P}_r(n)$ is a poset under the **dominance partial order** “ \succeq ”, where $\lambda \succeq \mu$ if

$$\sum_{a=1}^{s-1} |\lambda^{(a)}| + \sum_{j=1}^i \lambda_j^{(s)} \geq \sum_{a=1}^{s-1} |\mu^{(a)}| + \sum_{j=1}^i \mu_j^{(s)},$$

for all $1 \leq s \leq r$ and all $i \geq 1$.

Let $\lambda \in \mathcal{P}_r(n)$. Recall that the **Young diagram** of λ is the set

$$[\lambda] = \{(i, j, s) \mid 1 \leq j \leq \lambda_i^{(s)}\}.$$

The elements of $[\lambda]$ are called the nodes of λ . A **λ -tableau** is a bijection $\mathbf{t}: [\lambda] \rightarrow \{1, 2, \dots, n\}$. The λ -tableau \mathbf{t} is **standard** if $\mathbf{t}(i, j, s) \leq \mathbf{t}(i', j', s)$ whenever $i \leq i'$, $j \leq j'$. Let $\text{Std}(\lambda)$ be the set of standard λ -tableaux. For any two nodes $\gamma =$

$(a, b, c), \gamma' = (a', b', c')$ of λ , say that γ is *below* γ' , or γ' is *above* γ , if either $c > c'$ or $c = c'$ and $a > a'$. If γ' is above γ then we write $\gamma' > \gamma$. A removable node of λ is a triple $(i, j, s) \in [\lambda]$ such that $[\lambda] - \{(i, j, s)\}$ is the Young diagram of a multipartition, while an addable node of λ is a triple (i, j, s) which does not lie in $[\lambda]$ but is such that $[\lambda] \cup \{(i, j, s)\}$ is the Young diagram of a multipartition.

Now let e be 0 or an integer bigger than 1. Let $v_1, \dots, v_r \in \mathbb{Z}/e\mathbb{Z}$. Let $\mathbf{Q} := (v_1, \dots, v_r)$. The **residue** of the node $\gamma = (a, b, c)$ is defined to be

$$\text{res}(\gamma) := b - a + v_c + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z},$$

In this case, we say that γ is a $\text{res}(\gamma)$ -node.

If $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ is an r -partition of $n + 1$ with $[\mu] = [\lambda] \cup \{\gamma\}$ for some removable node γ of μ , we write $\lambda \rightarrow \mu$ or $\mu/\lambda = \gamma$. If in addition $\text{res}(\gamma) = x$, we also write $\lambda \xrightarrow{x} \mu$.

2.1. Definition. ([7]) *Let $x \in \mathbb{Z}/e\mathbb{Z}$. Let $\lambda \in \mathcal{P}_r(n)$ and η be a removable x -node of λ . If whenever γ is an addable x -node of λ which is below η , there are more removable x -nodes between γ and η than there are addable x -nodes, then we call η a **normal** x -node of λ . The unique highest normal x -node of λ is called the **good** x -node of λ ;*

For example, suppose $n = 19, r = 3, e = 4, v_1 = 4\mathbb{Z}, v_2 = 2 + 4\mathbb{Z}, v_3 = 4\mathbb{Z}$. The nodes of $\lambda = ((2), (4, 2, 2), (5, 2, 1, 1))$ have the following residues

$$\lambda = \left((0 \ 1), \begin{pmatrix} 2 & 3 & 0 & 1 \\ 1 & 2 & & \\ 0 & 1 & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 0 \\ 3 & 0 & & & \\ 2 & & & & \\ 1 & & & & \end{pmatrix} \right).$$

λ has six removable nodes. Fix a residue x and consider the sequence of removable and addable x -nodes obtained by reading the boundary of λ from the bottom up. In the above example, we consider the residue $x = 1$, then we get a sequence “RAARRR”, where each “A” corresponds to an addable x -node and each “R” corresponds to a removable x -node. Given such a sequence of letters A,R, we remove all occurrences of the string “AR” and keep on doing this until no such string “AR” is left. The *normal* x -nodes of λ are those that correspond to the remaining “R” and the highest of these is the *good* x -node. In the above example, there are two normal 1-nodes $(1, 2, 1)$ and $(4, 1, 3)$. The removable 1-node $(1, 2, 1)$ is a good 1-node. If γ is a good x -node of μ and λ is the multipartition such that $[\mu] = [\lambda] \cup \gamma$, we write $\lambda \xrightarrow{x} \mu$.

2.2. Definition. ([7]) *The set $\mathcal{K}_r(n)$ of Kleshchev r -partitions with respect to $(e; \mathbf{Q})$ is defined inductively as follows:*

- (1) $\mathcal{K}_r(0) := \left\{ \emptyset := \underbrace{(\emptyset, \dots, \emptyset)}_{r \text{ copies}} \right\}$;
- (2) $\mathcal{K}_r(n+1) := \left\{ \mu \in \mathcal{P}_r(n+1) \mid \lambda \xrightarrow{x} \mu \text{ for some } \lambda \in \mathcal{K}_r(n) \text{ and } x \in \mathbb{Z}/e\mathbb{Z} \right\}$.

Kleshchev’s good lattice with respect to $(e; \mathbf{Q})$ is the infinite graph whose vertices are the Kleshchev r -partitions with respect to $(e; \mathbf{Q})$ and whose arrows are given by

$$\lambda \xrightarrow{x} \mu \iff \lambda \text{ is obtained from } \mu \text{ by removing a good } x\text{-node.}$$

Let K be a field. Let q be a primitive e -th root of unity if $e > 1$; or not a root of unity if $e = 0$. The Ariki–Koike algebra $\mathcal{H}_{r,n}(q; q^{v_1}, \dots, q^{v_r})$ (or the cyclotomic Hecke algebra of type $G(r, 1, n)$) is the associative unital K -algebra with generators

T_0, T_1, \dots, T_{n-1} and relations

$$\begin{aligned} (T_0 - q^{v_1}) \cdots (T_0 - q^{v_r}) &= 0, \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ (T_i + 1)(T_i - q) &= 0, \quad \text{for } 1 \leq i \leq n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n-2, \\ T_i T_j &= T_j T_i, \quad \text{for } 0 \leq i < j-1 \leq n-2. \end{aligned}$$

These algebras were introduced in the work of Broué and Malle [10] and of Ariki and Koike [5]. They include the Iwahori–Hecke algebras of types A and B as special cases. Conjecturally, they have an intimate relationship with the representation theory of finite reductive groups. The modular representation theory of these algebras was studied in [14, Section 5] and [12], where $\mathcal{H}_{r,n}(q; q^{v_1}, \dots, q^{v_r})$ was shown to be a cellular algebra in the sense of [14]. Using the cellular basis constructed in [12], we know that the resulting cell modules (i.e., Specht modules) $\{S_\lambda\}_{\lambda \vdash n}$ are indexed by the set of r -partitions of n . By the theory of cellular algebras, each Specht module S_λ is equipped with a bilinear form \langle, \rangle . Let $D_\lambda := S_\lambda / \text{rad}\langle, \rangle$. The set $\{D_\lambda \mid D_\lambda \neq 0, \lambda \vdash n\}$ is a complete set of pairwise non-isomorphic absolutely simple $\mathcal{H}_{r,n}(q; q^{v_1}, \dots, q^{v_r})$ -modules. The significance of the notion of Kleshchev multipartition can be seen from the following remarkable result of Ariki.

2.3. Theorem. ([3, Theorem 4.2]) *Let $\lambda \in \mathcal{P}_r(n)$. Then, $D_\lambda \neq 0$ if and only if $\lambda \in \mathcal{K}_r(n)$.*

2.4. Definition. *Let $\lambda \in \mathcal{P}_r(n)$ and $\mathfrak{t} \in \text{Std}(\lambda)$. The residue sequence of \mathfrak{t} is defined to be the sequence*

$$(\text{res}(\mathfrak{t}^{-1}(1)), \dots, \text{res}(\mathfrak{t}^{-1}(n))).$$

The following definitions are natural extensions of the corresponding definitions given in the case where $r = 1, 2$, see [6], [13] and [14].

2.5. Definition. *Let $\lambda \in \mathcal{P}_r(n)$. λ is said to be (\mathbf{Q}, e) -restricted if there exists $\mathfrak{t} \in \text{Std}(\lambda)$ such that the residue sequence of any standard tableau of shape $\mu \triangleleft \lambda$ is not the same as the residue sequence of \mathfrak{t} .*

Note that if $r = 1$, by [20, Corollary 3.41], (\mathbf{Q}, e) -restricted partitions are the same as e -restricted partitions. In particular, $\mathcal{K}_1(n)$ is the same as the set of e -restricted partitions. If $r = 2$, the above definition appeared in the paper [13] of Dipper–James–Murphy. They proved that if λ is (\mathbf{Q}, e) -restricted, then $D_\lambda \neq 0$, and they conjectured the converse is also true, i.e., $\lambda \in \mathcal{K}_2(n)$ if and only if λ is (\mathbf{Q}, e) -restricted. This conjecture was recently proved by Ariki–Jacon [6], using a new characterization of Kleshchev bipartitions obtained in [8]. The general case (i.e., when $r > 2$) remains open. That is

2.6. Generalised DJM Conjecture. *Let $\lambda \in \mathcal{P}_r(n)$. Then $\lambda \in \mathcal{K}_r(n)$ if and only if λ is (\mathbf{Q}, e) -restricted.*

Note that the generalised DJM conjecture can be understood as a criterion for D_λ to be non-zero, where D_λ is defined using the Dipper–James–Mathas cellular basis of $\mathcal{H}_{r,n}(q; q^{v_1}, \dots, q^{v_r})$. With respect to a different cellular basis, Graham and Lehrer proposed a similar conjecture in [14, (5.9),(5.10)]. Since we do not know whether the two set of cellular datum give rise to equivalent cell modules and labeling of simple modules when $r > 2$, it is not clear to us whether the two conjectures are equivalent or not.

In fact, the “if” part of Conjecture 2.6 is easy, as we shall describe in the following. The definition of (\mathbf{Q}, e) -restricted multipartition can be reformulated in terms of

the action of the affine quantum group on a Fock space (cf. [6]). To recall this, we need some more notations. Let v be an indeterminate over \mathbb{Q} . Let $\mathfrak{g} := \widehat{\mathfrak{sl}}_e$ be the affine Lie algebra of type $A_{e-1}^{(1)}$ if $e > 1$; or $\mathfrak{g} := \mathfrak{g}_\infty$ be the affine Lie algebra of type A_∞ if $e = 0$. Let $U_v(\mathfrak{g})$ be the corresponding affine quantum group with Chevalley generators e_i, f_i, k_i and k_d for $i \in \mathbb{Z}/e\mathbb{Z}$. Let $\{\Lambda_i \mid i \in \mathbb{Z}/e\mathbb{Z}\}$ be the set of fundamental weights of \mathfrak{g} . Let \mathcal{F} be the level r v -deformed Fock space associated to $(e; v_1, \dots, v_r)$ which was used in [7]. Our space \mathcal{F} was denoted by \mathcal{F}_v in [7] and one should understand the r -tuple (Q_1, \dots, Q_r) in [7] as the r -tuple $(q^{v_1}, \dots, q^{v_r})$ in this paper, where q is a primitive e -th root of unity in \mathbb{C} if $e > 1$; or not a root of unity if $e = 0$. By definition¹, \mathcal{F} is a $\mathbb{Q}(v)$ -vector space with the basis given by the set of all r -partitions, i.e.,

$$\mathcal{F} = \bigoplus_{n \geq 0, \lambda \in \mathcal{P}_r(n)} \mathbb{Q}(v)\lambda.$$

By [22] and [7], there is an action of $U_v(\mathfrak{g})$ on \mathcal{F} which quantizes the classical action of \mathfrak{g} on the \mathbb{Q} -vector space $\bigoplus_{n \geq 0, \lambda \in \mathcal{P}_r(n)} \mathbb{Q}\lambda$. That is, for each $i \in \mathbb{Z}/e\mathbb{Z}$ and $\lambda \in \mathcal{P}_r(n)$,

$$\begin{aligned} e_i \lambda &= \sum_{\mu \stackrel{i}{\rightarrow} \lambda} v^{-N_i^r(\mu, \lambda)} \mu, & f_i \lambda &= \sum_{\lambda \stackrel{i}{\rightarrow} \mu} v^{N_i^l(\lambda, \mu)} \mu, \\ k_i \lambda &= v^{N_i(\lambda)} \lambda, & k_d \lambda &= v^{-N_d(\lambda)} \lambda, \end{aligned}$$

where

$$\begin{aligned} N_i^r(\mu, \lambda) &:= \#\left\{ \gamma \mid \begin{array}{l} \gamma \text{ is an addable } i\text{-node} \\ \text{for } \lambda, \gamma > \lambda/\mu \end{array} \right\} - \#\left\{ \gamma \mid \begin{array}{l} \gamma \text{ is a removable } i\text{-node} \\ \text{for } \lambda, \gamma > \lambda/\mu \end{array} \right\}, \\ N_i^l(\lambda, \mu) &:= \#\left\{ \gamma \mid \begin{array}{l} \gamma \text{ is an addable } i\text{-node} \\ \text{for } \lambda, \gamma < \mu/\lambda \end{array} \right\} - \#\left\{ \gamma \mid \begin{array}{l} \gamma \text{ is a removable } i\text{-node} \\ \text{for } \lambda, \gamma < \mu/\lambda \end{array} \right\}, \\ N_i(\lambda) &= \#\left\{ \gamma \mid \begin{array}{l} \gamma \text{ is an addable} \\ i\text{-node for } \lambda \end{array} \right\} - \#\left\{ \gamma \mid \begin{array}{l} \gamma \text{ is a removable} \\ i\text{-node for } \lambda \end{array} \right\}, \\ N_d(\lambda) &:= \#\left\{ \gamma \in [\lambda] \mid \text{res}(\gamma) = 0 \right\}. \end{aligned}$$

Note that the empty multipartition \emptyset is a highest weight vector of weight $\sum_{j=1}^r \Lambda_{v_j}$. One can also identify \mathcal{F} with a tensor product of r level one Fock spaces. We refer the reader to the proof of [7, Proposition 2.6] for more details.

2.7. Lemma. *Let $\lambda \in \mathcal{P}_r(n)$. Then λ is (\mathbf{Q}, e) -restricted if and only if there exists a sequence (i_1, \dots, i_n) of residues such that*

$$f_{i_n} \cdots f_{i_1} \emptyset = A_0 \lambda + \sum_{\mu \not\leq \lambda} A_{\lambda, \mu}(v) \mu,$$

for some $A_0, A_{\lambda, \mu}(v) \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$ with $A_0 \neq 0$, where f_{i_1}, \dots, f_{i_n} are the Chevalley generators of $U_v(\mathfrak{g})$.

Proof. For any residue j and any $\mu \in \mathcal{P}_r(n)$, by definition, we have that

$$f_j \mu = \sum_{\text{res}(\nu/\mu)=j} C_{\mu, \nu}(v) \nu,$$

for some $C_{\mu, \nu}(v) \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$ satisfying $C_{\mu, \nu}(1) \neq 0$. The lemma follows directly from this fact and the definition of standard tableaux. \square

2.8. Corollary. *Let $\lambda \in \mathcal{P}_r(n)$. If λ is (\mathbf{Q}, e) -restricted, then $\lambda \in \mathcal{K}_r(n)$.*

¹Although in [7] the ground field of \mathcal{F}_v is $\mathbb{C}(v)$, it does no harm to replace it by $\mathbb{Q}(v)$.

Proof. Let $\Lambda := \sum_{j=1}^r \Lambda_{v_j}$. Then the $U_v(\mathfrak{g})$ -submodule of \mathcal{F} generated by \emptyset is the irreducible highest weight module $V(\Lambda)$ of highest weight Λ . It is well-known that $V(\Lambda)$ has a canonical basis $\{G(\mu)\}$, which is indexed by the set $\mathcal{K}_r := \bigsqcup_{n \geq 0} \mathcal{K}_r(n)$. Combining [2, Theorem 4.4] with [9, Theorem 3.14, (5.3), Theorem 5.6, Theorem 5.14, Corollary 5.15], we see that for any $\mu \in \mathcal{K}_r(n)$,

$$G(\mu) = \mu + \sum_{\substack{\nu \in \mathcal{P}_r(n) \\ \nu \triangleright \mu}} d_{\nu, \mu}(v) \nu,$$

where $d_{\nu, \mu}(v) \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$ for each $\nu \in \mathcal{P}_r(n)$ satisfying $\nu \triangleright \mu$.

Since λ is (\mathbf{Q}, e) -restricted, by Lemma 2.7, we deduce that there exists a sequence (i_1, \dots, i_n) of residues such that

$$f_{i_n} \cdots f_{i_1} \emptyset = A_0 \lambda + \sum_{\mu \not\leq \lambda} A_{\lambda, \mu}(v) \mu,$$

for some $A_0, A_{\lambda, \mu}(v) \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$ with $A_0 \neq 0$.

On the other hand, since $f_{i_n} \cdots f_{i_1} \emptyset \in L(\Lambda)$, hence we can write

$$A_0 \lambda + \sum_{\mu \not\leq \lambda} A_{\lambda, \mu}(v) \mu = \sum_{\nu \in \mathcal{K}_r(n)} A'_\nu(v) G(\nu),$$

for some $A'_\nu(v) \in \mathbb{Z}[v, v^{-1}]$. It follows from the induction on the dominance partial order “ \triangleleft ” that $\lambda \in \mathcal{K}_r(n)$, as required. \square

Therefore, we have proved that “if” part of the Conjecture 2.6. It remains to consider the “only if” part of that conjecture. To this end, we need a result of Littelmann.

We need some more notations. Let $P^+ := \{\sum_{i \in \mathbb{Z}/e\mathbb{Z}} a_i \Lambda_i \mid a_i \in \mathbb{Z}_{\geq 0}, \forall i\}$ be the set of dominant weights. Let $\{\alpha_i\}_{i \in \mathbb{Z}/e\mathbb{Z}}$ (resp., $\{h_i\}_{i \in \mathbb{Z}/e\mathbb{Z}}$) be the set of simple roots (resp., simple coroots). For each dominant weight Λ , let $V(\Lambda)$ be the irreducible $U_v(\mathfrak{g})$ -module with highest weight Λ . We assume that the reader is familiar with the theory of Kashiwara crystals. It is well-known that $V(\Lambda)$ has a crystal basis. We denote by $B(\Lambda)$ its crystal graph. Note that $B(\Lambda)$ is equipped with additional data wt , ε_i , φ_i , \tilde{e}_i and \tilde{f}_i . We refer the readers to [4], [15] and [16] for details. We use u_Λ to denote the unique element in $B(\Lambda)$ satisfying $\text{wt}(u_\Lambda) = \Lambda$. For each $i \in \mathbb{Z}/e\mathbb{Z}$, there are two important realizations of the crystal graph $B(\Lambda_i)$, one by e -restricted partitions (cf. [22]), the other by Littelmann’s path model (cf. [17]). Let W be the affine Weyl group with standard Coxeter generators s_i , $i \in \mathbb{Z}/e\mathbb{Z}$. By definition, W is presented by the generators s_i , $i \in \mathbb{Z}/e\mathbb{Z}$ and the following relations:

$$\begin{aligned} s_i^2 &= 1, \quad \forall i \in \mathbb{Z}/e\mathbb{Z}; \\ s_i s_j &= s_j s_i, \quad \text{if } i \neq j \pm 1; \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \quad \forall i \in \mathbb{Z}/e\mathbb{Z}. \end{aligned}$$

With these two realizations in mind, we can associate each e -restricted partition λ with an Lakshmibai-Seshadri path $(w_1 \Lambda_i, \dots, w_s \Lambda_i; a_0, \dots, a_s)$, where w_1, \dots, w_s are elements in W such that $w_1 \Lambda_i, \dots, w_s \Lambda_i$ are distinct and $0 = a_0 < a_1 < \dots < a_s = 1$ are some rational numbers. We refer the readers to [17] for the precise definition of Lakshmibai-Seshadri paths and related notions. For simplicity, we shall often abbreviate “Lakshmibai-Seshadri paths” to “LS paths”. Note that there is a canonical crystal embedding $B(\Lambda_{v_1} + \dots + \Lambda_{v_r}) \hookrightarrow B(\Lambda_{v_1}) \otimes \dots \otimes B(\Lambda_{v_r})$. We identify $B(\Lambda_{v_1} + \dots + \Lambda_{v_r})$ with the image by this embedding. We have the following result of Littelmann, which was reformulated in [8, Theorem 5.7].

2.9. Theorem. ([18, Theorem 10.1]) *Let*

$$\pi = \pi^{(1)} \otimes \cdots \otimes \pi^{(r)} \in B(\Lambda_{v_1}) \otimes \cdots \otimes B(\Lambda_{v_r}).$$

Then π belongs to $B(\Lambda_{v_1} + \cdots + \Lambda_{v_r})$ if and only if there exists a sequence

$$w_1^{(1)} \geq \cdots \geq w_{N_1}^{(1)} \geq w_1^{(2)} \geq \cdots \geq w_{N_2}^{(2)} \geq \cdots \geq w_{N_r}^{(r)}$$

in W such that

$$\pi^{(k)} = (w_1^{(k)} \Lambda_{v_k}, \cdots, w_{N_k}^{(k)} \Lambda_{v_k}; a_0^{(k)}, \cdots, a_{N_k}^{(k)}),$$

for any integer $1 \leq k \leq r$, where “ \geq ” is the Bruhat order and $0 = a_0^{(k)} < a_1^{(k)} < \cdots < a_{N_k}^{(k)} = 1$ are some rational numbers.

2.10. Lemma. ([21, Proposition 4.10]) *If $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)})$ is a Kleshchev r -partition with respect to $(e; v_1, \cdots, v_r)$, then for each $1 \leq i \leq r$, $\lambda^{(i)}$ is an e -restricted partition.*

The combinatorial realization of $B(\Lambda_i)$ in terms of e -restricted partitions (cf. [22]) allows a natural generalization to the higher level case (cf. [7]) as we now recall. Let $\Lambda := \Lambda_{v_1} + \cdots + \Lambda_{v_r}$. Set $\mathcal{K}_r := \sqcup_{n \geq 0} \mathcal{K}_r(n)$. For each $i \in \mathbb{Z}/e\mathbb{Z}$ and $\lambda \in \mathcal{K}_r(n)$, we define

$$\begin{aligned} \tilde{e}_i \lambda &= \begin{cases} \lambda - \{\gamma\}, & \text{if } \lambda \text{ has a good } i\text{-node } \gamma; \\ 0, & \text{otherwise;} \end{cases} \\ \tilde{f}_i \lambda &= \begin{cases} \lambda \cup \{\gamma\}, & \text{if } \gamma \text{ is a good } i\text{-node of } \lambda \cup \{\gamma\}; \\ 0, & \text{otherwise.} \end{cases} \\ \varepsilon_i(\lambda) &= \max\{n \geq 0 \mid \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(\lambda) = \max\{n \geq 0 \mid \tilde{f}_i^n b \neq 0\}; \\ \text{wt}(\lambda) &= \Lambda - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \tilde{N}_i(\lambda) \alpha_i, \end{aligned}$$

where $\tilde{N}_i(\lambda)$ is the number of i -nodes in $[\lambda]$. By a result of Misra–Miwa [22] and Ariki–Mathas [7], the data \mathcal{K}_r , wt , ε_i , φ_i , \tilde{e}_i and \tilde{f}_i define a realization of the crystal $B(\Lambda)$ in terms of Kleshchev’s good lattice with respect to (e, \mathbf{Q}) . Henceforth, we make this identification. In particular, taking $r = 1$ and $i \in \mathbb{Z}/e\mathbb{Z}$, we can identify any element in $B(\Lambda_i)$ with an e -restricted partition.

Recall that for any two \mathfrak{g} -crystals B_1, B_2 . The tensor product $B_1 \otimes B_2$ is the set $B_1 \times B_2$ equipped with the crystal structure defined by

- a) $\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2);$
b)

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2); \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2); \end{cases}$$

- c)

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2); \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

- d) $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle\};$
e) $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle, \varphi_i(b_2)\}.$

Let $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)}) \in \mathcal{P}_r(n)$. By Lemma 2.10, λ is a Kleshchev multipartition with respect to $(e; \mathbf{Q})$ only if each $\lambda^{(s)}$ is an e -restricted partition for $1 \leq s \leq r$.

2.11. Lemma. *The map which sends each $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{K}_r(n)$ to $\lambda^{(r)} \otimes \lambda^{(r-1)} \otimes \dots \otimes \lambda^{(1)}$ coincides with the canonical crystal embedding $B(\Lambda) = B(\Lambda_{v_r} + \dots + \Lambda_{v_1}) \hookrightarrow B(\Lambda_{v_r}) \otimes \dots \otimes B(\Lambda_{v_1})$. In particular, if each $\lambda^{(s)}$ is e -restricted for $1 \leq s \leq r$, then $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a Kleshchev r -partition with respect to $(e; \mathbf{Q})$ if and only if $\lambda^{(r)} \otimes \dots \otimes \lambda^{(1)}$ belongs to $B(\Lambda_{v_r} + \dots + \Lambda_{v_1})$.*

Proof. This follows directly from the fact that if we reverse the order of the components, then the action of the Kashiwara operator \tilde{f}_i on tensor product of crystals coincides with the operator of adding good i -node on Kleshchev multipartitions. \square

The above lemma implies that the problem of characterizing Kleshchev multipartition in terms of its components is essentially a purely crystal theoretic question. For the latter, Theorem 2.9 gives an answer in the language of Littelmann's path model. The following lemma is a direct consequence of Theorem 2.9.

2.12. Corollary. *If $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a Kleshchev r -partition with respect to $(e; v_1, \dots, v_r)$, then $(\lambda^{(j_1)}, \dots, \lambda^{(j_t)})$ is a Kleshchev t -partition with respect to $(e; v_{j_1}, \dots, v_{j_t})$ for any integers $1 \leq t < r$ and $1 \leq j_1 < \dots < j_t \leq r$.*

2.13. Remark. We note that the converse of Corollary 2.12 is in general false. For example, let $e = 5$, $(v_1, v_2, v_3) = (3 + 5\mathbb{Z}, 2 + 5\mathbb{Z}, 1 + 5\mathbb{Z})$. Let

$$\lambda^{(1)} := (5, 1), \quad \lambda^{(2)} := (3), \quad \lambda^{(3)} := (2).$$

Then it is easy to check that

- (1) $(\lambda^{(1)}, \lambda^{(2)})$ is a Kleshchev bipartition with respect to $(5; v_1, v_2)$; and
- (2) $(\lambda^{(2)}, \lambda^{(3)})$ is a Kleshchev bipartition with respect to $(5; v_2, v_3)$; and
- (3) $(\lambda^{(1)}, \lambda^{(3)})$ is a Kleshchev bipartition with respect to $(5; v_1, v_3)$;

but $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ is not a Kleshchev 3-partition with respect to $(5; v_1, v_2, v_3)$.

Let Λ be any dominant weight. By [16, Corollaire 8.1.5], there exists a unique crystal morphism $K_h : B(\Lambda) \hookrightarrow B(h\Lambda)$ of amplitude h , for all $h \in \mathbb{N}$. In other words,

- (i) $K_h(u_\Lambda) = u_{h\Lambda}$;
- (ii) $\text{wt}(K_h(b)) = h \text{wt}(b)$, $\varepsilon_i(K_h(b)) = h\varepsilon_i(b)$ and $\varphi_i(K_h(b)) = h\varphi_i(b)$;
- (iii) $K_h(\tilde{e}_i b) = \tilde{e}_i^h K_h(b)$ and $K_h(\tilde{f}_i b) = \tilde{f}_i^h K_h(b)$ for all $b \in B(\Lambda)$.

Composing K_h with the natural embedding $B(h\Lambda) \hookrightarrow B(\Lambda)^{\otimes h}$, we get a crystal morphism $S_h : B(\Lambda) \hookrightarrow B(\Lambda)^{\otimes h}$ of amplitude h .

Recall that, for each $w \in W$, the weight space $V(\Lambda)_{w\Lambda}$ is one-dimensional. We use $u_{w\Lambda}$ to denote the unique element in $B(\Lambda)$ satisfying $\text{wt}(u_{w\Lambda}) = w\Lambda$.

2.14. Lemma. ([16, Proposition 8.3.2]) *Let $b \in B(\Lambda)$. Then there is an integer $s > 0$, and rational numbers $0 = a_0 < a_1 < \dots < a_s = 1$ and elements w_1, \dots, w_s of W such that $w_1\Lambda, \dots, w_s\Lambda$ are pairwise distinct and whenever h satisfies $(a_{i+1} - a_i)h \in \mathbb{Z}_{\geq 0}$ for all i we have*

$$S_h(b) = u_{w_1\Lambda}^{\otimes (a_1 - a_0)h} \otimes u_{w_2\Lambda}^{\otimes (a_2 - a_1)h} \otimes \dots \otimes u_{w_s\Lambda}^{\otimes (a_s - a_{s-1})h}.$$

Furthermore, the map

$$b \mapsto (w_1\Lambda, \dots, w_s\Lambda; a_0, a_1, \dots, a_s)$$

coincides with Littelmann's path model.

Henceforth, we assume that $\Lambda = \Lambda_k$ is a fundamental weight. We use W_k to denote the symmetric group generated by $s_i, i \in \mathbb{Z}/e\mathbb{Z} - \{k + e\mathbb{Z}\}$. Recall that the crystal $B(\Lambda_k)$ has a realization in terms of the set of e -restricted partitions. We denote by \emptyset_k the empty partition in $B(\Lambda_k)$. Let W/W_k be the set of distinguished coset representatives of W_k in W . For any $i \in \mathbb{Z}/e\mathbb{Z}$ and any e -core ν , let $s_i\nu$ be

defined as in [8, Lemma 3.3(2)]. For any $w \in W$, $w\nu := s_{i_1} \cdots s_{i_t} \nu$ if $s_{i_1} \cdots s_{i_t}$ is a reduced expression of w . This is well-defined, i.e., independent of the choice of the reduced expression. By [8, Proposition 3.5], if $w \in W/W_k$, then $w\emptyset_k$ is an e -core, and this gives rise to a natural bijection between the set of e -cores and the set $\{w\Lambda \mid w \in W\}$. Note that the empty partition \emptyset_k corresponds to u_Λ , while the e -core $w\emptyset$ corresponds to $u_{w\Lambda}$. Therefore, translating into the language of e -cores, we can write an LS-path $(w_1\Lambda, \dots, w_s\Lambda; a_0, a_1, \dots, a_s)$ as $(\nu_1, \dots, \nu_s; a_0, a_1, \dots, a_s)$, where $\nu_i = w_i\emptyset_k$ is an e -core for each i , and we can rephrase Lemma 2.14 as follows: for each e -restricted partition λ , there exist an integer s , distinct e -cores ν_1, \dots, ν_s and rational numbers $0 = a_0 < a_1 < \dots < a_s = 1$ such that whenever h satisfies $(a_{i+1} - a_i)h \in \mathbb{Z}_{\geq 0}$ for all i we have

$$S_h(\lambda) = \nu_1^{\otimes (a_1 - a_0)h} \otimes \nu_2^{\otimes (a_2 - a_1)h} \otimes \dots \otimes \nu_s^{\otimes (a_s - a_{s-1})h}.$$

Furthermore, the map $\lambda \mapsto (\nu_1, \dots, \nu_s; a_0, a_1, \dots, a_s)$ coincides with Littelmann's path model.

In the remaining part of this paper, we shall always write an LS-path

$$(w_1\Lambda_k, \dots, w_s\Lambda_k; a_0, a_1, \dots, a_s)$$

as $(\nu_1, \dots, \nu_s; a_0, a_1, \dots, a_s)$, where ν_i is the unique e -core such that $\nu_i = w_i\emptyset$ for each $1 \leq i \leq s$.

2.15. Lemma. ([8], [16]) *With the notations as above, the map which sends each λ to $(\nu_1, \dots, \nu_s; a_0, a_1, \dots, a_s)$ defines an isomorphism of crystals between the two realizations of $B(\Lambda_k)$, the one by e -restricted partitions and the one by LS-paths. Furthermore, if λ is an e -core, then $s = 1$ and $\nu_1 = \lambda$.*

Proof. The first part of the lemma follows from [8, Theorem 5.14] and [16, Theorem 8.2.3], while the second part of the lemma is a direct consequence of [16, Proposition 8.3.2 (1)]. \square

In the above lemma, whenever λ is mapped to $(\nu_1, \dots, \nu_s; a_0, a_1, \dots, a_s)$, we then write

$$\pi(\lambda) = (\nu_1, \nu_2, \dots, \nu_s).$$

2.16. Lemma. *With the notations as above, we have that $\nu_1 \supset \nu_2 \supset \dots \supset \nu_s$. In particular, (ν_s, \dots, ν_1) is a Kleshchev s -partition with respect to $(e; k, \dots, k)$.*

Proof. Let $\Lambda = \Lambda_k$. By Lemma 2.14, there exist integers n_1, \dots, n_s , such that

$$\nu_1^{\otimes n_1} \otimes \nu_2^{\otimes n_2} \otimes \dots \otimes \nu_s^{\otimes n_s} \in B(m\Lambda) \subset B(\Lambda)^{\otimes m},$$

where $m = n_1 + \dots + n_s$. For each integer $1 \leq j \leq s$, we write $\nu_j = d_j W_k$ for a unique $d_j \in W/W_k$. Applying Theorem 2.9, we deduce that $d_1 \geq d_2 \geq \dots \geq d_s$. Finally, applying [8, Proposition 4.4] and Lemma 2.15, we get that

$$\nu_1 \supset \nu_2 \supset \dots \supset \nu_s,$$

and (ν_s, \dots, ν_1) is a Kleshchev s -partition with respect to $(e; k, \dots, k)$. \square

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be an r -partition. Suppose that for each integer $1 \leq i \leq r$, the component $\lambda^{(i)}$ is an e -restricted partition. For each integer $1 \leq i \leq r$, we identify $\lambda^{(i)}$ as an element in $B(\Lambda_{v_i})$ and we write $\pi(\lambda^{(i)}) = (\nu_1^{(i)}, \dots, \nu_{s(i)}^{(i)})$ for some integer $s(i)$ and some pairwise distinct e -cores $\nu_1^{(i)}, \dots, \nu_{s(i)}^{(i)}$. We define

$$\tilde{\pi}(\boldsymbol{\lambda}) = (\nu_1^{(1)} \otimes \dots \otimes \nu_{s(1)}^{(1)}) \otimes \dots \otimes (\nu_1^{(r)} \otimes \dots \otimes \nu_{s(r)}^{(r)}).$$

We identify $B(\Lambda_{v_1} + \dots + \Lambda_{v_r})$ with its image by the natural embedding $B(\Lambda_{v_1} + \dots + \Lambda_{v_r}) \hookrightarrow B(\Lambda_{v_1}) \otimes \dots \otimes B(\Lambda_{v_r})$ and $B(s(1)\Lambda_{v_1} + \dots + s(r)\Lambda_{v_r})$ with its image by the natural embedding $B(s(1)\Lambda_{v_1} + \dots + s(r)\Lambda_{v_r}) \hookrightarrow B(\Lambda_{v_1})^{\otimes s(1)} \otimes \dots \otimes B(\Lambda_{v_r})^{\otimes s(r)}$.

2.17. Corollary. *With the notations as above, we have that*

$$\lambda^{(1)} \otimes \cdots \otimes \lambda^{(r)} \in B(\Lambda_{v_1} + \cdots + \Lambda_{v_r})$$

if and only if

$$\tilde{\pi}(\boldsymbol{\lambda}) \in B(s(1)\Lambda_{v_1} + \cdots + s(r)\Lambda_{v_r}).$$

Proof. For each $1 \leq i \leq r$, we identify $\lambda^{(i)}$ with an LS-path

$$\pi^{(i)} = (w_1^{(i)}\Lambda_{v_i}, \cdots, w_{s(i)}^{(i)}\Lambda_{v_i}; a_0^{(i)}, \cdots, a_{s(i)}^{(i)}).$$

If $\lambda^{(1)} \otimes \cdots \otimes \lambda^{(r)} \in B(\Lambda_{v_1} + \cdots + \Lambda_{v_r})$, then Theorem 2.9 implies that we can choose those elements $w_1^{(i)}, \cdots, w_{s(i)}^{(i)}$, $1 \leq i \leq r$, in a way such that

$$w_1^{(1)} \geq \cdots \geq w_{s(1)}^{(1)} \geq w_1^{(2)} \geq \cdots \geq w_{s(2)}^{(2)} \geq \cdots \geq w_{s(r)}^{(r)}.$$

By Lemma 2.15, we know that for each $1 \leq i \leq r$ and $1 \leq j \leq s(i)$, $(w_j^{(i)}; 0, 1)$ is an LS-path for the e -core $\nu_j^{(i)}$. Applying Theorem 2.9 again, we prove that $\tilde{\pi}(\boldsymbol{\lambda}) \in B(s(1)\Lambda_{v_1} + \cdots + s(r)\Lambda_{v_r})$.

Conversely, assume that $\tilde{\pi}(\boldsymbol{\lambda}) \in B(s(1)\Lambda_{v_1} + \cdots + s(r)\Lambda_{v_r})$. Then Theorem 2.9 implies that we can find elements $\hat{w}_1^{(i)}, \cdots, \hat{w}_{s(i)}^{(i)}$, $1 \leq i \leq r$ such that $\hat{w}_j^{(i)}\Lambda_{v_i} = w_j^{(i)}\Lambda_{v_i}$ for each $1 \leq i \leq r$, $1 \leq j \leq s(i)$, and

$$\hat{w}_1^{(1)} \geq \cdots \geq \hat{w}_{s(1)}^{(1)} \geq \hat{w}_1^{(2)} \geq \cdots \geq \hat{w}_{s(2)}^{(2)} \geq \cdots \geq \hat{w}_{s(r)}^{(r)}.$$

Since $\hat{w}_j^{(i)}\Lambda_{v_i} = w_j^{(i)}\Lambda_{v_i}$, we have that

$$\pi^{(i)} = (\hat{w}_1^{(i)}\Lambda_{v_i}, \cdots, \hat{w}_{s(i)}^{(i)}\Lambda_{v_i}; a_0^{(i)}, \cdots, a_{s(i)}^{(i)}).$$

Applying Theorem 2.9 again, we deduce that $\lambda^{(1)} \otimes \cdots \otimes \lambda^{(r)} \in B(\Lambda_{v_1} + \cdots + \Lambda_{v_r})$ as required. \square

2.18. Corollary. *With the notations as above, we have that $(\lambda^{(r)}, \cdots, \lambda^{(1)})$ is a Kleshchev r -partition with respect to $(e; v_r, \cdots, v_1)$ if and only if*

$$\left(\nu_{s(r)}^{(r)}, \cdots, \nu_1^{(r)}, \cdots, \nu_{s(1)}^{(1)}, \cdots, \nu_1^{(1)} \right)$$

is a Kleshchev $(\sum_{i=1}^r s(i))$ -partition with respect to

$$\left(\underbrace{e; v_r, \cdots, v_r}_{s(r) \text{ copies}}, \cdots, \underbrace{v_1, \cdots, v_1}_{s(1) \text{ copies}} \right).$$

3. THE MULTI-CORE CASE

The purpose of this section is to give a proof of the ‘‘only if’’ part of the Conjecture 2.6 in the multi-core case. In particular, we prove that the generalised Dipper–James–Murphy conjecture is true if $e = 0$.²

3.1. Definition. *Let $\boldsymbol{\lambda} \in \mathcal{P}_r(n)$ and $\gamma \in [\boldsymbol{\lambda}]$. We call γ a semi-ladder node of $\boldsymbol{\lambda}$ if γ is a removable node of $\boldsymbol{\lambda}$ and there is no lower addable node of the same residue. We call γ a ladder node of $\boldsymbol{\lambda}$ if γ is a semi-ladder node such that there is no higher semi-ladder node of the same residue.*

²Note that in the case $e = 0$, i.e., q is not a root of unity, the Ariki–Koike algebra $\mathcal{H}_{r,n}(q; q^{v_1}, \cdots, q^{v_r})$ is NOT necessarily semisimple whenever $r \geq 2$. We refer the reader to [1] for the semisimplicity criterion of Ariki–Koike algebra.

A semi-ladder node of λ is necessarily a normal node of λ . In general, a multipartition may have no semi-ladder nodes. Let $x \in \mathbb{Z}/e\mathbb{Z}$. By an x -sequence of λ we mean a sequence of removable x -nodes of λ , arraying in decreasing order, i.e., $\alpha_1 > \alpha_2 > \cdots > \alpha_s$. If λ has a semi-ladder x -node for some $x \in \mathbb{Z}/e\mathbb{Z}$, then we call the sequence of all the semi-ladder x -nodes, arraying in decreasing order, i.e., $\alpha_1 > \alpha_2 > \cdots > \alpha_s$, as a ladder x -sequence of λ . It is readily seen that every node in the ladder x -sequence of λ is necessarily a normal x -node of λ .

3.2. Lemma. *Assume that λ is a non-empty Kleshchev multipartition with respect to $(e; v_1, \dots, v_r)$. Then λ has at least one ladder node.*

Proof. By assumption, $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{K}_r(n)$. Let α be the lowest removable node of λ . Suppose that α is in the a th row of the c th component of λ . Let $x := \text{res}(\alpha) \in \mathbb{Z}/e\mathbb{Z}$. Then $\lambda^{(t)} = \emptyset$ for any integer $t > c$, and $\lambda_s^{(c)} = 0$ for any integer $s > a$.

By Lemma 2.10, $\lambda \in \mathcal{K}_r(n)$ implies that each component $\lambda^{(t)}$ is an e -restricted partition. In particular, $\lambda^{(c)}$ is a non-empty e -restricted partition. It follows that the residue of the unique addable node below α in $\lambda^{(c)}$ is different from x . In other words, there are no addable x -node in $\lambda^{(c)}$ which is below α .

We claim that for any integer $t > c$ we must have that $x \neq v_t$. Since $\lambda \in \mathcal{K}_r(n)$, by definition, we can find $\lambda(i) \in \mathcal{K}_r(i)$ for each $0 \leq i \leq n$ such that

- a) $\lambda(0) = \emptyset, \lambda(n) = \lambda$;
- b) $\lambda(i) \subset \lambda(i+1)$ for each $0 \leq i \leq n-1$;
- c) $\gamma(i) := \lambda(i+1)/\lambda(i)$ is a good node of $\lambda(i+1)$.

Let $1 \leq j \leq n$ be the unique integer such that $\alpha = \gamma(j)$. Since α is the lowest removable node of λ , α must also be the lowest removable node of $\lambda(i+1)$, it follows from the definition of good nodes that $x \neq v_t$ for any $x > t$. This proves our claim.

Therefore there are actually no addable x -nodes in $[\lambda]$ that can be lower than α . Hence α is a semi-ladder x -node and λ must have a ladder x -node. \square

3.3. Example. *Suppose that $e = 3$ and $v_1 = 2 + 3\mathbb{Z}, v_2 = 1 + 3\mathbb{Z}, v_3 = 3\mathbb{Z}$. Let*

$$\lambda := \left((1), (2, 1), (3, 1) \right), \quad \mu := \left((1), (3, 1), \emptyset \right).$$

Then it is easy to see that λ is a Kleshchev 3-partition with respect to $(3; v_1, v_2, v_3)$, while μ is not a Kleshchev 3-partition with respect to $(3; v_1, v_2, v_3)$. Furthermore, λ has a unique good node, that is, the good 2-node $(1, 1, 1)$ and λ has only semi-ladder 2-nodes, $(1, 3, 3) > (2, 1, 3)$ is the ladder 2-sequence of λ and $(1, 3, 3)$ is the ladder 2-node of λ . Note that $(1, 1, 1) > (1, 3, 3) > (2, 1, 3)$ are all the normal 2-nodes of λ . Finally, μ has no semi-ladder nodes.

3.4. Definition. *Let $\lambda \in \mathcal{P}_r(n)$. We call λ a ladder multipartition if there is a sequence of residues i_1, \dots, i_n and a sequence of multipartitions $\lambda[0] = \emptyset, \dots, \lambda[n] = \lambda$ such that for each t , $\lambda[t-1]$ is obtained from $\lambda[t]$ by removing the ladder i_t -node of $\lambda[t]$.*

3.5. Definition. *Let $\lambda \in \mathcal{P}_r(n)$. We call λ a strong ladder multipartition if there is a sequence of residues i_1, \dots, i_p and a sequence of multipartitions $\emptyset = \lambda[0], \dots, \lambda[p] = \lambda$ such that for each t , $\lambda[t-1]$ is obtained from $\lambda[t]$ by removing all the nodes in the ladder i_t -sequence of $\lambda[t]$.*

It is clear that a strong ladder multipartition is necessarily a ladder multipartition. But it is not obvious that the converse is also true. We conjecture that the converse is also true. Indeed we shall prove that the converse is true in the case

where $e = 0$. Note that in the case $r = 2$, a strong ladder multipartition is the same as a bipartition which has an optimal sequence in the sense of [6].

The next definition is a natural generalization of [6, Definition 4.1] to the arbitrary multipartition case.

3.6. Definition. Let $\lambda, \mu \in \mathcal{P}_r(n)$. We write $\lambda \prec \mu$ if there exist integers $1 \leq s \leq r$ and $t \geq 0$ such that

- a) $\lambda^{(j)} = \mu^{(j)}$ for any $s + 1 \leq j \leq r$ and $\lambda_j^{(s)} = \mu_j^{(s)}$ for any $j > t$;
- b) $\lambda_t^{(s)} < \mu_t^{(s)}$.

It is clear that if $\mu \triangleleft \lambda$ then $\lambda \prec \mu$.

3.7. Lemma. Any strong ladder multipartition is (\mathbf{Q}, e) -restricted, and hence is a Kleshchev multipartition with respect to $(e; \mathbf{Q})$.

Proof. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be a strong ladder multipartition in $\mathcal{P}_r(n)$. By Definition 3.5, we can find a sequence of residues i_1, \dots, i_p and a sequence of multipartitions $\emptyset = \lambda[0], \dots, \lambda[p] = \lambda$ such that for each t , $\lambda[t-1]$ is obtained from $\lambda[t]$ by removing all the nodes in the ladder i_t -sequence of $\lambda[t]$. By the definition of ladder sequence, we know that $i_s \neq i_t$ whenever $|s-t|=1$. For each $1 \leq t \leq p$, we use a_t to denote the number of semi-ladder i_t -nodes in the ladder i_t -sequence of $\lambda[t]$ and set $l_t := \sum_{j=1}^t a_j$, $l_0 = 0$. Let $\gamma_{l_{t-1}+1}, \gamma_{l_{t-1}+2}, \dots, \gamma_{l_t}$ be the ladder i_t -sequence of $\lambda[t]$.

We claim that

$$(3.8) \quad f_{i_p}^{(a_p)} \dots f_{i_1}^{(a_1)} \emptyset = \lambda + \sum_{\mu \triangleleft \lambda} C_{\lambda, \mu}(v) \mu,$$

for some $C_{\lambda, \mu}(v) \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$, where $f_{i_j}^{(a_j)}$ denotes the quantum dividing power (cf. [19, (1.4.1), (3.1.1)]).

The proof is similar to the proof [6, Proposition 4.2] in the bipartition case. We use induction on p . By definition, $l_p = n$. Let $c := n - a_p$. Then $\gamma_{c+1}, \dots, \gamma_{c+a_p} = \gamma_n$ is the ladder i_p -sequence for λ . Let $\lambda' := \lambda - \{\gamma_{c+1}, \dots, \gamma_n\}$, which is again a strong ladder multipartition and hence (\mathbf{Q}, e) -restricted, and hence a Kleshchev multipartition with respect to $(e; \mathbf{Q})$.

By induction hypothesis, we have that

$$f_{i_{p-1}}^{(a_{p-1})} \dots f_{i_1}^{(a_1)} \emptyset = \lambda' + \sum_{\mu' \triangleleft \lambda'} C_{\lambda', \mu'}(v) \mu'.$$

Let $\mu \neq \lambda$ be a multipartition which appears in $f_{i_p}^{(a_p)} \dots f_{i_1}^{(a_1)} \emptyset$ with nonzero coefficient. Then there exist removable i_p -nodes $\alpha_1, \dots, \alpha_{a_p}$ of μ and a multipartition μ' such that $\mu = \mu' \sqcup \{\alpha_1, \dots, \alpha_{a_p}\}$.

By the definition of ladder sequence of λ , it is clear that $\mu' = \lambda'$ implies that $\mu \prec \lambda$. Hence we can assume that $\mu' \triangleleft \lambda'$. Suppose that $\mu \not\prec \lambda$, i.e., $\lambda \prec \mu$. Then we can find integers $1 \leq s \leq r$ and $t \geq 0$ such that

- a) $\mu'^{(l)} = \lambda'^{(l)}$ for any $s + 1 \leq l \leq r$ and $\mu'_j^{(s)} = \lambda'_j^{(s)}$ for any $j > t$;
- b) $\mu'_t^{(s)} < \lambda'_t^{(s)}$.

We claim that

- c) $\mu_j^{(l)} = \lambda_j^{(l)}$ for any $(j, l) \in \{(j, l) | s + 1 \leq l \leq r\} \sqcup \{(j, s) | j > t\}$;
- d) $\mu_{t+1}^{(s)} < \mu_t^{(s)} = \mu'_t^{(s)} + 1 = \lambda'_t^{(s)} = \lambda_t^{(s)}$;
- e) at least one of the nodes $\gamma_{c+1}, \dots, \gamma_{c+a_p}$ is above $(t, \lambda_t^{(s)}, s)$.

In fact, all of these statements follow from the fact that $\mu' \triangleleft \lambda'$, $\lambda \prec \mu$ and there are no addable i_p -nodes below α_{a_p} . Now d) implies that $(t, \lambda_t^{(s)}, s)$ is an i_p -node

of $\lambda^{(s)}$. Hence it is not a removable node of λ otherwise it has to be removed to obtain λ' (by the definition of ladder i_p -sequence of λ). It follows that $\lambda_{t+1}^{(s)} = \lambda_t^{(s)}$. Thus $\mu_{t+1}^{(s)} < \lambda_t^{(s)} = \lambda_{t+1}^{(s)}$, which is a contradiction. This completes the proof of (3.8).

Since $\mu \prec \lambda$ (i.e., $\lambda \not\prec \mu$) implies that $\mu \not\prec \lambda$, it follows from Lemma 2.7 that λ is (\mathbf{Q}, e) -restricted. Applying Lemma 2.8, we deduce that λ is a Kleshchev multipartition with respect to $(e; \mathbf{Q})$. \square

Let m be an arbitrary integer. We use W_m to denote the symmetric group generated by $s_i, i \in \mathbb{Z}/e\mathbb{Z} - \{m + e\mathbb{Z}\}$. Let W/W_m be the set of distinguished coset representatives of W_m in W .

3.9. Lemma. ([11, Lemma 3.2]) *Let $i \in \mathbb{Z}/e\mathbb{Z}$. Let $x \in W_m, d \in W/W_m$ and $w := dx$. Suppose that $s_i w < w$ and $s_i d \notin W/W_m$. Then $d^{-1}s_i d = s_l$ for some $l \in \mathbb{Z}/e\mathbb{Z}$ with $l \neq m + e\mathbb{Z}$ and such that $s_l x < x$.*

Recall that a multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is said to be a multi-core if $\lambda^{(j)}$ is an e -core for each integer $1 \leq j \leq r$. For each multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_r(n)$, we write

$$\lambda^\diamond := (\lambda^{(r)}, \dots, \lambda^{(1)}).$$

Clearly, λ^\diamond is a multi-core if and only if λ is a multi-core. The next theorem is the main result of this paper.

3.10. Proposition. *Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_r(n)$. Suppose λ^\diamond is a Kleshchev r -partition of n with respect to (e, v_r, \dots, v_1) and also a multi-core. Then λ^\diamond is a strong ladder multipartition and hence (\mathbf{Q}, e) -restricted. Furthermore, for each ladder sequence $\gamma_1 > \dots > \gamma_a$ of λ^\diamond , $\lambda^\diamond - \{\gamma_1, \dots, \gamma_a\}$ is again a strong ladder multipartition and hence a Kleshchev multipartition with respect to $(e; v_r, \dots, v_1)$.*

Proof. We argue by induction on n . Suppose that the conclusion is true for any integer $0 \leq n' < n$. In other words, for any multi-core Kleshchev r -partition λ' of n' with respect to $(e; v_r, \dots, v_1)$, λ' is a strong ladder multipartition, and for any ladder sequence $\gamma'_1 > \dots > \gamma'_a$ of λ' , $\lambda' - \{\gamma'_1, \dots, \gamma'_a\}$ is again a strong ladder multipartition.

We now look at the multi-core Kleshchev r -partition $\lambda^\diamond := (\lambda^{(r)}, \dots, \lambda^{(1)})$ of n with respect to $(e; v_r, \dots, v_1)$. By Lemma 3.2, there exists a ladder sequence $\gamma_1 > \dots > \gamma_a$ with $\text{res}(\gamma_1) = i \in \mathbb{Z}/e\mathbb{Z}$. Suppose that the nodes $\gamma_1 > \dots > \gamma_c$ are located in the component $\lambda^{(t)}$ and $\gamma_{c+1} \notin \lambda^{(t)}$ for some $1 \leq c < a$. Let μ be the r -partition which is obtained from λ by deleting the nodes $\{\gamma_1, \dots, \gamma_c\}$. Since $\lambda^{(t)}$ is an e -core, $\lambda^{(t)}$ has no addable i -nodes. By definition, $\gamma_1, \dots, \gamma_c$ are all the removable i -nodes on the e -core partition $\lambda^{(t)}$. It follows (by considering the abacus display of partition) that $\lambda^{(t)} - \{\gamma_1, \dots, \gamma_t\}$ is again an e -core, and hence μ^\diamond is again a multi-core. We are going to show that μ^\diamond is Kleshchev. Note that $\gamma_{c+1} > \dots > \gamma_a$ is a ladder i -sequence of μ^\diamond . Once we can prove μ^\diamond is Kleshchev, then by induction hypothesis that

$$\lambda^\diamond - \{\gamma_1, \dots, \gamma_a\} = \mu^\diamond - \{\gamma_{c+1}, \dots, \gamma_a\}$$

is a strong ladder multipartition, and then by definition, λ^\diamond must be a strong ladder multipartition as well, which completes the proof of the proposition.

Let $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$. By Theorem 2.9 and Lemma 2.15, we can find elements $w_1 \geq w_2 \geq \dots \geq w_r$ in W , such that $\lambda^{(j)} = w_j \emptyset_{v_j}$, where the subscript is used to indicate the charge v_j . There is a unique way to write each w_j in the form $d_j x_j$ where $d_j \in W/W_{v_j}, x_j \in W_{v_j}$. For later use, we choose these elements w_j in a way such that $\sum_{j=1}^r \ell(x_j)$ is as small as possible.

Note that $\mu^{(j)} = \lambda^{(j)}$ for any $j \neq t$. Since both $\lambda^{(t)}$ and $\mu^{(t)}$ are e -cores, and $\mu^{(t)}$ is obtained from $\lambda^{(t)}$ by removing all its removable i -nodes, we deduce that $\mu^{(t)} = (s_i d_t) \emptyset_{v_t}$ with $d_t > s_i d_t \in W/W_{v_t}$. Now for each integer $1 \leq j \leq r$, we define

$$w'_j = \begin{cases} w_j, & \text{if } j \neq t; \\ s_i w_j, & \text{if } j = t. \end{cases}$$

Note that $\ell(s_i w_t) = \ell(s_i d_t) + \ell(x_t) = \ell(w_t) - 1$. Hence $w'_t < w_t$. Let $t+1 \leq k \leq r$ be the smallest integer such that $\lambda^{(k)}$ contains addable i -nodes. Then for each integer $t+1 \leq l \leq k-1$, $\lambda^{(l)}$ contains neither removable i -nodes nor addable i -nodes.

We claim that

$$(3.11) \quad w'_1 \geq w'_2 \geq \cdots \geq w'_r.$$

In fact, it suffices to prove that $w'_t \geq w'_{t+1}$. Suppose this is not the case, then we can deduce that w_{t+1} must have a reduced expression which starts from s_i (otherwise the inequality $s_i w'_t = w_t \geq w_{t+1}$ already implies that $w'_t \geq w_{t+1}$). Hence $s_i w_{t+1} < w_{t+1}$. Note that since the e -core $\lambda^{(t+1)}$ has no removable i -nodes, d_{t+1} has no reduced expression which starts from s_i . It follows that $s_i d_{t+1} > d_{t+1}$ and hence $s_i d_{t+1} \notin W/W_{v_{t+1}}$ (otherwise $s_i w_{t+1} = s_i d_{t+1} x_{t+1} > d_{t+1} x_{t+1} = w_{t+1}$, a contradiction). Applying Lemma 3.9, we get that $d_{t+1}^{-1} s_i d_{t+1} = s_l$ for some $l \neq v_{t+1} + e\mathbb{Z}$ and such that $\ell(s_l x_{t+1}) < \ell(x_{t+1})$. In particular, we see that

$$(3.12) \quad s_i w_{t+1} = d_{t+1} (s_l x_{t+1}) < d_{t+1} x_{t+1} = w_{t+1}.$$

For each integer $t+1 \leq j \leq k-1$, we define

$$\tilde{w}_j = \begin{cases} w_j, & \text{if } s_i w_j > w_j; \\ s_i w_j, & \text{if } s_i w_j < w_j. \end{cases}$$

We write $\tilde{w}_j = \tilde{d}_j \tilde{x}_j$, where $\tilde{d}_j \in W/W_{v_j}$, $\tilde{x}_j \in W_{v_j}$. Then from (3.12) we see that $\tilde{w}_{t+1} = s_i w_{t+1}$, $\tilde{d}_{t+1} = d_{t+1}$ and $\tilde{x}_{t+1} = s_l x_{t+1} < x_{t+1}$. In general, for each integer $t+1 \leq j \leq k-1$, if $s_i w_j > w_j$, then by definition $\tilde{w}_j = w_j$, $\tilde{d}_j = d_j$ and $\tilde{x}_j = x_j$; while if $s_i w_j < w_j$, then as $\lambda^{(j)}$ has no removable i -nodes, it follows that $s_i d_j \notin W/W_{v_j}$ (otherwise $s_i w_j = s_i d_j x_j > d_j x_j = w_j$, a contradiction). Applying Lemma 3.9, we get that $d_j^{-1} s_i d_j = s_l$ for some $l \neq v_j + e\mathbb{Z}$ and such that $\tilde{d}_j = d_j$, $\tilde{x}_j = s_l x_j < x_j$. In particular, we see that $\tilde{w}_j W_{v_j} = w_j W_{v_j}$ for any integer $t+1 \leq j \leq k-1$. We claim that

$$(3.13) \quad w_1 \geq \cdots \geq w_t \geq \tilde{w}_{t+1} \geq \cdots \geq \tilde{w}_{k-1} \geq w_k \geq \cdots \geq w_r.$$

It is enough to show that $\tilde{w}_{t+1} \geq \cdots \geq \tilde{w}_{k-1} \geq w_k$. For each integer $t+1 \leq j \leq k-2$, there are only the following three possibilities:

Case 1. $\tilde{w}_j = w_j$. In this case, it is trivial to see that $\tilde{w}_j \geq \tilde{w}_{j+1}$.

Case 2. $\tilde{w}_j = s_i w_j < w_j$, $\tilde{w}_{j+1} = s_i w_{j+1} < w_{j+1}$. From the inequality $w_j \geq w_{j+1}$ it is also clear that $\tilde{w}_j \geq \tilde{w}_{j+1}$.

Case 3. $\tilde{w}_j = s_i w_j < w_j$, $\tilde{w}_{j+1} = w_{j+1}$. By definition, we know that $s_i w_{j+1} > w_{j+1}$. In particular, w_{j+1} has no reduced expression starting from s_i . From the inequality $s_i \tilde{w}_j = w_j \geq w_{j+1}$ it follows that $\tilde{w}_j \geq w_{j+1} = \tilde{w}_{j+1}$.

It remains to show that $\tilde{w}_{k-1} \geq w_k$. If $\tilde{w}_{k-1} = w_{k-1}$, there is nothing to prove. Assume $\tilde{w}_{k-1} = s_i w_{k-1} < w_{k-1}$. Since $\lambda^{(k)}$ is an e -core which contains addable i -nodes. We deduce that $d_k < s_i d_k \in W/W_{v_k}$. In particular, $\ell(s_i w_k) = \ell(s_i d_k x_k) = \ell(s_i d_k) + \ell(x_k) = \ell(w_k) + 1$, which implies that w_k has no reduced expression which starts from s_i . Therefore, from the inequality $s_i \tilde{w}_{k-1} = w_{k-1} \geq w_k$ we can deduce that $\tilde{w}_{k-1} \geq w_k$, as required. This completes the proof of the claim (3.13).

Since

$$\sum_{j=t+1}^{k-1} \ell(\tilde{x}_j) < \sum_{j=t+1}^{k-1} \ell(x_j),$$

we get a contradiction to our previous choice of x_j . Therefore, we must have that $w'_t \geq w'_{t+1}$. This proves the claim (3.11). Now applying Theorem 2.9, we deduce that μ^\diamond is Kleshchev, as required. This completes the proof of the proposition. \square

3.14. Corollary. *Let $\lambda := (\lambda^{(1)}, \dots, \lambda^{(r)})$ be a Kleshchev r -partition of n with respect to (e, v_1, \dots, v_r) . Let γ be an arbitrary ladder node of λ , where $\text{res}(\gamma) = i \in \mathbb{Z}/e\mathbb{Z}$. Suppose that λ is a multi-core. Then $\lambda - \{\gamma\}$ is again a Kleshchev multipartition with respect to $(e; v_1, \dots, v_r)$.*

Proof. Let $\gamma = \gamma_1 > \dots > \gamma_a$ be the ladder i -sequence in λ . It is clear that $\mu := \lambda - \{\gamma_1, \dots, \gamma_a\}$ is again a multi-core. By Proposition 3.10, we know that both λ and $\mu := \lambda - \{\gamma_1, \dots, \gamma_a\}$ are strong ladder multipartitions. Since $\{\gamma_2, \dots, \gamma_a\}$ is the ladder i -sequence of $\lambda - \{\gamma\} = \mu \sqcup \{\gamma_2, \dots, \gamma_a\}$, it follows directly from definition that $\lambda - \{\gamma\}$ is a strong ladder multipartition. Now using Lemma 3.7, we see that $\lambda - \{\gamma\}$ must be a Kleshchev multipartition with respect to $(e; v_1, \dots, v_r)$ as well. \square

To sum up, we have the following inclusion relations:

$$\left\{ \begin{array}{l} \text{Strong ladder} \\ r\text{-partitions of } n \end{array} \right\} \subseteq \left\{ \begin{array}{l} (\mathbf{Q}, e)\text{-restricted} \\ r\text{-partitions of } n \end{array} \right\} \subseteq \mathcal{K}_r(n),$$

$$\left\{ \begin{array}{l} \text{Strong ladder} \\ r\text{-partitions of } n \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Ladder} \\ r\text{-partitions of } n \end{array} \right\}.$$

We conjecture they are actually all equalities. Proposition 3.10 says that

$$\mathcal{K}_r(n) \cap \{\text{multi-cores}\} \subseteq \left\{ \begin{array}{l} \text{Strong ladder} \\ r\text{-partitions of } n \end{array} \right\}.$$

In the remaining part of this section, we shall show that our conjecture is indeed true in the case $e = 0$. In particular, we shall show that the ‘‘only if’’ part of Conjecture 2.6 is always true if $e = 0$ and the notion of ladder multipartition coincides with the notion of strong ladder multipartition in that case.

3.15. Proposition. *Suppose that $e = 0$. Then any Kleshchev multipartition in $\mathcal{K}_r(n)$ is a strong ladder multipartition and hence is (\mathbf{Q}, e) -restricted. In that case, for any ladder node γ of a strong ladder multipartition λ , $\lambda - \{\gamma\}$ is again a strong ladder multipartition.*

Proof. Since in the case $e = 0$, every multipartition is an e -core. The proposition follows immediately from Proposition 3.10 and Corollary 3.14. \square

3.16. Theorem. *Suppose that $e = 0$. Let $\lambda \in \mathcal{P}_r(n)$. Then λ is a ladder multipartition if and only if λ is a strong ladder multipartition.*

Proof. It suffices to show that if λ is a ladder multipartition, then λ is a strong ladder multipartition.

We make induction on n . By definition, λ has a ladder node γ such that $\lambda - \{\gamma\}$ is again a ladder multipartition. Write $\text{res}(\gamma) = i \in \mathbb{Z}/e\mathbb{Z}$. Suppose that $\gamma \in \lambda^{(e)}$. Since $e = 0$, γ must be the unique i -node of $[\lambda^{(e)}]$. By induction hypothesis, $\mu := \lambda - \{\gamma\}$ is a strong ladder multipartition. In particular, $\mu \in \mathcal{K}_r(n-1)$. If μ has no ladder i -node, then γ is already a ladder i -sequence of λ . In that case it follows from definition that λ is a strong ladder multipartition. Therefore, we can assume that μ does have ladder i -nodes. Let $\gamma_1 > \dots > \gamma_a$ be the ladder i -sequence of μ . By Proposition 3.15, $\mu - \{\gamma_1, \dots, \gamma_a\}$ is again a strong ladder multipartition.

Since $\gamma > \gamma_1 > \cdots > \gamma_a$ is the ladder i -sequence of $\lambda = \mu \sqcup \{\gamma, \gamma_1, \dots, \gamma_a\}$, it follows directly from definition that λ must be a strong ladder multipartition as well. This completes the proof of the theorem. \square

ACKNOWLEDGMENTS

The research was supported by an Australian Research Council discovery grant and partly by the National Natural Science Foundation of China (Project 10771014 & 10871023). The author was grateful to Professor Andrew Mathas for many helpful discussion. He also thanks the referees for their helpful comments.

REFERENCES

- [1] S. ARIKI, *On the semi-simplicity of the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$* , J. Alg., **169** (1994), 216–225.
- [2] S. ARIKI, *On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$* , J. Math. Kyoto Univ., **36** (1996), 789–808.
- [3] S. ARIKI, *On the classification of simple modules for cyclotomic Hecke algebras of type $G(m, 1, n)$ and Kleshchev multi-partitions*, Osaka J. Math., (4) **38** (2001), 827–837.
- [4] S. ARIKI, *Representations of quantum algebras and combinatorics of Young tableaux*, Translated from the 2000 Japanese edition and revised by the author, University Lecture Series, **26**, American Mathematical Society, Providence, RI, 2002.
- [5] S. ARIKI AND K. KOIKE, *A Hecke algebra of $\mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$ and construction of its irreducible representations*, Adv. Math., **106** (1994), 216–243.
- [6] S. ARIKI AND N. JACON, *Dipper–James–Murphy’s conjecture for Hecke algebras of type B*, preprint, arXiv: math/0703447, (2007).
- [7] S. ARIKI AND A. MATHAS, *The number of simple modules of the Hecke algebras of type $G(r, 1, n)$* , Math. Z., **233** (2000), 601–623.
- [8] S. ARIKI, V. KREIMAN AND S. TSUCHIOKA, *On the tensor product of two basic representations of $U_v(\widehat{\mathfrak{sl}}_e)$* , Adv. Math., **218** (2008), 28–86.
- [9] J. BRUNDAN AND A. KLESHCHEV, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math., **222** (2009), 1883–1942.
- [10] M. BROUÉ AND G. MALLE, *Zyklotomische Heckealgebren*, Astérisque, **212** (1993), 119–189.
- [11] V.V. DEODHAR, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math., **39** (1977), 187–198.
- [12] R. DIPPER, G. JAMES, AND A. MATHAS, *Cyclotomic q -Schur algebras*, Math. Z., **229** (1999), 385–416.
- [13] R. DIPPER, G. JAMES, AND E. MURPHY, *Hecke algebras of type B_n at roots of unity*, Proc. London Math. Soc., (3) **70** (1995), 505–528.
- [14] J. J. GRAHAM AND G. I. LEHRER, *Cellular algebras*, Invent. Math., **123** (1996), 1–34.
- [15] M. KASHIWARA, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J., **63**, (1991), 465–516.
- [16] M. KASHIWARA, *Base cristallines des groupes quantiques*, Cours Spéc., vol. **9**, Soc. Math. France, 2002.
- [17] P. LITTELMANN, *A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras*, Invent. Math., **116** (1994), 329–346.
- [18] P. LITTELMANN, *A plactic algebra for semisimple Lie algebras*, Adv. Math., **124** (1996), 312–331.
- [19] G. LUSZTIG, *Introduction to Quantum Groups*, Progress in Math. **110** Birkhäuser, Boston, 1990.
- [20] A. MATHAS, *Hecke algebras and Schur algebras of the symmetric group*, Univ. Lecture Notes, **15**, Amer. Math. Soc., 1999.
- [21] A. MATHAS, *Simple modules of Ariki-Koike algebras*, Proc. Pure Symp. Math., **63**, (1998), 383–396.
- [22] T. MISRA AND K.C. MIWA, *Crystal bases for the basic representations of $U_q(\widehat{\mathfrak{sl}}(n))$* , Comm. Math. Phys., **134** (1990), 79–88.

SCHOOL OF MATHEMATICS AND STATISTICS,
UNIVERSITY OF SYDNEY,
NSW 2006, AUSTRALIA

&

DEPARTMENT OF MATHEMATICS,
BEIJING INSTITUTE OF TECHNOLOGY,
BEIJING 100081, P.R. CHINA,

E-mail address: junhu303@yahoo.com.cn